CERTAIN SUBGROUPS OF SELF-HOMOTOPY EQUIVALENCES
OF THE WEDGE OF TWO MOORE SPACES II.

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Abstract. In the previous work [5] we have determined the group $\mathcal{E}_{\#}^{dim+r}(X)$ for $X = M(\mathbb{Z}_q, n+1) \vee M(\mathbb{Z}_q, n)$ for all integers $q > 1$. In this paper, we investigate the group $\mathcal{E}_{\#}^{dim+r}(X)$ for $X = M(\mathbb{Z} \oplus \mathbb{Z}_q, n+1) \vee M(\mathbb{Z} \oplus \mathbb{Z}_q, n)$ for all odd numbers $q > 1$.

1. Introduction

For a based topological space $X$ the set $\mathcal{E}(X)$ of homotopy classes of self-homotopy equivalences forms a group under composition of maps.

For a based, 1-connected, finite CW-complex $X$, let $\mathcal{E}_{\#}^{dim+r}(X)$ be the subgroup of homotopy classes which induces the identity on the homotopy groups of $X$ in dimensions $\leq dim X + r$. The group $\mathcal{E}(X)$ and the subgroup $\mathcal{E}_{\#}^{dim+r}(X)$ have been studied extensively. For a survey of known results and applications of $\mathcal{E}(X)$, see [2], and for a list of references on the subgroups mentioned above, see [3]. In particular, Arkowitz and Maruyama examined $\mathcal{E}_{\#}^{dim+r}(X)$ for Moore spaces $X$ in [4], and we have extended their computation to the case $X = M(\mathbb{Z}_q, n+1) \vee M(\mathbb{Z}_q, n)$ for all positive integers $q > 1$ in [5].

In this paper we calculate the subgroup $\mathcal{E}_{\#}^{dim+r}(X)$ for $X$ the wedge of two Moore spaces $X = M(\mathbb{Z} \oplus \mathbb{Z}_q, n+1) \vee M(\mathbb{Z} \oplus \mathbb{Z}_q, n)$ for all odd numbers $q > 1$.

We fix some notations and conventions. We shall work in the category of spaces with base points and maps preserving the base points. If $f : X \rightarrow Y$ is a map, then $f_* : H_n(X) \rightarrow H_n(Y)$ and $f_\# : \pi_n(X) \rightarrow \pi_n(Y)$ denote the induced homology and homotopy homomorphism in dimension $n$, respectively. In this paper we do not distinguish notationally between a map $X \rightarrow Y$ and its homotopy class in $[X, Y]$.

Received by the editors May 31, 2008. Revised February 16, 2009. Accepted May 17, 2009.

2000 Mathematics Subject Classification. 55P10.

Key words and phrases. self-homotopy equivalences, Moore spaces.
For a finitely generated abelian group $G$ write $G = F \oplus T$, to indicate that $F$ is a free part of $G$ and $T$ is the torsion subgroup of $G$. Consequently $M(G, n) = M(F, n) \vee M(T, n)$. If $G$ is free-abelian, $M(G, n)$ is just a wedge of $n$-spheres. Note that when $G$ is finitely-generated, $M(G, n)$ is a finite CW-complex of dim $n$ if $G$ is free-abelian and of dim $n + 1$ if $G$ is not free-abelian. Since $M(G, n)$ is a double suspension, the set of homotopy classes $[M(G, n), X]$ can be given abelian group structure with binary operation $'+'$.

Finally, if $A$ is an abelian group, we write

$$
\bigoplus_{r} A = A \oplus \cdots \oplus A \ (r \text{ summands}).
$$

We also use $'\otimes'$ to denote cartesian product of sets.

2. Preliminaries

We begin with some well-known results. The first is the universal coefficient theorem for homotopy with coefficients.

**Theorem 2.1** ([6, p. 30]). There is a short exact sequence:

$$
0 \to \text{Ext}(G; \pi_{n+1}(X)) \to \pi_n(G; X) \to \text{Hom}(G; \pi_n(X)) \to 0,
$$

where $\lambda : \pi_n(G; X) \to \text{Hom}(G; \pi_n(X))$ is the homomorphism defined by $\lambda(f) = f_{\# n} : G \approx \pi_n(M(G, n)) \to \pi_n(X)$.

**Proposition 2.2.** If $X$ is $(k - 1)$-connected and $Y$ is $(l - 1)$-connected, $k, l \geq 2$, and $\dim P < k + l - 1$, then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection

$$
[P, X \vee Y] \to [P, X] \otimes [P, Y].
$$

Proposition 2.2 is a consequence of [7, p. 405] since the inclusion $X \vee Y \to X \times Y$ is a $(k + l - 1)$-equivalence.

