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CHARACTERIZING THE MINIMALITY AND MAXIMALITY OF ORDERED LATERAL IDEALS IN ORDERED TERNARY SEMIGROUPS

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Abstract. In 1932, Lehmer [4] gave the definition of a ternary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. In this paper, we give some auxiliary results which are also necessary for our considerations and characterize the relationship between the (0-)minimal and maximal ordered lateral ideals and the lateral simple and lateral 0-simple ordered ternary semigroups analogous to the characterizations of minimal and maximal left ideals in ordered semigroups considered by Cao and Xu [2].

1. Introduction and preliminaries

In 1995, Dixit and Dewan [3] introduced and studied the properties of (quasi-, bi-, left, right) lateral ideals in ternary semigroups. In 2000, Cao and Xu [2] characterized the minimal and maximal left ideals in ordered semigroups and gave some characterizations of minimal and maximal left ideals in ordered semigroups. In 2002, Arslanov and Kehayopulu [1] characterized the minimal and maximal ideals in ordered semigroups.

The concept of the minimal and maximal (left) ideals is the really interested and important thing about ordered semigroups. Now we characterize the (0-)minimal and maximal ordered lateral ideals in ordered ternary semigroups and give some characterizations of the (0-)minimal and maximal ordered lateral ideals in ordered ternary semigroups analogous to the characterizations of the minimal and maximal left ideals in ordered semigroups considered by Cao and Xu.

Our aim in this paper is fivefold.

(1) To give the definition of an ordered ternary semigroup.
(2) To introduce the concept of lateral simple and lateral 0-simple ordered ternary semigroups.

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(3) To characterize the properties of ordered lateral ideals in ordered ternary semigroups.
(4) To characterize the relationship between the minimal and 0-minimal ordered lateral ideals and the lateral simple and lateral 0-simple ordered ternary semigroups.
(5) To characterize the relationship between the maximal ordered lateral ideals and the lateral simple and lateral 0-simple ordered ternary semigroups.

To present the main theorems we first recall the definition of a ternary semigroup which is important here.

A nonempty set T is called a *ternary semigroup* [3] if there exists a ternary operation $T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]].$$

Hence we can see that any semigroup can be considered as a ternary semigroup. For nonempty subsets A, B and C of a ternary semigroup T, let

$$[ABC] := \{[abc] : a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, then we also write $[\{a\}BC]$ as $[aBC]$, and similarly if $B = \{b\}$ or $C = \{c\}$ or $A = \{a\}$ and $B = \{b\}$ or $A = \{a\}$ and $C = \{c\}$ or $B = \{b\}$ and $C = \{c\}$. A nonempty subset S of a ternary semigroup T is called a *ternary subsemigroup* [3] of T if $[SSS] \subseteq S$.

Example 1 ([3])**.** Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.

Example 2 ([3])**.** Let $O = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_4 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $T = \{O, I, A_1, A_2, A_3, A_4\}$ is a ternary semigroup under matrix multiplication.

A partially ordered ternary semigroup T is called an *ordered ternary semigroup* if for any $x_1, x_2, x_3, x_4 \in T$,

$$x_1 \leq x_2 \text{ implies } [x_1x_3x_4] \leq [x_2x_3x_4] \text{ and } [x_4x_3x_1] \leq [x_4x_3x_2].$$

If $(T, \cdot, \leq)$ is an ordered ternary semigroup and S is a ternary subsemigroup of T, then $(S, \cdot, \leq)$ is an ordered ternary semigroup. For a subset H of an ordered ternary semigroup T, we denote $(H] := \{t \in T : t \leq h$ for some $h \in H\}$ and $H \cup a := H \cup \{a\}$ for all $a \in T$. If $H = \{a\}$, we also write $(\{a\}]$ as $(a]$. We see that $H \subseteq (H], ((H]] = (H]$. For subsets A and B of an ordered ternary semigroup T, we have $(A] \subseteq (B]$, if $A \subseteq B$ and $(A \cup B] = (A] \cup (B]$. A nonempty subset M of an ordered ternary semigroup T is called a *lateral ideal* of T if $[TMT] \subseteq M$. A lateral ideal M of an ordered ternary semigroup T is called an *ordered lateral ideal* of T if for any $b \in T$ and $a \in M, b \leq a$ implies $b \in M$. The intersection of all ordered lateral ideals of a ternary subsemigroup S of an ordered ternary semigroup T containing a nonempty subset A of S is the *ordered lateral ideal of S generated by A*. For $A = \{a\}$, let $M_S(a)$ denote the ordered lateral ideal of S generated by $\{a\}$. If $S = T$, then we also write $M_T(a)$ as $M(a)$. An element a of an ordered ternary semigroup T with at least two elements is called a *zero element* of T if $[at_1t_2] = [t_1at_2] = [t_1t_2a] = a$ for all $t_1, t_2 \in T$ and $a \leq t$ for all $t \in T$ and we denote it by 0. If T is an ordered ternary semigroup with zero, then every ordered lateral ideal of T contains a zero element. An ordered ternary semigroup T without zero is called *lateral simple* if it has no proper ordered lateral ideals. An ordered ternary semigroup T with zero is called *lateral* 0*-simple* if it has no nonzero proper ordered lateral ideals and $[TTT] \neq \{0\}$.

