EXISTENCE OF SOLUTIONS OF QUASILINEAR
INTEGRODIFFERENTIAL EVOLUTION EQUATIONS
IN BANACH SPACES

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ABSTRACT. We prove the local existence of classical solutions of quasi-
linear integrodifferential equations in Banach spaces. The results are
obtained by using fractional powers of operators and the Schauder fixed-
point theorem. An example is provided to illustrate the theory.

1. Introduction

The problem of existence of solutions of quasilinear evolution equations in
Banach spaces has been studied by many authors [1, 2, 5-7, 12, 15-24, 26],
Crandall and Souganidis [8] have proved the existence, uniqueness and con-
tinuous dependence of a continuously differentiable solution to the quasilinear

\begin{align*}
    \frac{d}{dt}u(t) + A(u(t))u(t) &= 0, \quad 0 < t \leq a, \\
    u(0) &= u_0
\end{align*}

under the assumptions similar to one considered by Kato [14]. Pazy [21] con-
sidered the following quasilinear equation

\begin{align*}
    \frac{d}{dt}u(t) + A(t, u(t))u(t) &= 0, \quad 0 < t \leq a, \\
    u(0) &= u_0
\end{align*}

and discussed the mild and classical solutions by using a fixed point argument.
The same problem has been studied to the nonhomogeneous quasilinear evolu-

tion equation

\begin{align*}
    \frac{d}{dt}u(t) + A(t, u(t))u(t) &= f(t, u), \quad 0 < t \leq a, \\
    u(0) &= u_0
\end{align*}

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\[ u'(t) + A(t, u)u(t) = \int_0^t a(t-s)k(s, u(s))ds + f(t), \quad 0 \leq t \leq a, \]
\[ u(0) = u_0 \]
by using the method of lines. He also established a local classical solution for the same equation in [4]. Oka [19] and Oka and Tanaka [20] investigated the existence of classical solutions of abstract quasilinear integrodifferential equations. An equation of this type occurs in a nonlinear conservation law with memory
\[ u_t(t, x) + \Psi(u(t, x))_x = \int_0^t b(t-s)\Psi(u(t, x))_x ds + f(t, x), t \in [0, a], x \in \mathbb{R}, \]
\[ u(0, x) = \phi(x), x \in \mathbb{R}. \]

It is interesting to investigate the existence problem for these type of equations in Banach spaces. The aim of this paper is to study the existence of solutions of quasilinear integrodifferential equations in Banach spaces by using fractional powers of operators and the Schauder fixed-point theorem. The results generalize the results of [4, 13, 21, 25].

2. Preliminaries

Consider the initial value problem
\[ x'(t) + A(t)x(t) = f(t) \quad 0 \leq s < t \leq a \]
\[ x(s) = y \]
with the following assumptions:

(P1) The domain \( D(A(t)) = D \) of \( A(t) \), \( 0 \leq t \leq a \) is dense in \( X \) and independent of \( t \);

(P2) For \( t \in [0, a] \), the resolvent \( R(\lambda; A(t)) = (\lambda I - A(t))^{-1} \) of \( A(t) \) exists for all \( \lambda \) with \( \text{Re} \lambda \leq 0 \) and there is a constant \( C \) such that
\[ \|R(\lambda; A(t))\| \leq C|\lambda|^{-1} \quad \text{for} \ \text{Re} \lambda \leq 0, t \in [0, a]; \]

(P3) There exist constants \( L \) and \( 0 < \alpha \leq 1 \) such that
\[ \|(A(t) - A(s))A(\tau)\| \leq L|t-s|^\alpha \quad \text{for} \ t, s, \tau \in [0, a]. \]

Theorem 2.1. Under the assumptions (P1) – (P3) there is a unique evolution system \( U(t, s) \) on \( 0 \leq s \leq t \leq a \), satisfying
\[ (i) \quad \|U(t, s)\| \leq M_0 \quad \text{for} \ 0 \leq s \leq t \leq a \]
(ii) For \(0 \leq s \leq t \leq a\), \(U(t, s) : X \to D\) and \(t \to U(t, s)\) is strongly differentiable in \(X\). The derivative \(\frac{\partial}{\partial t}U(t, s) \in B(X)\) and it is strongly continuous on \(0 \leq s < t \leq a\). Moreover,

\[
\frac{\partial}{\partial t}U(t, s) + A(t)U(t, s) = 0 \quad \text{for} \quad 0 \leq s < t \leq a,
\]

and

\[
\|\frac{\partial}{\partial t}U(t, s)\| = \|A(t)U(t, s)\| \leq M_0(t - s)^{-1}
\]

and

\[
\|A(t)U(t, s)A^{-1}(s)\| < M_0 \quad \text{for} \quad 0 < s < t < a.
\]