We consider abelian groups $G_1$ and $G_2$ and Moore spaces $Y_1 = M(G_1, n_1)$ and $Y_2 = M(G_2, n_2)$ . Let $X = Y_1 \vee Y_2 = M(G_1, n_1) \vee M(G_2, n_2)$ and denote by $i_j : Y_j \to X$ the inclusions and by $p_j : X \to Y_j$ the projections, $j = 1, 2$. If $f : X \to X$, then we define $f_{jk} : Y_k \to Y_j$ by $f_{jk} = p_j f i_k$ for $j, k = 1, 2$.

**Proposition 2.3.** The function $\theta$ which assigns to each $f \in [X, X]$, the $2 \times 2$ matrix

$$
\theta(f) = \begin{pmatrix}
    f_{11} & f_{12} \\
    f_{21} & f_{22}
\end{pmatrix},
$$

where $f_{jk} \in [Y_1, Y_2]$, is a bijection. In addition,
(1) $\theta(f + g) = \theta(f) + \theta(g)$, so $\theta$ is an isomorphism $[X, X] \rightarrow \bigoplus_{j,k=1,2}[Y_1, Y_2]$.  
(2) $\theta(fg) = \theta(f)\theta(g)$, where $fg$ denotes composition in $[X, X]$ and $\theta(f)\theta(g)$ denotes matrix multiplication.  
(3) If $\alpha_r : \pi(Y_1) \oplus \pi_r(Y_2) \rightarrow \pi_r(Y_1 \oplus Y_2)$ and $\beta_r : \pi(Y_1) \vee \pi_r(Y_2) \rightarrow \pi_r(Y_1 \oplus Y_2)$ are the homomorphisms induced by the inclusions and projections, respectively, then

$$
\beta_r f_\# \alpha_r(x, y) = (f_{11\#r}(x) + f_{12\#r}(x), f_{21\#r}(x) + f_{22\#r}(x)),
$$

for $x \in \pi_r(Y_1)$ and $y \in \pi_r(Y_2)$.

The homotopy groups $\pi_{n+k}(M(G,n))$ and the groups of homotopy classes $[M(G, n+k), M(G,k)]$ have been determined by Araki and Toda [1] when $G$ is the cyclic group $\mathbb{Z}_q, (q > 1)$ in stable homotopy category. They obtained the following results. See [1] if you want to know that in details.

**Proposition 2.4** ([1]). Let $q > 1$ be an odd number. Then

1. $\pi_n(M(\mathbb{Z}_q, n)) \approx \mathbb{Z}_q$.
2. $\pi_{n+1}(M(\mathbb{Z}_q, n)) = 0$.
3. $\pi_{n+2}(M(\mathbb{Z}_q, n)) = 0$.
4. $\pi_{n+3}(M(\mathbb{Z}_q, n)) \approx \mathbb{Z}_{(q,24)}$.

**Proposition 2.5** ([1]). Let $q > 1$ be an odd number. Then

1. $[(M(\mathbb{Z}_q, n - 1)), (M(\mathbb{Z}_q, n))] \approx \mathbb{Z}_q$.
2. $[(M(\mathbb{Z}_q, n)), (M(\mathbb{Z}_q, n))] \approx \mathbb{Z}_q$.
3. $[(M(\mathbb{Z}_q, n + 1)), (M(\mathbb{Z}_q, n))] = 0$.
4. $[(M(\mathbb{Z}_q, n + 2)), (M(\mathbb{Z}_q, n))] \approx \mathbb{Z}_{(q,24)}$.

**Proposition 2.6.** Let $q > 1$ be an odd number. Then

1. $[(M(\mathbb{Z}_q, n - 2)), S^n)] = 0$.
2. $[(M(\mathbb{Z}_q, n - 1)), S^n)] = 0$.
3. $[(M(\mathbb{Z}_q, n)), S^n)] = 0$.
4. $[(M(\mathbb{Z}_q, n + 1)), S^n)] = 0$.
5. $[(M(\mathbb{Z}_q, n + 2)), S^n)] \approx \mathbb{Z}_{(q,24)}$.

**Proof.** (1) We know that $[(M(\mathbb{Z}_q, n - 2)), S^n)] \approx \pi_{n-2}(\mathbb{Z}_q, S^n)$.

By Theorem 2.1, we obtain the short exact sequence:

$$0 \rightarrow Ext(\mathbb{Z}_q, \pi_{n-1}(S^n)) \rightarrow \pi_{n-2}(\mathbb{Z}_q, S^n) \rightarrow Hom(\mathbb{Z}_q, \pi_{n-2}(S^n)) \rightarrow 0.$$

And $Ext(\mathbb{Z}_q, \pi_{n-1}(S^n)) = 0$ and $Hom(\mathbb{Z}_q, \pi_{n-2}(S^n)) = 0$. 
Therefore \([(M(Z_q, n - 2)), S^n)] = 0.