We shall give an example of an ordered ternary semigroup without zero which there exists a ternary subsemigroup with zero.

Example 3. Let $\mathbb{Z}$ be the set of all integers. Define multiplication on $\mathbb{Z}$ by $[xyz] = \min\{x, y, z\}$ for all $x, y, z \in \mathbb{Z}$. Then $\mathbb{Z}$ is an ordered ternary semigroup without zero under usual partial order. Let $\mathbb{N}$ be the set of all positive integers. Then $\mathbb{N}$ is a ternary subsemigroup of $\mathbb{Z}$ with a zero element 1.

For any positive integers m and n with $m \leq n$ and any elements $x_1, x_2, ..., x_{2n}$ and x_{2n+1} of a ternary semigroup T [5], we can write

$$\begin{aligned}[x_1x_2\cdots x_{2n+1}] &= [x_1\cdots x_mx_{m+1}x_{m+2}\cdots x_{2n+1}]\\ &= [x_1\cdots[[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}]\cdots x_{2n+1}].\end{aligned}$$

We shall assume throughout this paper that T stands for an ordered ternary semigroup.

The following two lemmas are also necessary for our considerations and easy to verify.

Lemma 1.1. *For any nonempty subset A of T, $([TTATT] \cup [TAT] \cup A]$ is the smallest ordered lateral ideal of T containing A.*

Furthermore, for any $a \in T$,

$$M(a) = ([TTaTT] \cup [TaT] \cup a].$$

Lemma 1.2. *For any nonempty subset A of T, $([TTATT] \cup [TAT]]$ is an ordered lateral ideal of T.*

Lemma 1.3. *If T has no zero element, then the following statements are equivalent.*

(a) *T is lateral simple.*
(b) *$([TTaTT] \cup [TaT]] = T$ for all $a \in T$.*
(c) *$M(a) = T$ for all $a \in T$.*

Proof. By Lemma 1.2 and T is lateral simple, we have $([TTaTT] \cup [TaT]] = T$ for all $a \in T$. Therefore (a) implies (b). By Lemma 1.1, we have $M(a) = ([TTaTT] \cup [TaT] \cup a] = ([TTaTT] \cup [TaT]] \cup (a] = T \cup (a] = T$. Thus (b) implies (c). Now let M be an ordered lateral ideal of T and let $a \in M$. Then $T = M(a) \subseteq M \subseteq T$, so $M = T$. Hence T is lateral simple, we have that (c) implies (a). Hence the proof is completed. $\square$

Lemma 1.4. *If T has a zero element, then the following statements hold.*

(a) *If T is lateral 0-simple, then $M(a) = T$ for all $a \in T \setminus \{0\}$.*
(b) *If $M(a) = T$ for all $a \in T \setminus \{0\}$, then either $[TTT] = \{0\}$ or T is lateral 0-simple.*

Proof. (a) Assume that T is lateral 0-simple. Then $M(a)$ is a nonzero ordered lateral ideal of T for all $a \in T \setminus \{0\}$. Hence $M(a) = T$ for all $a \in T \setminus \{0\}$.

(b) Assume that $M(a) = T$ for all $a \in T \setminus \{0\}$ and let $[TTT] \neq \{0\}$. Now let M be a nonzero ordered lateral ideal of T and put $a \in M \setminus \{0\}$. Then $T = M(a) \subseteq M \subseteq T$, so $M = T$. Therefore T is lateral 0-simple.