(iii) For every \(v \in D\) and \(t \in [0, a]\), \(U(t, s)v\) is differentiable with respect to \(s\) on \(0 \leq s \leq t \leq a\) and

\[
\frac{\partial}{\partial t}U(t, s)v = U(t, s)A(s)v.
\]

Note that \((P_2)\) and the fact that \(D\) is dense in \(X\) imply that for every \(t \in [0, a]\), \(-A(t)\) is the infinitesimal generator of an analytic semigroup. We define the classical solutions of (1) as functions \(x : [s, a] \to X\) which are continuous for \(s \leq t \leq a\), continuously differentiable for \(s < t \leq a\), \(x(s) = y\) and \(x'(t) + A(t)x(t) = f(t)\) holds for \(s < t \leq a\). We will call a function \(x(t)\) a solution of the initial value problem (1) if it is a classical solution of this problem.

**Theorem 2.2.** Let \(A(t), 0 \leq t \leq a\) satisfy the conditions \((P_1) - (P_3)\) and let \(U(t, s)\) be the evolution system in Theorem 2.1. If \(f\) is Holder continuous on \([0, a]\), then the initial value problem (1) has, for every \(y \in X\), a unique solution \(x(t)\) given by

\[
x(t) = U(t, s)y + \int_s^t U(t, \tau)f(\tau)d\tau.
\]

The proofs of the above theorems can be found in \([9, 21]\).

Now consider the quasilinear integrodifferential evolution equations of the form

\[
x'(t) + A(t, x(t))x(t) = f(t, x(t)) + \int_0^t k(t, s)g(s, x(s))ds,\]

where \(-A(t, x)\) is the infinitesimal generator of an analytic semigroup in a Banach space \(X\). The nonlinear operators \(f, g : J \times X \to X\) are uniformly bounded and continuous in all of its arguments and \(k : \Delta \to J\) is continuous. Here \(J = [0, a]\) and \(\Delta = \{(t, s) : 0 \leq s \leq t \leq a\}\). Throughout the paper \(C_i's\) are positive constants.

Let \(r > 0\) and take \(B_r = \{x \in X : \|x\| < r\}\), and assume the following conditions:
(i) The operator $A_0 = A(0, x_0)$ is a closed operator with domain $D$ dense in $X$ and
\[ \|(\lambda I - A_0)^{-1}\| \leq C|\lambda| + 1 \]
for all $\lambda$ with $\Re \lambda \leq 0$ and $C > 0$.
(ii) The operator $A_0^{-1}$ is a completely continuous operator in $X$.
(iii) For some $\alpha \in [0, 1)$ and for any $y \in B_r$, the operator $A(t, A_0^{-\alpha}y)$ is well defined on $D$ for all $t \in J$. Further more for any $t, \tau \in J$ and for $y, z \in B_r$,
\[ \|A(t, A_0^{-\alpha}y) - A(\tau, A_0^{-\alpha}z)\|A^{-1}(\tau, A_0^{-\alpha}z)\| \leq C_1|t - \tau|^\varepsilon + \|y - z\|_0, \]
where $0 < \varepsilon \leq 1$, $0 < \rho \leq 1$.
(iv) For every $t, \tau \in J$ and $y, z \in B_r$,
\[ \|f(t, A_0^{-\alpha}y) - f(\tau, A_0^{-\alpha}z)\| \leq C_2|t - \tau|^\varepsilon + \|y - z\|_0. \]
(v) For every $t \in J$ and $y, z \in B_r$,
\[ \|g(s, A_0^{-\alpha}y) - g(s, A_0^{-\alpha}z)\| \leq C_3\|y - z\|_0. \]
(vi) For every $t, s, \tau \in J$,
\[ |k(t, s) - k(\tau, s)| \leq C_4|t - \tau|^\varepsilon. \]
(vii) $x_0 \in D(A_0^\beta)$ for some $\beta > \alpha$ and
\[ \|A_0^\beta x_0\| < r. \]

3. Main result

**Theorem 3.1.** If the hypotheses (i)-(vii) are satisfied, then there exists at least one continuously differentiable solution of the equation (3) on $(0, T]$ for some $T \leq a$.

