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이펙트 집합에서 확률측도로서 시그마 모르피즘 개념

The concept of σ-morphism as a probability measure on the set of effects

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요 약

이 논문에서는 사건과 확률변수를 각각 일반화한 이펙트와 옵저버블을 소개하였다. 그리고 σ-함수의 개념을 소개하고 이펙트 집합위에서의 확률측도로서의 σ-함수의 성질을 조사하였다.

Abstract

In this paper, we introduce the concepts of effects and observable as generalizations of event and random variable, respectively. Also, we introduce the concept of σ-morphism and we investigate some results on σ-morphism as a probability measure on the set of effects.

Key Words : σ-morphism, fuzzy probability

1. 서 론

The imprecision in probability theory comes from our incomplete knowledge of the system but the random variables (measurements) still have precise values. But, in fuzzy theory, we also have an imprecision in our measurements, and so random variables must be replaced by fuzzy random variables and events by fuzzy events. In this sense, S. Gudder introduced the concepts of effects (fuzzy events), observable(fuzzy random variables) and their distribution. Also, he introduced the concept of σ-morphism on the set of effects. In this paper, we have some results on σ-morphism as a probability measure on the set of effects.

For general fuzzy theoretical background, we refer to L. A. Zadeh [5].

2. Preliminaries

Let Ω be a non-empty set. Let $\mathcal{F}$ be a σ-field of subsets of Ω, that is, a non-empty class of subsets of Ω which is closed under countable union and complementation. The basic structure is a measurable space $(\Omega, \mathcal{F})$ where Ω is a sample space consisting of *outcomes* and $\mathcal{F}$ is a σ-field of *events* in Ω corresponding to some probabilistic experiment. If μ is a probability measure on $(\Omega, \mathcal{F})$, then $\mu(A)$ is interpreted as the probability that the event A occurs. A measurable function $f : \Omega \to R$ is called a *random variable*. The *expectation* of f is defined by $E[f] = \int f d\mu$. Denoting the Borel σ-algebra on the real line R by $\mathcal{B}(R)$, the *distribution* of f is the probability measure μ_f on $(R, \mathcal{B}(R))$ given by $\mu_f(B) = \mu(f^{-1}(B))$. We interpret $\mu_f(B)$ as the probability that f has a value in the set B.

A random variable $f : \Omega \to [0, 1]$ is called an *effect* or *fuzzy event*. Thus, an effect is just a measurable fuzzy subset of Ω. The set of effects is denoted by $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{F})$. If μ is a probability measure on $(\Omega, \mathcal{F})$ and $f \in \mathcal{E}$, we define the *probability* of f to be its expectation $E[f] = \int f d\mu$. If (f_i) is an increasing sequence in $\mathcal{E}$, then by the monotone convergence theorem, $E[\lim f_i] = \lim E[f_i]$ so E is countably additive. Stated in another

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way, if a sequence (f_i) in $\mathcal{E}$ satisfies $\sum f_i \in \mathcal{E}$, then

$E[\sum f_i] = \sum E[f_i]$.

Definition 2.1 Let $\mathcal{B}$ be a σ-field of Λ. An *observable* is a map $X: \mathcal{B} \to \mathcal{E}(\Omega, \mathcal{F})$ such that $X(\Lambda) = 1_\Omega$ and if $B_i \in \mathcal{B}(i = 1, 2, 3 \cdots)$ are mutually disjoint, then $X(\cup B_i) = \sum X(B_i)$ where the convergence of the summation is pointwise.

Example 2.2 If $f: (\Lambda, \mathcal{B}) \to (\Omega, \mathcal{F})$ is a measurable function, the corresponding sharp observable $X_f: \mathcal{F} \to \mathcal{E}(\Lambda, \mathcal{B})$ is given by $X_f(B) = I_{f^{-1}(B)}$.

