International Journal of Management Science
Vol 15, No 1, May 2009

# The Effect of (Q, r) Policy in Production-Inventory Systems

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(Received: April 10, 2008 / Revised: April 17, 2009 / Accepted: April 19, 2009)

**ABSTRACT**

We examine the effectiveness of the conventional (Q, r) model in managing production-inventory systems with finite capacity, stochastic demand, and stochastic order processing times. We show that, for systems with finite production capacity, order replenishment lead times are highly sensitive to loading and order quantity. Consequently, the choice of optimal order quantity and optimal reorder point can vary significantly from those obtained under the usual assumption of a load-independent lead time. More importantly, we show that for a given (Q, r) policy the conventional model can grossly under or over-estimate the actual cost of the policy. In cases where a setup time is associated with placing a production order, we show that the optimal (Q, r) policy derived from the conventional model can, in fact, be infeasible.

Keywords: (Q, r) Inventory Policy, Production-Inventory Systems, Make-To-Stock Queue

## 1. Introduction

One of the most common methods for managing inventory systems with stochastic demand is the reorder point/order quantity policy, also known as the ($Q$, $r$) policy. Under a ($Q$, $r$) policy, finished goods inventory is continuously reviewed and a new production order is placed each time inventory position (on-hand inventory+outstanding orders-backorders) falls to a reorder point $r$. Values for $Q$ and $r$ are, typically,

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selected so that inventory costs (ordering costs+holding costs+backordering costs) are minimized. Policies of the $(Q, r)$ type have been widely studied, with a rich body of literature on $(Q, r)$ models dating back to the late 50's (see, for example, Galliher *et al.* [8] and Hadley and Whittin [9]). For single item inventory systems under the standard assumptions, an optimal policy can, in fact, be shown to exist within the class of $(Q, r)$ policies [6]. The applicability of $(Q, r)$ policies has been extended to inventory systems with multiple items and multiple echelons (see for example Atkins and Iyogun [1] and Chen and Zheng [7] and the references therein).

In determining optimal values for $Q$ and $r$, it is assumed in most of the existing literature that order replenishment lead times are not sensitive to the loading of the production facility or to the order quantity. In fact, the most common assumption is a fixed positive lead time [13]. This assumption is realistic when the inventory system is fully decoupled from the production system through large inventory holding at the production facility or at subsequent stages of the supply chain (for example, when a local retailer is replenished from a large regional warehouse). It may also be realistic when non-production lead time is long (for example, when transportation lead times are significantly longer than manufacturing lead times). However, for most integrated production-inventory systems, these assumptions rarely hold. In fact, with increased emphasis on lean manufacturing, finished goods inventory at factories has become much more tightly controlled and material handling time between the production facility and the finished-goods warehouse is often minimal. Similar principles are being applied to the entire supply chain with much tighter coupling between retailers, distributors and suppliers taking place. This increased coupling means that distributors and retailers are often immediately affected by congestion and delays on the factory floor [5].

In this paper, we examine the impact of explicitly modeling production lead times on inventory costs, order replenishment lead times, and the optimal $(Q, r)$ policy. We contrast results obtained using our model with those obtained using the conventional or *standard* $(Q, r)$ model. We show that, for systems with finite production capacity, order replenishment lead times are highly sensitive to loading and order quantity. Consequently, the choice of optimal order quantity and optimal reorder point can vary significantly from those obtained under the usual assumption of a load-independent lead time. More importantly, we show that for a given $(Q, r)$ policy the conventional model can grossly under or over-estimate the actual cost of the policy depending on the value of lead time used in the standard model. Equally significant, we show that the order quantity that minimizes order replenishment lead time is not necessarily the one that minimizes total cost. In cases where a setup time as well as a setup cost, is associated with placing a production order, the optimal $(Q, r)$ policy derived from the standard model can be, in fact, infeasible. To illustrate our results we restrict our discussion initially to the case where demand is Poisson distributed since this is one of the few cases for which the optimal solution to the standard model can be easily obtained. A model for systems with general demand distributions is presented later in the paper.

## 2. The Standard (Q, r) Model

In this paper, the *standard* $(Q, r)$ model refers to a single item continuous review inventory model where demand is stationary and occurs one unit at a time with rate. An order of fixed quantity $Q$ is placed whenever inventory position drops to a fixed reorder point $r$. Orders are delivered after a fixed positive lead time $L$. All stockouts are backordered. The relevant costs include a fixed setup cost $K$ for each placed order, a holding cost $h$ per unit of held inventory per unit time, and a backordering cost $b$ per unit backordered per unit time. The long run average cost $C_S(Q, r)$ under the above conditions is given by the following (see, for example, Federgruen and Zheng [6]):

$$C_S(Q, r) = \frac{\lambda}{Q}K + \sum_{y=r+1}^{r+Q} \{h\sum_{i=0}^{y}(y-i)p_i + b\sum_{i=y+1}^{\infty}(i-y)p_i\}/Q \tag{1}$$

where $p_i$ is the probability that total demand during lead time is $i$. The above expression is exact when the inventory position in steady state is uniformly distributed on $\{r + 1, r + 2, \cdots, r + Q\}$ and is independent of demand during lead time. This condition is satisfied when demand is Poisson, in which case $p_i$ is given by $p_i = e^{-\lambda L}(\lambda L)^i / i!$. Although closed form expressions for the optimal order quantity and optimal reorder point $(Q^*, r^*)$ are difficult to obtain, an efficient algorithm for calculating $Q^*$ and $r^*$ has been proposed by Federgruen and Zheng [6].