**Proof.** In order to study the existence problem, we must introduce a set $S$ of functions $x(t), t \in [0, T]$ and a transformation $z_x = \Phi x$ defined by $z_x = A_0^\alpha z$, where $z$ is the unique solution of
\[ \frac{dz}{dt} + A_0 z = f(t, A_0^{-\alpha}x(t)) + \int_0^t k(t, s)g(s, A_0^{-\alpha}x(s))ds, \]
\[ z(0) = x_0. \]
We then show that $\Phi$ has a fixed point, that is, there is a function $y \in S$ such that $\Phi y = y$, and so $x = A_0^{-\beta}y$ is the required solution of our problem (3).

Define the set
\[ S = \{ x \in Y : \|x(t) - x(\tau)\| \leq K|t - \tau|^{\eta} \text{ for } t, \tau \in [0, T], x(0) = A_0^\alpha x_0 \}, \]
where $K$ is a positive constant and $\eta$ is any number satisfying $0 < \eta < \beta - \alpha$ and $Y$ is a Banach space $C(J, X)$ with usual supnorm. From hypothesis (vii), and the definition of $S$ it follows that if $T$ is sufficiently small (depending on $K, \eta, \|A_0^\alpha x_0\|$), then
\[ \|x(t)\| < r \text{ for } t \in [0, T]. \]
Hence the operator $A_x(t) = A(t, A_0^\alpha x(t))$ is well defined and satisfies the conditions
\[
\| (A_x(t) - A_x(\tau)) A_0^{-1} \| \leq C_5 |t - \tau|^\epsilon + \| x(t) - x(\tau) \|^\mu
\]
\[
\leq C_6 |t - \tau|^{\mu},
\]
where $\mu = \min\{\epsilon, \rho\}$. Further, if $x(0) = A_0^\alpha x_0$,
\[
A_x(0) = A(0, A_0^\alpha x(0)) = A(0, A_0^\alpha A_0^\alpha x_0) = A(0, x_0) = A_0,
\]
and it follows that the function $\Phi$ itself. Obviously $\Phi$ is closed convex and bounded subset of $S$. From (v) and (vi), we can see that there exist constants $M_1 > 0$, $M_2 > 0$ such that
\[
\| g(t, A_0^{-\alpha} x(t)) \| \leq M_1 \text{ and } |k(t, s)| \leq M_2.
\]
Let us take
\[
f_x(t) = f(t, A_0^{-\alpha} x(t)), \quad g_x(t) = \int_0^t k(t, s) g(s, A_0^{-\alpha} x(s)) ds.
\]
Then, it follows that the function $f_x(t)$ is Holder continuous such that
\[
\| f_x(t) - f_x(\tau) \| \leq C_{10} |t - \tau|^{\mu}, \quad \| g_x(t) - g_x(\tau) \| \leq C_{11} |t - \tau|^{\mu}.
\]
Since $f_x(0) = f(0, A_0^{-\alpha} x(0))$ and $g_x(0) = 0$ are independent of $x$, we have from the above inequalities
\[
\| f_x(t) \| \leq M_3, \quad \| g_x(t) \| \leq M_4, \quad M_3 > 0, \quad M_4 > 0
\]
and
\[
\| A_0^\alpha \left[ \int_0^{t_1} U_x(t_1, s)(f_x(s) + g_x(s)) ds \right] \| \leq C_{12} |t_1 - t_2|^{1-\alpha}.
\]
We shall show that the operator $\Phi : S \rightarrow Y$ defined by
\[
\Phi x(t) = A_0^\alpha U_x(t, 0)x_0 + A_0^\alpha \int_0^t U_x(t, s)(f_x(s) + g_x(s)) ds
\]
has a fixed point. This fixed point is the solution of equation (3). Clearly $S$ is closed convex and bounded subset of $Y$. First we show that $\Phi$ maps $S$ into itself. Obviously $\Phi x(0) = A_0^\alpha x_0$. 
For any $0 \leq \alpha < \beta \leq 1$ and $0 \leq t_1 \leq t_2 \leq T$, we have

$$\| \Phi(x(t_1)) - \Phi(x(t_2)) \| \leq \| A_0^\alpha [U_x(t_1, 0) - U_x(t_2, 0)] A_0^{-\beta} \| \| A_0^2 x_0 \|$$

$$+ \left| A_0^3 \int_0^{t_1} U_x(t_1, s)[f_x(s) + g_x(s)] ds - A_0^3 \int_0^{t_2} U_x(t_2, s)[f_x(s) + g_x(s)] ds \right| .$$

Thus, for $T$ sufficiently small,

$$\| \Phi(x(t_1)) - \Phi(x(t_2)) \| \leq rC_0|t_1 - t_2|^{\beta - \alpha} + C_{12}|t_1 - t_2|^{1 - \alpha}$$

$$\leq K|t_1 - t_2|^\eta \text{ for some } K > 0, \eta < \beta - \alpha.$$  

Hence $\Phi$ maps $S$ into itself.