Definition 2.3 A *state* on $\mathcal{E}(\Omega, \mathcal{F})$ is a map $s: \mathcal{E}(\Omega, \mathcal{F}) \to [0,1]$ that satisfies $s(1_\Omega) = 1$ and if (f_i) is a sequence in $\mathcal{E}$ such that $\sum f_i \in \mathcal{E}(\Omega, \mathcal{F})$,

then $s(\sum f_i) = \sum s(f_i)$.

Definition 2.4 $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism if $\widetilde{X}(1_\Omega) = 1_\Lambda$ and if (f_i) is a sequence in $\mathcal{E}$ such that $\sum f_i \in \mathcal{E}(\Omega, \mathcal{F})$, then

$\widetilde{X}(\sum f_i) = \sum \widetilde{X}(f_i)$.

Example 2.5 Let $\Omega = [0,1]$ and $\Lambda = [1,2]$. Let $\mathcal{F}$ and be σ-fields of Ω and Λ, respectively. Define $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ by $\widetilde{X}(f)(x) = f(x-1)$.

Then $\widetilde{X}$ is a σ-morphism. In fact,
$\widetilde{X}(1_\Omega)(x) = 1_\Omega(x-1) = 1_\Lambda(x)$ and

$$\widetilde{X}(\sum f_i)(x) = \sum f_i(x-1)$$

$$= \sum \widetilde{X}(f_i)(x).$$

Example 2.6 Let $\Omega = \Lambda = [0,1]$. Let $\mathcal{F}$ and $\mathcal{B}$ be σ-fields of Ω and Λ, respectively. Define $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ by $\widetilde{X}(f)(x) = \frac{1}{2}(f(x) + f(1-x))$.

Then $\widetilde{X}$ is a σ-morphism. In fact,
$\widetilde{X}(1_\Omega)(x) = \frac{1}{2}(1_\Omega(x) + 1_\Omega(1-x)) = 1_\Lambda(x)$
and

$$\widetilde{X}(\Sigma f_i)(x) = \frac{1}{2}(\Sigma f_i(x) + \Sigma f_i(1-x))$$

$$= \sum \frac{1}{2}(f_i(x) + f_i(1-x))$$

$$= \sum \widetilde{X}(f_i)(x).$$

3. Basic properties

Theorem 3.1([1]) We have the followings.

1. If $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then $\widetilde{X}(\lambda f) = \lambda \widetilde{X}(f)$ for every $\lambda \in [0,1]$ and $f \in \mathcal{E}(\Omega, \mathcal{F})$.
2. If $s: \mathcal{E}(\Omega, \mathcal{F}) \to [0,1]$ is a state, then there exists a unique probability measure μ on $(\Omega, \mathcal{F})$ such that $s(f) = \int f d\mu$ for every $f \in \mathcal{E}(\Omega, \mathcal{F})$.

The next result shows that there exists a natural one-to-one correspondence between observables and σ-morphisms.

Theorem 3.2 ([1]) If $X: \mathcal{F} \to \mathcal{E}(\Lambda, \mathcal{B})$ is an observable, then X has a unique extension to a σ-morphism $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$. If $Y: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then $Y|_F$ is an observable.

If $f: \Lambda \to \Omega$ is a measurable function, the corresponding sharp observable $X_f: \mathcal{F} \to \mathcal{E}(\Lambda, \mathcal{B})$ is given by $X_f(B) = I_{f^{-1}(B)}$. The next result shows that $\widetilde{X_f}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ has a simple form.

Corollary 3.3 ([1]) If $f: \Lambda \to \Omega$ is a measur- able function, then $\widetilde{X}_f(g) = g \circ f$ for every $g \in \mathcal{E}(\Omega, \mathcal{F})$, where $\widetilde{X}_f$ is an extension of X_f in Example 2.2.