Therefore we complete the proof of the lemma. $\square$

The next lemma is easy to verify.

Lemma 1.5. *Let $\{M_\gamma : \gamma \in \Gamma\}$ be a family of ordered lateral ideals of T. Then $\bigcup_{\gamma\in\Gamma} M_\gamma$ is an ordered lateral ideal of T and $\bigcap_{\gamma\in\Gamma} M_\gamma$ is also an ordered lateral ideal of T if $\bigcap_{\gamma\in\Gamma} M_\gamma \neq \emptyset$.*

Lemma 1.6. *If M is an ordered lateral ideal of T and S is a ternary subsemigroup of T, then the following statements hold.*

(a) *If S is lateral simple such that $S \cap M \neq \emptyset$, then $S \subseteq M$.*
(b) *If S is lateral 0-simple such that $S \setminus \{0\} \cap M \neq \emptyset$, then $S \subseteq M$.*

Proof. (a) Assume that S is lateral simple such that $S \cap M \neq \emptyset$. Then let $a \in S \cap M$. Since M is an ordered lateral ideal of T, $(M] \subseteq M$. By Lemma 1.2, we have $([SSaSS] \cup [SaS]] \cap S$ is an ordered lateral ideal of S. This implies that $([SSaSS] \cup [SaS]] \cap S = S$. Hence $S \subseteq ([SSaSS] \cup [SaS]] \subseteq ([TTMTT] \cup [TMT]] \subseteq ([TMT]] \subseteq (M] \subseteq M$, so $S \subseteq M$.

(b) Assume that S is lateral 0-simple such that $S \setminus \{0\} \cap M \neq \emptyset$. Then let $a \in S \setminus \{0\} \cap M$. By Lemmas 1.1 and 1.4(a), we have $S = M_S(a) = ([SSaSS] \cup [SaS] \cup a] \cap S \subseteq ([SSaSS] \cup [SaS] \cup a] \subseteq ([TTaTT] \cup [TaT] \cup a] = M(a) \subseteq M$. Therefore $S \subseteq M$.

Hence the proof of the lemma is completed. $\square$

2. (0-)Minimal ordered lateral ideals

For an ordered ternary semigroup T without zero, an ordered lateral ideal M of T is called a *minimal ordered lateral ideal* of T if there is no ordered lateral ideal A of T such that $A \subset M$. Equivalently, if for any ordered lateral ideal A of T such that $A \subseteq M$, we have $A = M$. For an ordered ternary semigroup T with zero, a nonzero ordered lateral ideal M of T is called a 0-*minimal ordered lateral ideal* of T if there is no nonzero ordered lateral ideal A of T such that $A \subset M$. Equivalently, if for any nonzero ordered lateral ideal A of T such that

$A \subseteq M$, we have $A = M$. Equivalently, if for any ordered lateral ideal A of T such that $A \subset M$, we have $A = \{0\}$.

In this section, we characterize the relationship between the minimal and 0-minimal ordered lateral ideals and the lateral simple and lateral 0-simple ordered ternary semigroups.

Theorem 2.1. *If T has no zero element and M is an ordered lateral ideal of T, then the following statements hold.*

(a) *If M is a minimal ordered lateral ideal of T without zero, then either there exists an ordered lateral ideal A of M such that $[MMAMM] \neq [MAM]$ or M is lateral simple.*
(b) *If M is lateral simple, then M is a minimal ordered lateral ideal of T.*
(c) *If M is a minimal ordered lateral ideal of T with zero, then either there exists a nonzero ordered lateral ideal A of M such that $[MMAMM] \neq [MAM]$ or M is lateral 0-simple.*