## 3. The Production-Inventory (Q, r) Model

In contrast to the standard ($Q$, $r$) model, we consider an integrated production-inventory system, where inventory is replenished from a production facility with finite capacity. Inventory is managed according to a ($Q$, $r$) policy similar to the standard policy. That is, a production order of size $Q$ is placed each time inventory position drops to the reorder point $r$. As in the standard model, the order is delivered once all $Q$ units have been produced (it is possible to extend the model to allow for unit by unit replenishment). As in the standard model, we assume that demand occurs one unit at a time according to a Poisson process with rate $\lambda$. Although customer orders arrive individually, a production order is placed only after $Q$ customer orders are received. Therefore, the inter-arrival time of orders to the production system is Erlang with phas $Q$.

In contrast to the standard model, we let production capacity be finite with a positive processing rate $\mu$. We consider the case where unit processing times are independent, identical and exponentially distributed random variables with mean $1/\mu$. Since the processing time of an order of size $Q$ is the sum of $Q$ independent, identical and exponentially distributed random variables, order processing time is Erlang with phase $Q$. Thus, the production facility can be viewed as an $E_Q/E_Q/1$ queueing system (i.e. a single-server queueing system with Erlang inter-arrival and processing times). If we let $i$ refer to inventory level, $n$ to the number of production orders in the production system (each order is of size $Q$), $z$ to the number of customer unit demands that have yet to be ordered (an order is placed only when inventory position reaches the reorder point), and $j$ to the number of backorders, then it is easy to see that $i = \{Q + r - nQ - z\}^+$ and $j = \{nQ + z - Q - r\}^+$. The number of inventory and the number of backorders are dependent upon $Q$, $r$, $n$ and $z$. For example, if we have $Q = 3$, $r = 10$, $n = 1$, and $z = 2$, the inventory level $i$ is 8 (3+10-1(3)-2 = 8) and the number of backorders is 0.

Noting that z is uniformly distributed on {0, 1, $\cdots$, Q –1} (this follows from the fact that inventory position is uniformly distributed on $\{r + 1, r + 2, \cdots, r + Q\}$[6]). Here, we assume that $z$ and $n$ are independent. Then it is straightforward to show that:

$$P_I(i) = \frac{1}{Q} P_N\left(\left\lfloor \frac{Q+r-i}{Q} \right\rfloor\right), \quad i = 1, 2, \cdots, Q+r \tag{2}$$

and

$$P_B(j) = \frac{1}{Q} P_N\left(\left\lfloor \frac{Q+r+j}{Q} \right\rfloor\right), \quad j = 1, 2, \cdots, \infty. \tag{3}$$

where $P_I(\cdot)$, $P_B(\cdot)$, and $P_N(\cdot)$ refer, respectively, to the steady state probability distribution of inventory level, number of backorders, and number of orders in the production facility (in queue + in service). For the steady state probabilities to exist, we require the stability condition $\rho = \lambda/\mu < 1$, where $\rho$ refers to the utilization of the production facility.

From 2 and 3, we can obtain average inventory $\overline{I}$, average number of backorders $\overline{B}$ and average order replenishment lead time $\overline{L}$ as follows:

$$\overline{I} = \sum_{i=1}^{Q+r} i P_I(i), \tag{4}$$

$$\overline{B} = \sum_{i=1}^{\infty} i P_B(i), \text{ and} \tag{5}$$

$$\overline{L} = \frac{Q}{\lambda} \sum_{i=1}^{\infty} i P_N(i). \tag{6}$$

A closed form expression for the probability distribution of the number of customers in an $E_Q/E_Q/1$ system is difficult to obtain. However, the probabilities can be computed numerically (see for example Neuts [10] for a matrix-geometric approach). Noting that $P_N(i)$ converges to zero as $i$ approaches infinity, $\overline{I}$, $\overline{B}$ and $\overline{L}$ can be computed to desired accuracy.

Our cost function can now be written as:

$$C_P(Q, r) = h \sum_{i=1}^{Q+r} \frac{i}{Q} P_N\left(\left\lceil \frac{Q+r-i}{Q} \right\rceil\right) + b \sum_{i=1}^{\infty} \frac{i}{Q} P_N\left(\left\lfloor \frac{Q+r+i}{Q} \right\rfloor\right) + k\lambda/Q \tag{7}$$

In lemma 1, we show that $C_P(Q, r)$ is convex in $r$.

**Lemma 1:** *Average inventory cost $C_P(Q, r)$ is convex in $r$.*

**Proof:** In order to show that $C_P(Q, r)$ is convex in $r$, it is sufficient to show that average inventory and average number of backorders are convex in $r$. Let $\bar{I}(Q, r)$ refer to average inventory for given $Q$ and $r$. In order to show that average inventory is convex in $r$, we need to show that $\bar{I}(Q, r+1)+ \bar{I}(Q, r-1)/2 \geq \bar{I}(Q, r)$. Noting that

$$\bar{I}(Q, r)=\frac{1}{Q}\{(Q+r)P_N(0)+(Q+r-1)P_N\left(\left\lfloor\frac{1}{Q}\right\rfloor\right)+\cdots+2P_N\left(\left\lfloor\frac{Q+r-2}{Q}\right\rfloor\right)+P_N\left(\left\lfloor\frac{Q+r-1}{Q}\right\rfloor\right),$$

it is not too difficult to show that

$$\frac{\bar{I}(Q, r+1)+\bar{I}(Q, r-1)}{2}-\bar{I}(Q, r)=\frac{1}{2Q}P_N\left(\left\lfloor\frac{Q+r}{Q}\right\rfloor\right)\geq 0.$$