Theorem 3.4 If $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ is a σ-morphism, then

1. $\widetilde{X}(0_\Omega) = 0_\Lambda$.
2. $\widetilde{X}\left(\sum_{i=1}^{n} f_i\right) = \sum_{i=1}^{n} \widetilde{X}(f_i)$.
3. If $f - g \in \mathcal{E}(\Omega, \mathcal{F})$, then $\widetilde{X}(f-g) = \widetilde{X}(f) - \widetilde{X}(g)$. In particular, $\widetilde{X}(1_\Omega - g) = 1_\Lambda - \widetilde{X}(g)$.
4. If $f \le g$, then $\widetilde{X}(f) \le \widetilde{X}(g)$.
5. $\widetilde{X}(f + g - fg) = \widetilde{X}(f) + \widetilde{X}(g) - \widetilde{X}(fg)$.

Proof. (1) Let $f_1 = 1_\Omega$, $f_i = 0_\Omega (i \ge 2)$. Since
$\widetilde{X}\left(\sum_{i=1}^{\infty} f_i\right) = \widetilde{X}(1_\Omega) = 1_\Lambda$ and

$\sum_{i=1}^{\infty} \widetilde{X}(f_i) = \widetilde{X}(f_1) + \sum_{i=2}^{\infty} \widetilde{X}(f_i)$

$$= 1_{\Lambda} + \sum_{i=2}^{\infty} \widetilde{X}(0_{\Omega}),$$

we have $1_{\Lambda} = 1_{\Lambda} + \sum_{i=2}^{\infty} \widetilde{X}(0_{\Omega})$ and thus $\widetilde{X}(0_{\Omega}) = 0_{\Lambda}$.

(2) Let $f_i = 0_{\Omega}$ $(i \geq n+1)$, then $\sum_{i=1}^{\infty} f_i = \sum_{i=1}^{n} f_i$. Thus

$$\widetilde{X}\left(\sum_{i=1}^{n} f_i\right) = \widetilde{X}\left(\sum_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} \widetilde{X}(f_i) = \sum_{i=1}^{n} \widetilde{X}(f_i).$$

(3) Since
$\widetilde{X}(f) = \widetilde{X}(f-g+g) = \widetilde{X}(f-g) + \widetilde{X}(g)$, we have
$\widetilde{X}(f-g) = \widetilde{X}(f) - \widetilde{X}(g)$.

(4) Since $\widetilde{X}(g) = \widetilde{X}(g-f) + \widetilde{X}(f)$, $\widetilde{X}(g-f) \geq 0$.

(5) It is trivial.

Theorem 3.5 If $f : (\Lambda_1, \mathcal{B}_1) \to (\Lambda_2, \mathcal{B}_2)$ and $g : (\Lambda_2, \mathcal{B}_2) \to (\Lambda_3, \mathcal{B}_3)$ are measurable functions, then $\widetilde{X}_{g \circ f} = \widetilde{X}_f \circ \widetilde{X}_g$.

Proof. Note that $\widetilde{X}_f : \mathcal{E}(\Lambda_2, \mathcal{B}_2) \to \mathcal{E}(\Lambda_1, \mathcal{B}_1)$, $\widetilde{X}_g : \mathcal{E}(\Lambda_3, \mathcal{B}_3) \to \mathcal{E}(\Lambda_2, \mathcal{B}_2)$ and $\widetilde{X}_{g \circ f} : \mathcal{E}(\Lambda_3, \mathcal{B}_3) \to \mathcal{E}(\Lambda_1, \mathcal{B}_1)$. Since $\widetilde{X}_{g \circ f}(h)(\omega) = h \circ (g \circ f)(\omega)$ for every $h \in \mathcal{E}(\Lambda_3, \mathcal{B}_3)$ and $\omega \in \Lambda_1$,

$$\begin{aligned} \widetilde{X}_{g \circ f}(h)(\omega) &= h \circ (g \circ f)(\omega) \\ &= (h \circ g) \circ f(\omega) \\ &= \widetilde{X}_f(h \circ g)(\omega) \\ &= \widetilde{X}_f \circ \widetilde{X}_g(h)(\omega). \end{aligned}$$

Hence $\widetilde{X}_{g \circ f} = \widetilde{X}_f \circ \widetilde{X}_g$.