Next we show that this operator is continuous on the space $Y$. Let $x_1, x_2 \in S$ and set $z_1 = A_0^{-\alpha}\Phi x_1, z_2 = A_0^{-\alpha}\Phi x_2$. Then,

$$\frac{dz_1}{dt} + A_{x_1}(t)z_1 = f_{x_1}(t) + g_{x_1}(s)$$

$$z_1(0) = x_0, \ i = 1, 2.$$  

Therefore,

$$\frac{d}{dt}(z_1 - z_2) + A_{x_1}(t)(z_1 - z_2)$$

$$= [A_{x_2}(t) - A_{x_1}(t)]z_2 + f_{x_1}(t) - f_{x_2}(t) + g_{x_1}(t) - g_{x_2}(t).$$

It is easy to see that the functions $A_{x_1}(t)z_2(t)$ and $A_0^{-1}A_{x_2}(t)$ are uniformly Holder continuous, and so $A_0 z_2(t) = [A_0^{-1}A_{x_2}(t)]A_{x_2}(t)z_2(t)$ is uniformly Holder continuous. Similarly the functions

$$f_{x_1}(t) - f_{x_2}(t), g_{x_1}(t) - g_{x_2}(t)$$

are also uniformly Holder continuous in $[\tau, T], \tau > 0$. Hence, we have

$$|z_1(t) - z_2(t)|$$

$$= U_{x_1}(t, \tau)[z_1(\tau) - z_2(\tau)] + \int_0^t U_{x_1}(t, s)[(A_{x_2}(s) - A_{x_1}(s)]z_2(s)$$

$$+ [f_{x_1}(s) - f_{x_2}(s)] + [g_{x_1}(s) - g_{x_2}(s)] ds.$$  

Since $A_0 \int_0^t U_{x_2}(t, s)[f_{x_2}(s) + g_{x_2}(s)] ds$ is a bounded function, it follows that

$$\| A_0 z_2(t) \| \leq C_{13}t^{\beta - 1}.$$  

Hence we can take $\tau \to 0$ in the above equation and we get

$$|z_1(t) - z_2(t)| = \int_0^t U_{x_1}(t, s)[(A_{x_2}(s) - A_{x_1}(s)]z_2(s)$$

$$+ [f_{x_1}(s) - f_{x_2}(s)] + [g_{x_1}(s) - g_{x_2}(s)] ds.$$
Since \( z_1 = A_0^{-\alpha}\Phi x_1 \) and \( z_2 = A_0^{-\alpha}\Phi x_2 \) and from (iii), (iv), (v) and (vi) it follows that

\[
\|\Phi x_1(t) - \Phi x_2(t)\| \leq \int_0^t \|A_0^{\alpha}U_{x_1}(t, s)\|\|A_x(s) - A_{x_1}(s)\|ds + \|f_{x_1}(s) - f_{x_2}(s)\| + \|g_{x_1}(s) - g_{x_2}(s)\|ds
\]

\[
\leq \int_0^t C_{14}|t - s|^{-\alpha}[C_{15}\|x_1(s) - x_2(s)\|^\rho s^{\beta - 1} + C_{16}\|x_1(s) - x_2(s)\|^\rho]ds.
\]

Hence

\[
\|\Phi x_1 - \Phi x_2\|_Y \leq K^*T^{\beta - \alpha}\|x_1 - x_2\|_Y^\rho \text{ for some } K^* > 0.
\]

This shows that \( \Phi : S \to Y \) is continuous. We shall show that this operator is completely continuous. We now claim that the set \( \Phi S \) is contained in a compact subset of \( Y \). Indeed, the functions \( x(t) \) of \( S \) are uniformly bounded and equicontinuous. By Arzela-Ascoli's theorem it is sufficient to show that for each \( t \) the set \( \{\Phi x(t) : x \in S\} \) is contained in a compact subset of \( X \). For each \( t \in [0, T] \), we can write \( \Phi x(t) = A_0^{-\gamma}A_0^{\alpha}\Phi x(t), (0 < \gamma < \beta - \alpha) \). Since \( \{A_0^{\gamma}\Phi x(t) : x \in S\} \) is a bounded subset of \( X \), and since \( A_0^{-\gamma} \) is completely continuous, it follows that the set \( \{\Phi x(t) : x \in S\} \) is contained in a compact subset of \( X \). Therefore by the Schauder fixed point theorem, \( \Phi \) has a fixed point \( z \in S \) such that \( \Phi z = z(t) \) which satisfies

\[
z(t) = A_0^{\alpha}U_z(t, 0)x_0 + A_0^{\alpha}\int_0^t U_z(t, s)[f_{z(s)} + g_{z(s)}]ds.
\]