Hence, average inventory is convex in $r$. Similarly, we can show that

$$\frac{\bar{B}(Q, r+1)+\bar{B}(Q, r-1)}{2}-\bar{B}(Q,r)=\frac{1}{2Q}P_N\left(\left\lfloor\frac{Q+r}{Q}\right\rfloor\right)\geq 0.$$

where $\bar{B}(Q, r)$ is the average number of backorders. Thus, $\bar{B}(Q, r)$ is also convex in $r$, which completes our proof. □

With lemma 1, for each order size $Q$, the optimal reorder point can be easily obtained using standard convex optimization. The optimal order quantity can be obtained by an exhaustive search over the range of feasible order sizes (although numerical results suggest that $C_P(Q, r)$ is jointly convex in $Q$ and $r$, showing joint convexity analytically is difficult).

## 4. Comparisons and Analysis

Since the order quantity affects both the distribution of the arrival process to and the processing time at the production facility, it easy to show that lead time is sensitive to our choice of order quantity (see expression 6). Also, since congestion at the production facility is affected by both the available capacity and demand level, lead time is affected by the utilization of the production facility. The behavior of average lead time, for varying order quantities and utilization levels, is illustrated in Figure 1. As we can see, lead time is increasing in both $Q$ and $\rho$.

Because lead time is affected by the choice of $Q$, which, in turn, affects our choice of $r$, the optimal values of $Q$ and $r$ obtained respectively by the standard and the production-inventory models are generally different. Depending on the value of lead time used in the standard model, the difference in these values can be significant. This is illustrated in Figures 2 and 3, where values of $Q^*$ and $r^*$ for both models are shown for varying assumptions of fixed lead time (we use the notation ($Q_s^*$, $r_s^*$) and ($Q_p^*$, $r_p^*$) to differentiate, respectively, between the optimal solution to the standard ($Q$, $r$) model and the production-inventory model). It is interesting to note that even in cases where the optimal order quantities are the same, the reorder points are generally different. More importantly, the value of fixed lead time for which the two optimal order quantities are the same does not generally correspond to the actual lead time experienced by the production-inventory system.

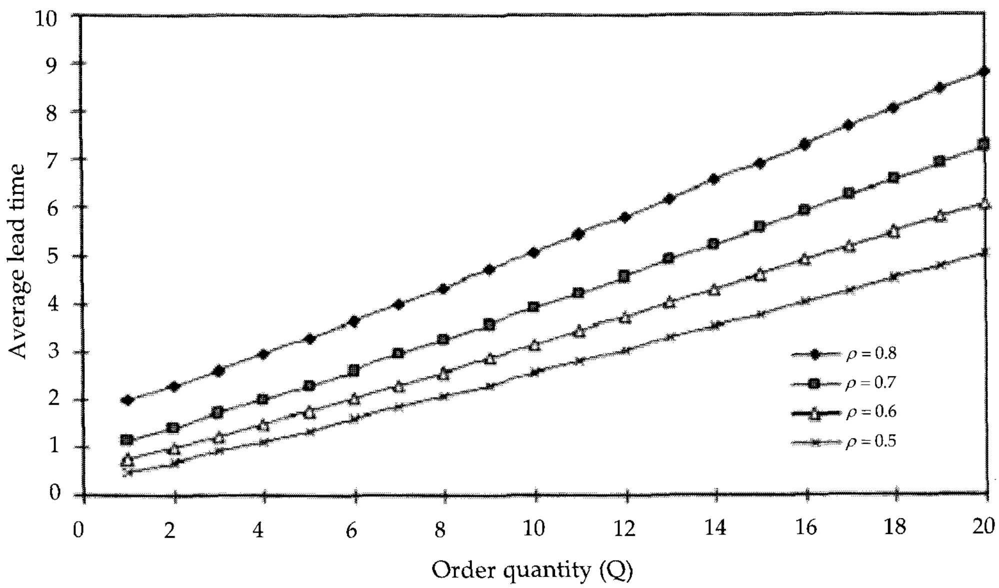


Figure 1. The effect of order quantity on average lead time
($\lambda$ = 2, $\mu$ = 4.0, 3.33, 2.8571, 2.5, $h$ = 1.0, $b$ = 1.0, $K$ = 10.0)