(2) We know also that

\[ [(M(Z_q, n - 1)), S^n)] \cong \pi_{n-1}(Z_q, S^n), \text{Ext}(Z_q, \pi_n(S^n)) \cong Z_q \]

and

\[ \text{Hom}(Z_q, \pi_{n-1}(S^n)) = 0. \]

By use of the short exact sequence in Theorem 2.1, \([(M(Z_q, n - 1)), S^n)] = 0.

(3) Since \( q \) is an odd number,

\[ \text{Ext}(Z_q, \pi_{n+1}(S^n)) \cong \text{Ext}(Z_q, Z_2) = 0 \]

and

\[ \text{Hom}(Z_q, \pi_n(S^n)) = 0. \]

We obtain \([(M(Z_q, n)), S^n)] = 0.

We can show the rest of the proof by the same manner. \( \square \)

We also need the following theorem.

**Theorem 2.7** ([4]). For the Moore space \( X = M(G, n) \),

1. \( \mathcal{E}_{\#}^{\dim}(X) \cong \bigoplus_{(r+s)s} Z_2 \), where \( r \) is the rank of \( G \) and \( s \) is the number of 2-torsion summands in \( G \).

2. \( \mathcal{E}_{\#}^{\dim + 1}(X) = 1 \) if \( n > 3 \).

3. MAIN THEOREM

In this section we determine the group \( \mathcal{E}_{\#}^{\dim + r}(X) \) for \( X = M(Z \oplus Z_q, n + 1) \lor M(Z \oplus Z_q, n), n \geq 5 \) and \( q > 1 \): odd.

We let \( M_1 = M(Z_q, n + 1) = S^{n+1} \cup_q e^{n+2} \) and \( M_2 = M(Z_q, n) = S^n \cup_q e^{n+1} \). We know that \( M(Z \oplus Z_q, n + 1) = M(Z, n + 1) \lor M(Z_q, n + 1) = S^{n+1} \lor (S^{n+1} \cup_q e^{n+2}) \) and \( M(Z \oplus Z_q, n) = M(Z, n) \lor M(Z_q, n) = S^n \lor (S^n \cup_q e^{n+1}) \). And we set \( Y_1 = S^{n+1} \lor M_1 \) and \( Y_2 = S^n \lor M_2 \). Then we can denote \( X = Y_1 \lor Y_2 \). We now let \( f \in [X, X] \) and use the notation of Section 2 so that \( f_{jk} = p_j f_{jk} \in [Y_k, Y_j] \) for \( j, k = 1, 2 \). By Proposition 2.2 and Proposition 2.3, we can identify \( f \in \mathcal{E}(X) \) with the \( 2 \times 2 \) matrix

\[ \theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} , \]

where \( f_{11} \in \mathcal{E}(Y_1) \), \( f_{12} \in [Y_2, Y_1] \), \( f_{21} \in [Y_1, Y_2] \), \( f_{22} \in \mathcal{E}(Y_2) \). The group structure in \( \mathcal{E}(X) \) is then given by matrix multiplication.

**Lemma 3.1.** \( \pi_{n+k}(Y_1 \lor Y_2) \cong \pi_{n+k}(Y_1) \oplus \pi_{n+k}(Y_2) \) for \( k = 0, 1, 2, 3, 4 \).
Proof. The Moore spaces $Y_1$ and $Y_2$ are $n$-connected and $(n - 1)$-connected, respectively and $n \geq 5$.

By Proposition 2.1, $[S^{n+k}, Y_1 \vee Y_2] \approx [S^{n+k}, Y_1] \oplus [S^{n+k}, Y_2]$, for $k < n$. □

From Lemma 3.1, it is clear that

$$f_{\#n+k}(x, y) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \forall x \in \pi_{n+k}(Y_1), \forall y \in \pi_{n+k}(Y_2),$$

$k = 0, 1, 2, 3, 4$.

The following theorem is the main result in this paper.