Proof. (a) Assume that M is a minimal ordered lateral ideal of T without zero and $[MMAMM] = [MAM]$ for all ordered lateral ideals A of M. Now let A be an ordered lateral ideal of M. Then $[MMAMM] = [MAM] \subseteq A \subseteq M$. Define $H := \{h \in A : h \leq [m_1am_2]$ for some $m_1, m_2 \in M$ and $a \in A\}$. Then $\emptyset \neq H \subseteq A \subseteq M$. To show that H is an ordered lateral ideal of T, let $t_1, t_2 \in T$ and $h \in H$. Then $h \leq [m_1am_2] = [m_1'[m_2'am_3']m_4']$ for some $m_1, m_2, m_1', m_2', m_3', m_4' \in M$ and $a \in A$, so $[t_1ht_2] \leq [t_1[m_1am_2]t_2] = [t_1[m_1'[m_2'am_3']m_4']t_2] = [[t_1m_1'm_2']a[m_3'm_4't_2]]$. Since M is an ordered lateral ideal of T, we have $[t_1ht_2], [t_1m_1'm_2'], [m_3'm_4't_2] \in M$. Since A is an ordered lateral ideal of M, we have $[[t_1m_1'm_2']a[m_3'm_4't_2]], [t_1ht_2] \in A$. Hence $[t_1ht_2] \in H$, so $[THT] \subseteq H$. Next let $t \in T$ and $h \in H$ be such that $t \leq h$. Then $h \leq [m_1am_2]$ for some $m_1, m_2 \in M$ and $a \in A$, so $t \leq [m_1am_2] \in [MAM] \subseteq A \subseteq M$. Since M is an ordered lateral ideal of T, we get $t \in M$. Thus $t \in A$ because A is an ordered lateral ideal of M. Hence $t \in H$, so H is an ordered lateral ideal of T. Since M is a minimal ordered lateral ideal of T, $H = M$. Therefore $A = M$, so we conclude that M is lateral simple.

(b) Assume that M is lateral simple. Let A be an ordered lateral ideal of T such that $A \subseteq M$. Then $A \cap M \neq \emptyset$, it follows from Lemma 1.6(a) that $M \subseteq A$. Hence $A = M$, so M is a minimal ordered lateral ideal of T.

(c) It is similar to the proof of statement (a).

Therefore we complete the proof of the theorem. □

Using the similar proof of Theorem 2.1(a) and the Lemma 1.6(b), we have Theorem 2.2.

Theorem 2.2. *If T has a zero element and M is a nonzero ordered lateral ideal of T, then the following statements hold.*

(a) *If M is a* 0*-minimal ordered lateral ideal of T, then either there exists a nonzero ordered lateral ideal A of M such that $[MMAMM] \neq [MAM] = \{0\}$ or M is lateral* 0*-simple.*
(b) *If M is lateral* 0*-simple, then M is a* 0*-minimal ordered lateral ideal of T.*

Theorem 2.3. *If T has no zero element but it has proper ordered lateral ideals, then every proper ordered lateral ideal of T is minimal if and only if T contains exactly one proper ordered lateral ideal or T contains exactly two proper ordered lateral ideals M_1 and M_2, $M_1 \cup M_2 = T$ and $M_1 \cap M_2 = \emptyset$.*

Proof. Assume that every proper ordered lateral ideal of T is minimal. Now let M be a proper ordered lateral ideal of T. Then M is a minimal ordered lateral ideal of T. We consider the following two cases:

Case 1: $T = M(a)$ for all $a \in T \setminus M$.

If K is also a proper ordered lateral ideal of T and $K \neq M$, then $K \setminus M \neq \emptyset$ because M is a minimal ordered lateral ideal of T. Thus there exists $a \in K \setminus M \subseteq T \setminus M$. Hence $T = M(a) \subseteq K \subseteq T$, so $K = T$. It is impossible, so $K = M$. In this case, M is the unique proper ordered lateral ideal of T.

Case 2: There exists $a \in T \setminus M$ such that $T \neq M(a)$.

Then $M(a) \neq M$ and $M(a)$ is a minimal ordered lateral ideal of T. By Lemma 1.5, $M(a) \cup M$ is an ordered lateral ideal of T. By hypothesis and $M \subset M(a) \cup M$, we get $M(a) \cup M = T$. Since $M(a) \cap M \subset M(a)$ and $M(a)$ is a minimal ordered lateral ideal of T, $M(a) \cap M = \emptyset$. Now let K be an arbitrary proper ordered lateral ideal of T. Then K is a minimal ordered lateral ideal of T. We observe that $K = K \cap T = K \cap (M(a) \cup M) = (K \cap M(a)) \cup (K \cap M)$. If $K \cap M \neq \emptyset$, then $K = M$ because K and M are minimal ordered lateral ideals of T. If $K \cap M(a) \neq \emptyset$, then $K = M(a)$ because K and $M(a)$ are minimal ordered lateral ideals of T. In this case, T contains exactly two proper ordered lateral ideals M and $M(a)$, $M(a) \cup M = T$ and $M(a) \cap M = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 2.3, we have Theorem 2.4.