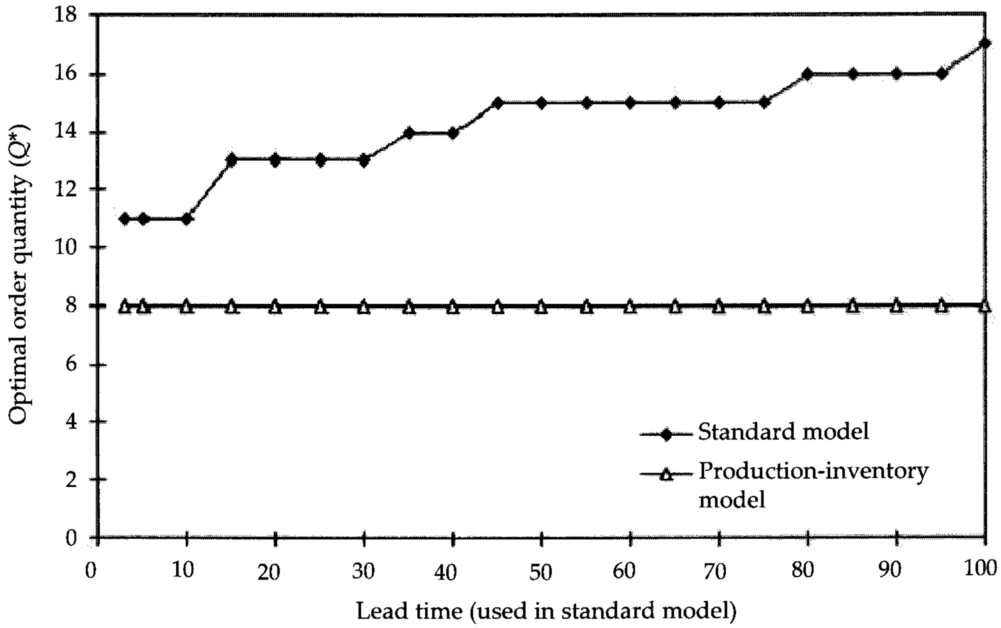


Figure 2. The effect of lead time on optimal order quantity
($\lambda = 2$, $\mu = 2.5$, $h = 1.0$, $b = 1.0$, $K = 10.0$)

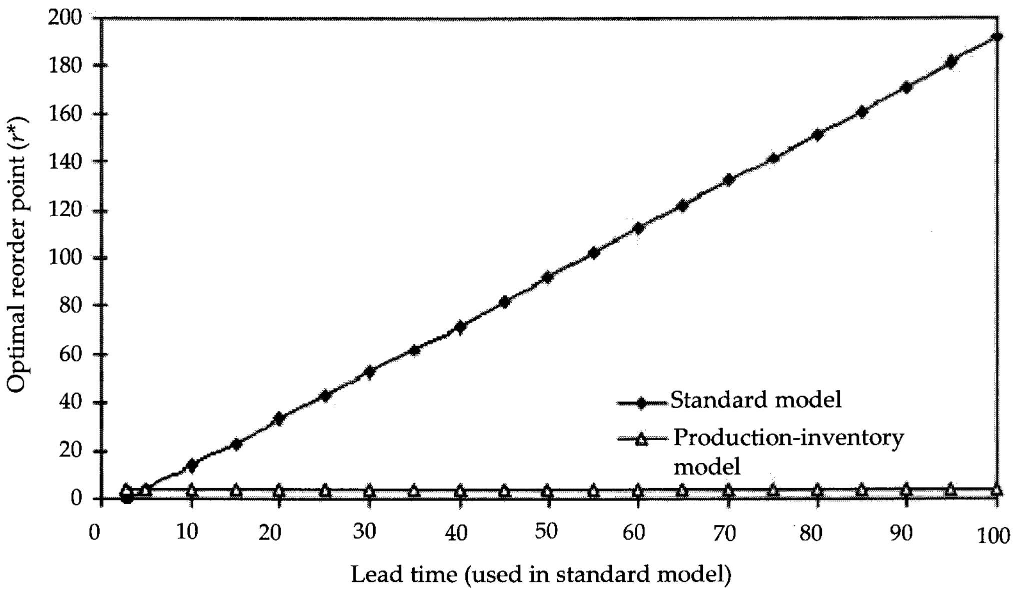


Figure 3. The effect of lead time on optimal reorder point
($\lambda = 2$, $\mu = 2.5$, $h = 1.0$, $b = 1.0$, $K = 10.0$)

The fact that $Q^*$ and $r^*$, as well as lead time, are different under the two models, means that average inventory and average number of backorders are also different. In turn, this means that the estimated costs of the optimal policies under the two models can also be very different. In fact, depending on the value of lead time used in the standard model, his difference can be quite significant. This is illustrated in Figure 4. Note that there may exist a fixed lead time for which the predicted optimal costs are the same for the two models. However, the corresponding values of $Q^*$ and $r^*$ are not necessarily the same, and the fixed lead time does not usually correspond to the actual lead time in the production inventory system.