**Theorem 3.2.** For the space $X = M(Z \oplus Z_q, n + 1) \vee M(Z \oplus Z_q, n)$,

$$\mathcal{E}_{\#}^{\dim X} \approx \mathcal{E}_{\#}^{\dim +1 X} \approx Z_q \oplus Z_q \ (\forall q > 1: \text{odd}).$$

Proof. By Proposition 2.2, $[X, X] = [Y_1, Y_1] \oplus [Y_1, Y_2] \oplus [Y_2, Y_1] \oplus [Y_2, Y_2]$. Now $G = Z \oplus Z_q$ has no 2-torsion, $\dim X = \dim Y_1 = n + 2$ and $\dim Y_2 = n + 1$. By Theorem 2.7, $E_{\#}^{\dim X}(Y_1) = 1$ and $E_{\#}^{\dim X}(Y_2) = 1$. Let $f \in E_{\#}^{\dim X}(X)$ be given a $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then $f_{11} = 1$ and $f_{22} = 1$. So it suffices that we consider just $f_{12}$ and $f_{21}$.

First $f_{12} \in [Y_2, Y_1] \approx [S^n, S^{n+1}] \oplus [M_2, S^{n+1}] \oplus [S^n, M_1] \oplus [M_2, M_1]$. So we can identify $f_{12} \in [Y_2, Y_1]$ with the $2 \times 2$ matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, where $g_{11} \in [S^n, S^{n+1}]$, $g_{12} \in [M_2, S^{n+1}]$, $g_{21} \in [S^n, M_1]$, $g_{22} \in [M_2, M_1]$. Then we see that $g_{11} = 0$ and $g_{21} = 0$ obviously.

Now for any element $g_{12} \in [M_2, S^{n+1}]$, $g_{12}(\pi_k(M_2)) = 0, \forall k \leq \dim X$. Because $\pi_k(S^{n+1}) = 0, \forall k \leq n$ and $\pi_k(M_2) = 0$, $k = n + 1, n + 2$.

And for any element $g_{22} \in [M_2, M_1]$, $g_{22}(\pi_k(M_2)) = 0, \forall k \leq \dim X$. Because $\pi_k(M_1) = 0, \forall k \leq n$ and $\pi_k(M_2) = 0$, $k = n + 1, n + 2$.

By the fact of $[M_2, S^{n+1}] \approx Z_q = < \pi >$ and $[M_2, M_1] \approx Z_q = < i \pi >$, we obtain

$$f_{12} \in \{ \begin{pmatrix} 0 & g_{12} \\ 0 & g_{22} \end{pmatrix} \ | g_{12} \in < \pi >, \ g_{22} \in < i \pi > \} \approx Z_q \oplus Z_q.$$  

Second $f_{21} \in [Y_1, Y_2] \approx [S^{n+1}, S^n] \oplus [M_1, S^n] \oplus [S^{n+1}, M_2] \oplus [M_1, M_2]$. So we can identify $f_{21} \in [Y_1, Y_2]$ with the $2 \times 2$ matrix $\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, where $h_{11} \in [S^{n+1}, S^n]$, $h_{12} \in [M_1, S^n]$, $h_{21} \in [S^{n+1}, M_2]$, $h_{22} \in [M_1, M_2]$. By Proposition 2.4, 2.5 and 2.6, $[M_1, S^n] = [S^{n+1}, M_2] = [M_1, M_2] = 0$. So $h_{12} = h_{21} = h_{22} = 0$.

Now $\eta_{\#n+1} : \pi_{n+1}(S^{n+1}) \rightarrow \pi_n(S^n)$, $\eta_{\#n+1}(1) = \eta \circ 1 = \eta \neq 0$. So $h_{11} = 0$.

Finally, $f_{21} = 0$. 

Therefore
\[ \mathcal{E}_\#^{\dim}(X) \approx \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & g_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \} \} |g_{12}, g_{22} \in \pi, i\pi \} \approx \mathbb{Z}_q \oplus \mathbb{Z}_q. \]

By Proposition 2.3, \( \pi_{n+3}(M_2) \approx \mathbb{Z}_{(q,24)} = \langle iv \rangle. \pi_{n+3}(iv) = \pi iv = 0 \) and \( (i\pi)_{n+3}(iv) = i\pi iv = 0. \) So \( \mathcal{E}_\#^{\dim}(X) \approx \mathcal{E}_\#^{\dim+1}(X). \]

We denote by \( \mathcal{Z}(X) \) the subset of \([X, X]\) consisting of all homotopy classes which induces the trivial homomorphism on homotopy groups in dimensions less than or equal to \( n. \)

**Corollary 3.3.** For the space \( X = Y_1 \vee Y_2 \) and \( q: \text{odd}, \)
\[ \mathcal{Z}(X) \approx \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & g_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \} \} |g_{12}, g_{22} \in \pi, i\pi \} \}

**Proof.** Consider the bijection map \( T: \mathcal{E}_\#^{\dim}(X) \rightarrow \mathcal{Z}(X) \) defined by the translation by the identity map, that is, \( T(f) = f - 1. \)

**References**