Theorem 2.4. *If T has a zero element and nonzero proper ordered lateral ideals, then every nonzero proper ordered lateral ideal of T is* 0*-minimal if and only if T contains exactly one nonzero proper ordered lateral ideal or T contains exactly two nonzero proper ordered lateral ideals M_1 and M_2, $M_1 \cup M_2 = T$ and $M_1 \cap M_2 = \{0\}$.*

3. Maximal ordered lateral ideals

A proper ordered lateral ideal M of T is called a *maximal ordered lateral ideal* of T if for any ordered lateral ideal A of T such that $M \subset A$, we have

$A = T$. Equivalently, if for any proper ordered lateral ideal A of T such that $M \subseteq A$, we have $A = M$.

In this section, we characterize the relationship between the maximality of ordered lateral ideals and the union $\mathcal{U}$ of all (nonzero) proper ordered lateral ideals in ordered ternary semigroups.

Theorem 3.1. *If T has no zero element but it has proper ordered lateral ideals, then every proper ordered lateral ideal of T is maximal if and only if T contains exactly one proper ordered lateral ideal or T contains exactly two proper ordered lateral ideals M_1 and M_2, $M_1 \cup M_2 = T$ and $M_1 \cap M_2 = \emptyset$.*

Proof. Assume that every proper ordered lateral ideal of T is maximal. Now let M be a proper ordered lateral ideal of T. Then M is a maximal ordered lateral ideal of T. We consider the following two cases:

Case 1: $T = M(a)$ for all $a \in T \setminus M$.

If K is also a proper ordered lateral ideal of T and $K \neq M$, then K is a maximal ordered lateral ideal of T. This implies that $K \setminus M \neq \emptyset$, so there exists $a \in K \setminus M \subseteq T \setminus M$. Thus $T = M(a) \subseteq K \subseteq T$, so $K = T$. It is impossible, so $K = M$. In this case, M is the unique proper ordered lateral ideal of T.

Case 2: There exists $a \in T \setminus M$ such that $T \neq M(a)$.

Then $M(a) \neq M$ and $M(a)$ is a maximal ordered lateral ideal of T. By Lemma 1.5, $M(a) \cup M$ is an ordered lateral ideal of T. Since $M \subset M(a) \cup M$ and M is a maximal ordered lateral ideal of T, $M(a) \cup M = T$. By hypothesis and $M(a) \cap M \subset M(a)$, we get $M(a) \cap M = \emptyset$. Now let K be an arbitrary proper ordered lateral ideal of T. Then K is a maximal ordered lateral ideal of T. We observe that $K = K \cap T = K \cap (M(a) \cup M) = (K \cap M(a)) \cup (K \cap M)$. If $K \cap M \neq \emptyset$, then $K = M$ because $K \cap M$ and M are maximal ordered lateral ideals of T. If $K \cap M(a) \neq \emptyset$, then $K = M(a)$ because $K \cap M(a)$ and $M(a)$ are maximal ordered lateral ideals of T. In this case, T contains exactly two proper ordered lateral ideals M and $M(a)$, $M(a) \cup M = T$ and $M(a) \cap M = \emptyset$.

The converse is obvious. $\square$

Using the same proof of Theorem 3.1, we have Theorem 3.2.

Theorem 3.2. *If T has a zero element and nonzero proper ordered lateral ideals, then every nonzero proper ordered lateral ideal of T is maximal if and only if T contains exactly one nonzero proper ordered lateral ideal or T contains exactly two nonzero proper ordered lateral ideals M_1 and M_2, $M_1 \cup M_2 = T$ and $M_1 \cap M_2 = \{0\}$.*

Theorem 3.3. *A proper ordered lateral ideal M of T is maximal if and only if*

(a) *$T \setminus M = \{a\}$ and $([TaT]] \subseteq M$ for some $a \in T$ or*
(b) *$T \setminus M \subseteq ([TTaTT] \cup [TaT]]$ for all $a \in T \setminus M$.*

Proof. Assume that M is a maximal ordered lateral ideal of T. Then we consider the following two cases:

Case 1: There exists $a \in T \setminus M$ such that $([TTaTT] \cup [TaT]] \subseteq M$.