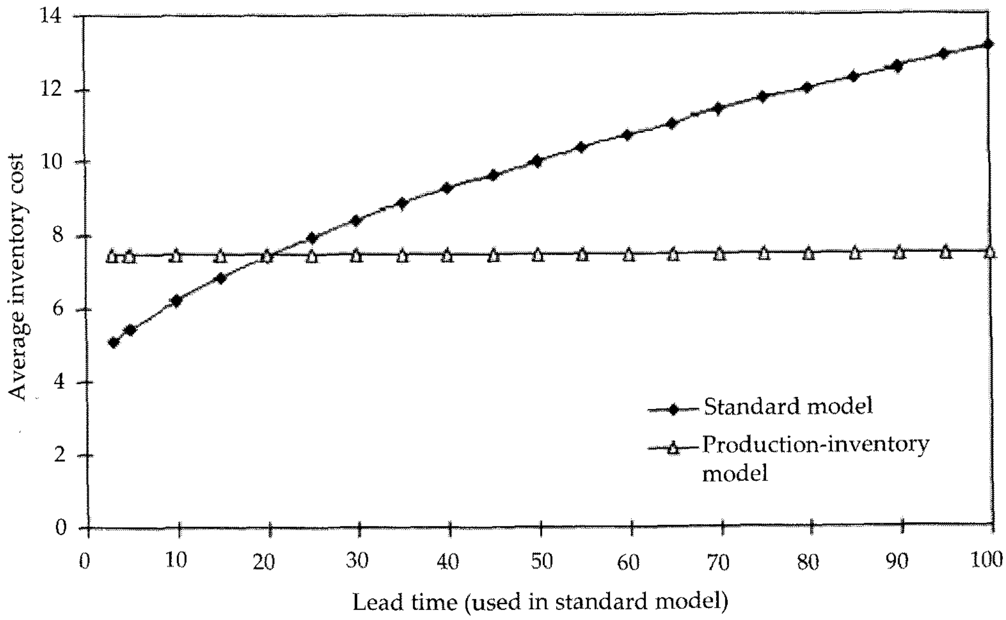


Figure 4. The effect of lead time on inventory cost
($\lambda = 2$, $\mu = 2.5$, $h = 1.0$, $b = 1.0$, $K = 10.0$)

More importantly, the standard ($Q$, $r$) model can significantly over or underestimate the true costs of implementing the corresponding optimal policy (or in fact, any ($Q$, $r$) policy). In Figure 5, we show the difference between the optimal costs, as estimated by the standard model, and the "true" costs of implementing this policy as obtained from the production inventory model. Again, we can see that this difference is dependent on the choice of lead time (in the standard model) and can be quite large.

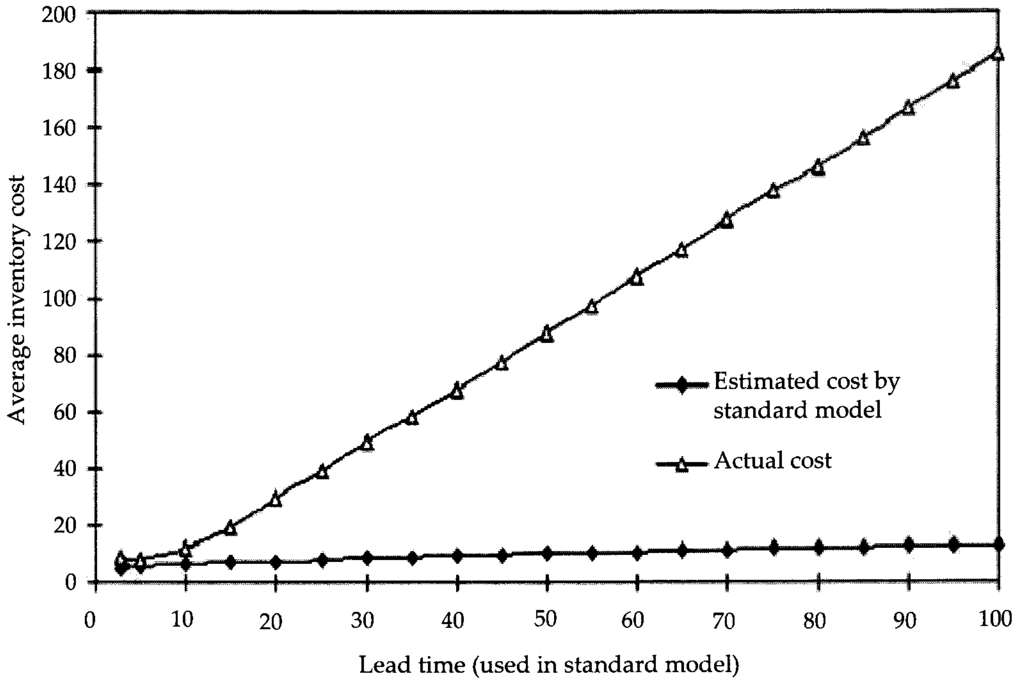


Figure 5. The Impact of lead time on estimated and actual inventory costs ($\lambda = 2$, $\mu = 2.5$, $h = 1.0$, $b = 1.0$, $K = 10.0$)

In addition to inducing errors in estimating inventory holding costs, backordering costs, and ordering costs, the standard model can lead to errors in estimating other performance metrics, such as customer order fulfillment time, fill rate, or the probability of stockout. For example, customer order fulfillment time (time from when a customer places an order to the time when the order is shipped) is given, respectively for the standard and production-inventory models, by the following:

$$\bar{F}_s = \frac{\bar{B}}{\lambda} = \sum_{y=r+1}^{r+Q} \sum_{i=y+1}^{\infty} (i-y)p_i\} / \lambda Q \quad (8)$$

$$\bar{F}_p = \frac{\bar{B}}{\lambda} \sum_{i=1}^{\infty} \frac{i}{Q} P_N(\left\lfloor \frac{Q+r+i}{Q} \right\rfloor) \quad (9)$$

Since average order fulfillment time is linearly increasing in the average number of backorders, order fulfillment is similarly sensitive to the choice of order quantity.