4. Main results

Theorem 4.1 Let $f : \Lambda_2 \to \Lambda_1$ be a measurable function and $\mu_i : (\Lambda_i, \mathcal{B}_i) \to [0,1]$ be a probability measure $(i = 1, 2)$. If $\mu_1 = (\mu_2)_f$, then $\mu_2 \circ \widetilde{X}_f = \mu_1$.

Proof. Let $g = \sum_{i=1}^{n} c_i I_{B_i}$ be a simple function in $\mathcal{E}(\Lambda_1, \mathcal{B}_1)$, then by Corollary 3.3,

$$\begin{aligned} \mu_2 \circ \widetilde{X}_f(g) &= \int \widetilde{X}_f(g)\, d\mu_2 \\ &= \int (g \circ f) d\mu_2 \\ &= \int \sum_{i=1}^{n} (c_i I_{B_i} \circ f)\, d\mu_2 \\ &= \int \sum_{i=1}^{n} c_i I_{f^{-1}(B_i)} d\mu_2. \end{aligned}$$

And, by the definition of expectation and distribution,

$$\begin{aligned} \int \sum_{i=1}^{n} c_i I_{f^{-1}(B_i)} d\mu_2 &= \sum_{i=1}^{n} c_i \mu_2(f^{-1}(B_i)) \\ &= \sum_{i=1}^{n} c_i (\mu_2)_f(B_i) \\ &= \sum_{i=1}^{n} c_i \mu_1(B_i) \\ &= \mu_1(g). \end{aligned}$$

Hence, $\mu_2 \circ \widetilde{X}_f(g) = \mu_1(g)$.

Now for an arbitrary $g \in \mathcal{E}(\Lambda_1, \mathcal{B}_1)$, there exists an increasing sequence of simple functions $g_n \in \mathcal{E}(\Lambda_1, \mathcal{B}_1)$ such that $\lim_{n\to\infty} g_n = g$. Then by Corollary 3.3,

$$\begin{aligned} \mu_2 \circ \widetilde{X}_f(g) &= \int \widetilde{X}_f(g)\, d\mu_2 \\ &= \int \widetilde{X}_f\left(\lim_{n\to\infty} g_n\right) d\mu_2 \\ &= \int (\lim_{n\to\infty} g_n \circ f) d\mu_2. \end{aligned}$$

By the monotone convergence theorem and the continuity of probability,

$$\begin{aligned} \int \left(\lim_{n\to\infty} g_n \circ f\right) d\mu_2 &= \lim_{n\to\infty} \int (g_n \circ f) d\mu_2 \\ &= \lim_{n\to\infty} \mu_2 \circ \widetilde{X}_f(g_n) \\ &= \lim_{n\to\infty} \mu_1(g_n) \\ &= \mu_1\left(\lim_{n\to\infty} g_n\right) \\ &= \mu_1(g). \end{aligned}$$

Therefore, $\mu_2 \circ \widetilde{X}_f = \mu_1$.

Theorem 4.2 Let $\widetilde{X} : \mathcal{E}(\Omega, \mathcal{F}) \to \mathcal{E}(\Lambda, \mathcal{B})$ be a σ-morphism. If (g_n) is an increasing sequence in $\mathcal{E}(\Omega, \mathcal{F})$ with $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$ in $\mathcal{E}(\Lambda, \mathcal{B})$.

Proof. Let $f_1 = g_1$ and $f_n = g_n - g_{n-1}$ $(n \geq 2)$. Then $f_n \in \mathcal{E}(\Omega, \mathcal{F})$ for all n and $g_n = \sum_{i=1}^{n} f_i$.

Since $g = \sum_{i=1}^{\infty} f_i$, we have

$$\begin{aligned} \widetilde{X}(g) &= \widetilde{X}\left(\sum_{i=1}^{\infty} f_i\right) \\ &= \sum_{i=1}^{\infty} \widetilde{X}(f_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \widetilde{X}(f_i) \\ &= \lim_{n\to\infty} \widetilde{X}\left(\sum_{i=1}^{n} f_i\right) \\ &= \lim_{n\to\infty} \widetilde{X}(g_n). \end{aligned}$$

Hence $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$.