Then $([TaT]] \subseteq M$. By Lemma 1.1, we have $M \cup (a] = (M \cup ([TTaTT] \cup [TaT]]) \cup (a] = M \cup (([TTaTT] \cup [TaT]] \cup (a]) = M \cup ([TTaTT] \cup [TaT] \cup a] = M \cup M(a)$. Thus $M \cup (a]$ is an ordered lateral ideal of T because $M \cup M(a)$ is an ordered lateral ideal of T. Since M is a maximal ordered lateral ideal of T and $M \subset M \cup (a]$, we have $M \cup (a] = T$. Hence $T \setminus M \subseteq (a]$. To show that $T \setminus M = \{a\}$, let $x \in T \setminus M$. Then $x \leq a$, so $([TTxTT] \cup [TxT]] \subseteq ([TTaTT] \cup [TaT]] \subseteq M$. From $([TTxTT] \cup [TxT]] \subseteq M$ and $x \in T \setminus M$, a similar argument shows that $T \setminus M \subseteq (x]$. Consequently $a \leq x$, so $x = a$. Hence $T \setminus M = \{a\}$. In this case, the condition (a) is satisfied.

Case 2: $([TTaTT] \cup [TaT]] \nsubseteq M$ for all $a \in T \setminus M$.

If $a \in T \setminus M$, then $([TTaTT] \cup [TaT]] \nsubseteq M$ and $([TTaTT] \cup [TaT]]$ is an ordered lateral ideal of T by Lemma 1.2. By Lemma 1.5, we have $M \cup ([TTaTT] \cup [TaT]]$ is an ordered lateral ideal of T and $M \subset M \cup ([TTaTT] \cup [TaT]]$. Since M is a maximal ordered lateral ideal of T, $M \cup ([TTaTT] \cup [TaT]] = T$. Hence we conclude that $T \setminus M \subseteq ([TTaTT] \cup [TaT]]$ for all $a \in T \setminus M$. In this case, the condition (b) is satisfied.

Conversely, let J be an ordered lateral ideal of T such that $M \subset J$. Then $J \setminus M \neq \emptyset$. If $T \setminus M = \{a\}$ and $([TaT]] \subseteq M$ for some $a \in T$, then $J \setminus M \subseteq T \setminus M = \{a\}$. Thus $J \setminus M = \{a\}$, so $J = M \cup a = T$. Hence M is a maximal ordered lateral ideal of T. If $T \setminus M \subseteq ([TTaTT] \cup [TaT]]$ for all $a \in T \setminus M$, then $T \setminus M \subseteq ([TTxTT] \cup [TxT]] \subseteq ([TTJTT] \cup [TJT]] \subseteq J$ for all $x \in J \setminus M$. Hence $T = (T \setminus M) \cup M \subseteq J \subseteq T$, so $J = T$. Therefore M is a maximal ordered lateral ideal of T.

Hence the theorem is now completed. □

For an ordered ternary semigroup T, let $\mathcal{U}$ denote the union of all nonzero proper ordered lateral ideals of T if T has a zero element and let $\mathcal{U}$ denote the union of all proper ordered lateral ideals of T if T has no zero element. Then it is easy to verify Lemma 3.4.

Lemma 3.4. *$\mathcal{U} = T$ if and only if $M(a) \neq T$ for all $a \in T$.*

As a consequence of Theorem 3.3 and Lemma 3.4, we obtain the next two theorems.

Theorem 3.5. *If T has no zero element, then one and only one of the following four conditions is satisfied.*

(a) *T is lateral simple.*
(b) *$M(a) \neq T$ for all $a \in T$.*
(c) *There exists $a \in T$ such that $M(a) = T$, $a \notin ([TTaTT] \cup [TaT]]$ and $([TaT]] \subseteq \mathcal{U} = T \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal ordered lateral ideal of T.*

(d) $T\setminus\mathcal{U} = \{x \in T : ([TTxTT] \cup [TxT]] = T\}$ *and* $\mathcal{U}$ *is the unique maximal ordered lateral ideal of* T.

Proof. Assume that T is not lateral simple. Then there exists a proper ordered lateral ideal of T, so $\mathcal{U}$ is an ordered lateral ideal of T. We consider the following two cases:

Case 1: $\mathcal{U} = T$.