## 5. The Impact of Setup Times

In many production systems, a setup time is required prior to the initiation of an order. A non-zero setup time increases order-processing time and, therefore, increases order lead time. Since both average inventory and number of backorders are sensitive to lead time, the introduction of setup time affects total cost. In turn, this affects the value of the optimal order quantity and the optimal reorder point. More importantly, a non-zero setup time affects the stability condition of the production system and places a minimum requirement on order size. This can be seen by noting that with non-zero setup times the stability condition is given by the following:

$$\lambda S/Q + \lambda/\mu < 1, \tag{10}$$

which can be rewritten as:

$$Q > \lambda S/(1- \lambda/\mu ), \tag{11}$$

where $S$ is average setup time. The right-hand side of (11) represents, for a given average setup time, the minimum feasible order size (i.e., a smaller order size would result in infinitely long lead times). We should note that, although a setup cost is included in the standard ($Q$, $r$) model, the model does not account for setup time nor does it account for the relationship between order size and frequency of setups, and their joint effect on production capacity. Therefore, the standard model could lead to the choice of an infeasible order quantity.

In order to examine the effect of setup time on lead time, optimal order quantity, and optimal reorder point, we considered the case where setup times are exponentially distributed. In this case, order processing time can be represented by a generalized Erlang distribution with $Q + 1$ phases where phase 1 represents setup time. A similar approach to the one described in section 4 can be used to compute average inventory level, average number of backorders, and average lead time. These can then be used to determine optimal order quantity and reorder point.

In Figure 6, we show numerical results that illustrate the effect of order quantity on lead time for varying values of setup time. It is interesting to note that, in contrast

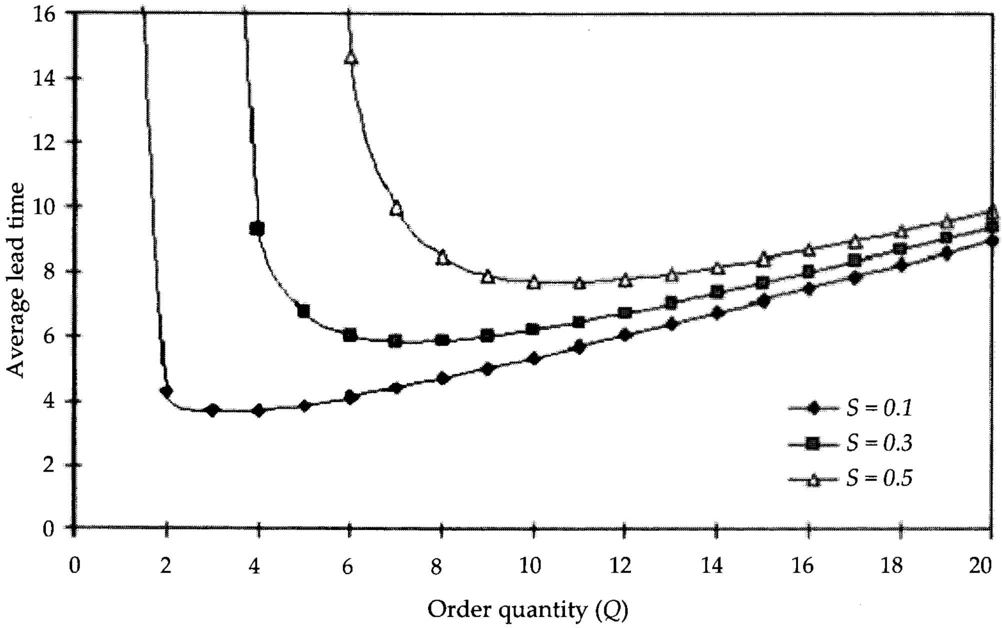


Figure 6. The effect of order quantity on average lead time ($\lambda$ = 2, $\mu$ = 2.5, $h$ = 1.0, $b$ = 1.0, $K$ = 10.0)

to the case with zero setup time, the effect of order quantity on lead time is not monotonic. Initial increases in $Q$, reduce lead time by reducing the frequency of setups. However, further increases in $Q$ result in sufficient increases in the order processing time leading to an overall increase in lead time. Similar observations were made by Karmarkar [9] and Benjaafar and Sheikhzadeh [2] for make to order systems. In Figures 7 and 8, we illustrate the impact of setup time on the value of the optimal order quantity and reorder point. As we can see, both quantities are highly sensitive to setup time. The effect of setup time is particularly pronounced when utilization is high. We should note that since these effects are ignored by the standard model, the values of $Q^*$ and $r^*$ obtained from the standard model could be again very different from those we obtain using the production-inventory model.

## 6. Extensions

We have so far limited our discussion to the case where demand is Poisson and processing and setup times are exponentially distributed. These cases allowed us to benchmark our results against those obtained by the standard model under similar assumptions using exact analysis. However, for many production-inventory systems, these distributional assumptions do not hold. Unfortunately, exact analysis of queueing and inventory systems with general distributions is difficult.

In this section, we present an approach, based on approximations, to model systems with general demand, processing time, and setup time distributions. This approach allows us to obtain closed form approximations for average inventory, average number of backorders and average lead time. These approximations are useful even for the case of Poisson demand and exponential processing times where the computational effort needed for the exact analysis could be significant for large Q.