Then \( x(t) = A_0^{-\alpha}z(t) \) satisfies

\[
x(t) = U_{A_0^{\alpha}z}(t, 0)x_0 + \int_0^t U_{A_0^{\alpha}z}(t, s)[f_{A_0^{\alpha}z}(s) + g_{A_0^{\alpha}z}(s)]ds.
\]

By Theorem 2.2, \( x(t) \) is a solution of (3).

**Theorem 3.2.** Let the assumptions (i), (iii)-(v) hold with \( \rho = 1 \). Then the assertion of Theorem 3.1 is valid and the solution is unique.

**Proof.** If \( \rho = 1 \), then from (6) shows that for \( T \) sufficiently small \( \Phi \) is a contraction, that is \( \|\Phi x_1 - \Phi x_2\| \leq \theta\|x_1 - x_2\| \) for some \( \theta < 1 \). Hence by the Banach fixed point theorem \( \Phi \) has a unique fixed point. \( \square \)
4. Example

Consider the following nonlinear parabolic integrodifferential equation

\[
\frac{\partial z}{\partial t} + \sum_{|\alpha|=2m} a_\alpha(x, t, z, Dz, \ldots, D^{2m-1}z)D^\alpha z = f(x, t, z, Dz, \ldots, D^{2m-1}z) + \int_0^t k(x, t, s)g(x, s, z, Dz, \ldots, D^{2m-1}z)ds,
\]

\[
\frac{\partial^j z}{\partial \nu^j} = 0 \text{ on } S_T = \{(x, t) : x \in \partial \Omega, 0 \leq t \leq T\}, 0 \leq j \leq m-1
\]

\[
u = 0 \text{ on } \Omega_0 = \{(x, 0) : x \in \partial \Omega\}
\]

in a cylinder \(Q_T = \Omega \times (0, T)\) with coefficients in \(\mathcal{Q}_T\), where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(\partial \Omega\) the boundary of \(\Omega\), \(\nu\) is the outward normal. Here the parabolicity means that for any vector \(y \neq 0\) and for arbitrary values of \(z, Dz, \ldots, D^{2m-1}z\),

\[
(-1)^m \text{Re}\{\sum_{|\alpha|=2m} a_\alpha(x, t, z, Dz, \ldots, D^{2m-1}z)y^\alpha\} \geq C|y|^{2m}, C > 0.
\]

If \(z_0(x) \in C^{2m-1}(\bar{\Omega})\), then

\[A_0 z = \sum_{|\alpha|=2m} a_\alpha(x, t, z, Dz, \ldots, D^{2m-1}z)D^\alpha z\]

is a strongly elliptic operator with continuous coefficients. So the condition (i) holds. Let us take \(X\) to be \(L^p(\Omega)\), \(1 < p < \infty\). Then \(A_0^{-1}\) maps bounded subsets of \(L^p(\Omega)\) into bounded subsets of \(W^{2m,p}(\Omega)\), so it is a completely continuous operator in \(L^p(\Omega)\). Further, if \((2m-1)/2m < \alpha < 1\), then [9]

\[
|D^\beta A_0 - \alpha z|_{0,p}^{\Omega} \leq C|z|_{0,p}^{\Omega}, 0 \leq |\beta| \leq 2m - 1,
\]

where \(C\) depends only on a bound on the coefficients \(A_0\), on a module of strong ellipticity and on a modulus of continuity of the leading coefficients. Here the norm is defined as

\[
|z|_{j,p}^{\Omega} = \left\{ \sum_{|\alpha| \leq j} \int_\Omega |D^\alpha z(x)|^p dx \right\}^{1/p}
\]

for any nonnegative integer \(j\) and a real number \(p\), \(1 \leq p < \infty\). It follows that if \(f\) and \(a_\alpha\) are continuously differentiable in all variables, then (iii) and (iv) hold with \(\sigma = p = 1\). Hence there exist fundamental operator solution \(U_z(t, s)\) for the equation (7). The nonlinear functions \(f, g\) satisfy the conditions (iv),(v) and \(k\) satisfies the condition (vi). Hence by the above theorem there exist a local solution for the equation (7).

References


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