Corollary 4.3 Let $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{J}) \to \mathcal{E}(\Lambda, \mathcal{B})$ be a σ-morphism. If (g_n) is a decreasing sequence in $\mathcal{E}(\Omega, \mathcal{J})$ with $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$ in $\mathcal{E}(\Lambda, \mathcal{B})$.

Theorem 4.4 Let $\widetilde{X}: \mathcal{E}(\Omega, \mathcal{J}) \to \mathcal{E}(\Lambda, \mathcal{B})$ be a σ-morphism. If (g_n) is sequence in $\mathcal{E}(\Omega, \mathcal{J})$ with $\lim_{n\to\infty} g_n = g$, then $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$ in $\mathcal{E}(\Lambda, \mathcal{B})$.

Proof. First, we prove that

$$\begin{aligned}\widetilde{X}\left(\underline{\lim}_{n\to\infty} g_n\right) &\le \underline{\lim}_{n\to\infty} \widetilde{X}(g_n)\\ &\le \overline{\lim}_{n\to\infty} \widetilde{X}(g_n)\\ &\le \widetilde{X}\left(\overline{\lim}_{n\to\infty} g_n\right).\end{aligned}$$

Let $f_n = \inf_{i \ge n} g_i$. Since (f_n) is an increasing sequence in $\mathcal{E}(\Omega, \mathcal{J})$, by Theorem 4.2, we have

$$\begin{aligned}\widetilde{X}\left(\underline{\lim}_{n\to\infty} g_n\right) &= \widetilde{X}(\sup_{n\ge1} \inf_{i \ge n} g_i)\\ &= \widetilde{X}(\sup_{n\ge1} f_n).\\ &= \widetilde{X}\left(\lim_{n\to\infty} f_n\right)\\ &= \lim_{n\to\infty} \widetilde{X}(f_n).\end{aligned}$$

Let $n \in N$. Then, for each $n \le i$, $f_n \le g_i$, we have $\widetilde{X}(f_n) \le \widetilde{X}(g_i)$ and hence $\widetilde{X}(f_n) \le \inf_{i\ge n} \widetilde{X}(g_i)$.
Therefore
$\sup_{n\ge1} \widetilde{X}(f_n) \le \sup_{n\ge1} \inf_{i \ge n} \widetilde{X}(g_i) = \underline{\lim}_{n\to\infty} \widetilde{X}(g_n)$.

But, since $\lim_{n\to\infty} \widetilde{X}(f_n) = \sup_{n\ge1} \widetilde{X}(f_n)$,
$\widetilde{X}\left(\underline{\lim}_{n\to\infty} g_n\right) \le \underline{\lim}_{n\to\infty} \widetilde{X}(g_n)$.

Similarly, $\overline{\lim}_{n\to\infty} \widetilde{X}(g_n) \le \widetilde{X}\left(\overline{\lim}_{n\to\infty} g_n\right)$.

For $g_n \in \mathcal{E}(\Omega, \mathcal{J})$, since

$$\begin{aligned}\overline{\lim}_{n\to\infty} \widetilde{X}(g_n) &\le \widetilde{X}\left(\overline{\lim}_{n\to\infty} g_n\right)\\ &= \widetilde{X}\left(\lim_{n\to\infty} g_n\right)\\ &= \widetilde{X}\ \underline{\lim}_{n\to\infty} g_n\\ &\le \underline{\lim}_{n\to\infty} \widetilde{X}(g_n).\end{aligned}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \widetilde{X}(g_n) &= \underline{\lim}_{n\to\infty} \widetilde{X}(g_n)\\ &= \overline{\lim}_{n\to\infty} \widetilde{X}(g_n)\\ &= \widetilde{X}\left(\lim_{n\to\infty} g_n\right)\\ &= \widetilde{X}(g).\end{aligned}$$

Hence $\lim_{n\to\infty} \widetilde{X}(g_n) = \widetilde{X}(g)$.

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