In general, if customer orders form a renewal process and the processing times are independent and identically distributed, we can model the production system as a GI/G/1 queue. The probability distribution of the number of customers in a GI/G/1 queue can be approximated as follows (see Buzacott and Shanthikumar [4]):

$$P_N(i) \approx \begin{cases} 1-\rho & i=0 \\ \rho(1-\sigma)\sigma^{i-1}, & i=1,2,\cdots \end{cases} \tag{12}$$

where $\sigma = (\hat{N}-\rho)/\hat{N}$ and $\hat{N}$ is the approximation for the number of customers in a GI/G/1 queue. By virtue of Little's law, we have:

$$\hat{N} = (\lambda/Q)\hat{W} + \rho \tag{13}$$

where $\hat{W}$ is the approximate waiting time in a GI/G/1 system. Average waiting time can be approximated as follows [4]:

$$\hat{W} = \left\{ \frac{\rho^2(1+C_s^2)}{1+\rho^2 C_s^2} \right\} \left\{ \frac{C_a^2 + \rho^2 C_s^2}{2\lambda(1-\rho)} \right\}, \tag{14}$$

where $C_a^2$ and $C_s^2$ refer, respectively, to the squared coefficient of variation (ratio of the variance over the squared mean) in order inter-arrival and processing times at the

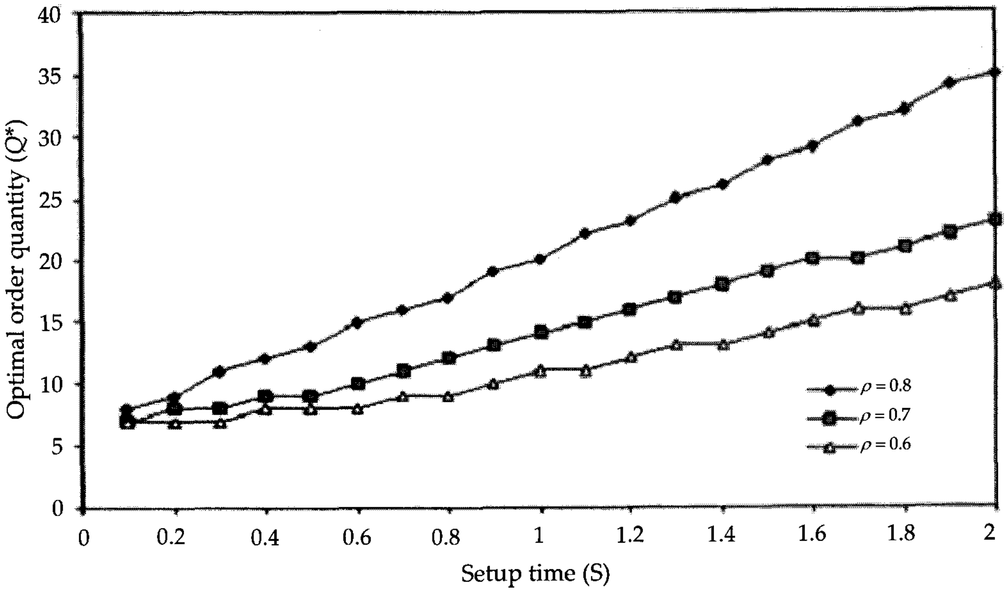


Figure 7. The effect of setup time on optimal order quantity
($\lambda$ = 2, $\mu$ = 3.333, 2.8571, 2.5, $h$ = 1.0, $b$ = 1.0, $K$ = 10.0)

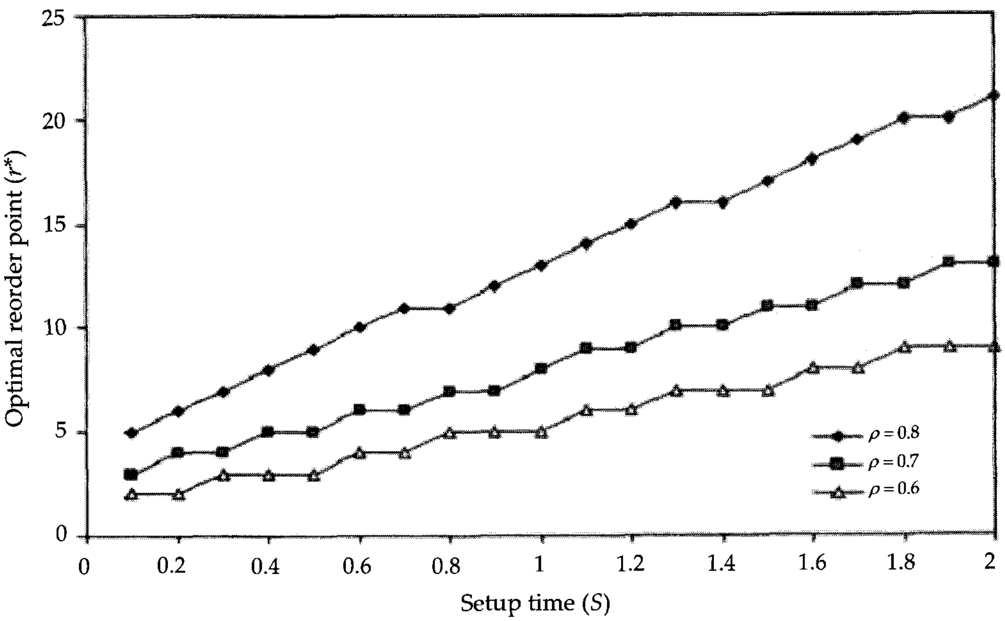


Figure 8. The effect of setup time on optimal reorder point
($\lambda$ = 2, $\mu$ = 3.333, 2.8571, 2.5, $h$ = 1.0, $b$ = 1.0, $K$ = 10.0)