By Lemma 3.4, we have $M(a) \neq T$ for all $a \in T$. In this case, the condition (b) is satisfied.

Case 2: $\mathcal{U} \neq T$.

Then $\mathcal{U}$ is a maximal ordered lateral ideal of T. Now assume that M is a maximal ordered lateral ideal of T. Then $M \subseteq \mathcal{U} \subset T$ because M is a proper ordered lateral ideal of T. Since M is a maximal ordered lateral ideal of T, we have $M = \mathcal{U}$. Hence $\mathcal{U}$ is the unique maximal ordered lateral ideal of T. By Theorem 3.3, we get

(i) $T \setminus \mathcal{U} = \{a\}$ and $([TaT]] \subseteq \mathcal{U}$ for some $a \in T$ or
(ii) $T \setminus \mathcal{U} \subseteq ([TTaTT] \cup [TaT]]$ for all $a \in T \setminus \mathcal{U}$.

Suppose that $T \setminus \mathcal{U} = \{a\}$ and $([TaT]] \subseteq \mathcal{U}$ for some $a \in T$. Then $([TaT]] \subseteq \mathcal{U} = T \setminus \{a\}$. Since $a \notin \mathcal{U}$, we have $M(a) = T$. If $a \in ([TTaTT] \cup [TaT]]$, then $(a] \subseteq ([TTaTT] \cup [TaT]]$. By Lemma 1.1, we have $T = M(a) = ([TTaTT] \cup [TaT] \cup a] = ([TTaTT] \cup [TaT]] \cup (a] = ([TTaTT] \cup [TaT]] = ([TTaTT]] \cup ([TaT]] \subseteq ([T\mathcal{U}T]] \cup \mathcal{U} = \mathcal{U} \subseteq T$. Thus $T = \mathcal{U}$, so it is impossible. Hence $a \notin ([TTaTT] \cup [TaT]]$. In this case, the condition (c) is satisfied.

Now suppose that $T \setminus \mathcal{U} \subseteq ([TTaTT] \cup [TaT]]$ for all $a \in T \setminus \mathcal{U}$. To show that $T \setminus \mathcal{U} = \{x \in T : ([TTxTT] \cup [TxT]] = T\}$, let $x \in T \setminus \mathcal{U}$. Then $x \in ([TTxTT] \cup [TxT]]$, so $(x] \subseteq ([TTxTT] \cup [TxT]]$. By Lemma 1.1, we have $M(x) = ([TTxTT] \cup [TxT] \cup x] = ([TTxTT] \cup [TxT]] \cup (x] = ([TTxTT] \cup [TxT]]$. Since $x \notin \mathcal{U}$, we have $M(x) = T$. Hence $T = M(x) = ([TTxTT] \cup [TxT]]$. Conversely, let $x \in T$ be such that $([TTxTT] \cup [TxT]] = T$. If $x \in \mathcal{U}$, then $M(x) \subseteq \mathcal{U} \subset T$. By Lemma 1.1, we have $M(x) = ([TTxTT] \cup [TxT] \cup x] = ([TTxTT] \cup [TxT]] \cup (x] = T \cup (x] = T$. It is impossible, so $x \in T \setminus \mathcal{U}$. Hence we conclude that $T \setminus \mathcal{U} = \{x \in T : ([TTxTT] \cup [TxT]] = T\}$. In this case, the condition (d) is satisfied.

Hence the proof of the theorem is completed. □

Using the same proof of Theorem 3.5, we have Theorem 3.6.

Theorem 3.6. *If* T *has a zero element and* $[TTT] \neq \{0\}$, *then one and only one of the following four conditions is satisfied.*

(a) T *is lateral* 0*-simple.*
(b) $M(a) \neq T$ *for all* $a \in T$.

(c) *There exists* $a \in T$ *such that* $M(a) = T, a \notin ([TTaTT] \cup [TaT]]$ *and* $([TaT]] \subseteq \mathcal{U} = T \setminus \{a\}$ *and* $\mathcal{U}$ *is the unique maximal ordered lateral ideal of* T.

(d) $T \setminus \mathcal{U} = \{x \in T : ([TTxTT] \cup [TxT]] = T\}$ *and* $\mathcal{U}$ *is the unique maximal ordered lateral ideal of* T.

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