production system (alternative approximations can be found in [3] and [12]). The probability distribution of inventory level and number of back orders can be obtained using expressions 2 and 3, from which average inventory, average number of backorders and average order lead time can be calculated as follows:

$$\begin{aligned}\overline{I} &= \sum_{i=1}^{Q+r} iP_I(i) = \sum_{i=r+1}^{Q+r} \frac{i}{Q}(1-\rho) + \sum_{i=1}^{r} \frac{i\rho}{Q}(1-\sigma)\sigma^{\left\lfloor \frac{r-i}{Q} \right\rfloor} \\ &= \frac{1}{2}(1-\rho)(2r+Q+1) + \frac{\rho(1-\sigma)}{Q}\sum_{i=1}^{r} i\sigma^{\left\lfloor \frac{r-i}{Q} \right\rfloor},\end{aligned} \tag{15}$$

$$\overline{B} = \sum_{i=1}^{\infty} iP_B(i) = \frac{\rho(1-\sigma)}{Q}\sum_{i=1}^{\infty} i\sigma^{\left\lfloor \frac{r+i}{Q} \right\rfloor}, \tag{16}$$

and

$$\overline{L} = \frac{\overline{N}}{\lambda} = \frac{Q}{\lambda}\sum_{i=1}^{\infty} iP_N(i) = \frac{\rho Q}{\lambda(1-\sigma)}. \tag{17}$$

Although these results are based on a series of approximations, comparisons with exact results for the $E_Q/E_Q/1$ model show that the difference between the exact and approximate estimates of average inventory and average number of backorders not to exceed 10% (for a wide range of parameter values). In most cases, we found that the optimal order quantities and reorder points to be the same as those obtained using the exact model.

## 7. Conclusion

In this paper, we demonstrated the importance of modeling the dependency of lead time on system loading in production-inventory systems. Specifically, we showed that the optimal order quantity and optimal reorder point could vary significantly from those obtained under the usual assumption of a fixed lead time. We also showed that the costs estimated using a fixed lead time (even if this fixed lead time is a good approximation of the actual lead time) could be significantly different from those actually experienced by the production-inventory system. For systems where the initiation of a production order is preceded by a setup, we found that system stability is dependent on order size. Therefore, ignoring this dependency could lead to system instability. These results highlight the fact that production and inventory systems cannot be managed (or modeled) separately. Unfortunately, the interaction between production and inventory, as noted by Buzacott and Shanthikumar [4], has gone largely under-studied. Therefore, there is a great opportunity for extending the existing literature on inventory to include this important interaction. Despite their importance in the context of the $(Q, r)$ inventory policy, our findings are limited to the cases under the assumption of independency between the number of production orders in the production system and the number of customer unit demands that have yet to be ordered.

Although we have focused on the $(Q, r)$ policy in this paper, it is worthwhile to carry out similar analysis with respect to other common policies such as periodic review policies. It is also worthwhile to extend the analysis to systems with multiple echelons. In that case, capturing the interaction between inventory and production one hand, and inventory and transportation on the other, becomes important.

## References

[1] Atkins, D. and P. Iyogun, "Periodic versus Can-Order Policies for Coordinated Multi-item Inventory Systems," *Management Science*, 34 (1988), 791-795.

[2] Benjaafar, S. and M. Sheikhzadeh, "Scheduling Policies, Batch Sizes, and Manufacturing Lead Times," *IIE Transactions*, 29 (1997), 159-166.

[3] Bitran, G. R. and D. Tirupati, "Multi-Product Queueing Networks with Deterministic Routing Decomposition Approach and the Notion of Interference," *Management Science*, 35 (1989), 851-878.

[4] Buzacott, J. and G. Shanthikumar, *Stochastic Models of Manufacturing Systems*, Prentice Hall, Englewood Cliffs, NJ, 1992.

[5] Benjaafar, S., W. L. Cooper and J. S. Kim, "Benefits of Pooling in Production-Inventory Systems," *Management Science*, 51 (2005), 548-565.

[6] Federgruen, A. and Y. S. Zheng, "An Efficient Algorithm for Computing an Optimal $(r, Q)$ Policy in Continuous Review Stochastic Inventory Systems," *Operations Research*, 40 (1992), 808-813.

[7] Chen, F. and Y. S. Zheng, "Evaluating Echelon Stock $(R, nQ)$ Policies in Serial Production/Inventory Systems with Stochastic Demand," *Management Science*, 40 (1994), 1262-1275.

[8] Galliher, H. P., P. M. Morse, and M. Somond, "Dynamics of Two Classes of Continuous Review Inventory Systems," *Operations Research*, 7 (1957), 362-384.

[9] Hadley, G. and T. Whitin, *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, NJ 1963.

[10] Karmarkar, U. S., "Lot Sizes, Lead Times and In-Process Inventories," *Management Science*, 33 (1987), 409-418.

[11] Luenberger, D. G., *Linear and Nonlinear Programming*, Addison-Wesley Publishing Company, Reading, MA, 1981.

[12] Neuts, M. F., *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, 1981.

[13] Silver, E. A., D. F. Pyke, and R. Peterson, *Inventory Management and Production Planning and Scheduling*, John Wiley, New York, NY, 1998.

[14] Whitt, W., "The Queueing Network Analyzer," *Bell Systems Technical Journal*, 63 (1983), 1911-1979.