On Self-commutator Approximants

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ABSTRACT. Let $B(X)$ denote the algebra of operators on a complex Banach space $X$, $H(X) = \{h \in B(X) : h$ is hermitian\}, and $J(X) = \{x \in B(X) : x = x_1 + ix_2, x_1$ and $x_2 \in H(X)\}$. Let $\delta_a \in B(B(X))$ denote the derivation $\delta_a(x) = ax - xa$. If $J(X)$ is an algebra and $\delta_a^{-1}(0) \subseteq \delta_a^{-1}(0)$ for some $a \in J(X)$, then $||a|| \leq ||a - (x'x - xx')||$ for all $x \in J(X) \cap \delta_a^{-1}(0)$. The cases $J(X) = B(H)$, the algebra of operators on a complex Hilbert space, and $J(X) = C_n$, the von Neumann–Schatten $p$-class, are considered.

1. Introduction

An element $h \in B(X)$, $B(X) = \{x \in B(X) : x$ is a subset of the set of reals $[3, Page 8]$. Let $H(X) = \{h \in B(X) : h$ is hermitian\}, and let $J(X) = \{x \in B(X) : x = x_1 + ix_2, x_1$ and $x_2 \in H(X)\}$. Then each $x \in J(X)$ has a unique representation $x = x_1 + ix_2, x_1$ and $x_2 \in H(X)$, and we may define a mapping $x \mapsto x^*$ from $J(X)$ into itself by $x^* = x_1 - ix_2$ ($= (x_1 + ix_2)^*$): $J(X)$ with the operator norm $||.||$ of $B(X)$ is a complex Banach space such that $\ast$ is a continuous linear involution on $J(X)$ $[3, Lemma 8, Page 50]$. Recall that an operator $a \in B(X)$ is normal if $a = a_1 + ia_2 \in J(X)$ and $[a_1, a_2] = a_1 a_2 - a_2 a_1 = 0$. We say that an operator $a \in J(X)$ satisfies the PF-property, short for the Putnam–Fuglede property, if $a^{-1}(0) \subseteq a^{-1}(0)$. Normal operators satisfy the PF-property: if $a = a_1 + ia_2$ is normal, then $ax = 0$ implies $a_1 x = a_2 x = 0 \Rightarrow a^* x = 0$ $[4, Page 124]$.

Let $\delta_a \in B(B(X))$ denote the derivation $\delta_a(x) = ax - xa = (L_a - R_a)x$, where $L_a$ and $R_a$ denote, respectively, the operators of left multiplication and right multiplication by $a$. If $a \in H(X)$, then $L_a$, $R_a$ and $L_a - R_a \in H(X)$. Evidently, if $a = a_1 + ia_2$, then $\delta_a = \delta_{a_1} + i \delta_{a_2}$, where $[\delta_{a_1}, \delta_{a_2}] = 0$ whenever $[a_1, a_2] = 0$.

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Hence, if $a$ is normal then $\delta_a$ is normal, and this by [12, Corollary 8] implies that
\[ -2\sqrt{||\delta_a(x)||y|| + ||x||} \leq ||x - \delta_a(y)|| \]
for all $x, y \in B(\mathcal{X})$. In particular, if $x \in \delta_a^{-1}(0)$, then (for all $y \in B(\mathcal{X})$
\begin{equation}
||x|| \leq ||x - \delta_a(y)||.
\end{equation}
i.e., the kernel $\delta_a^{-1}(0)$ of $\delta_a$ is orthogonal to the range $\delta_a(B(\mathcal{X}))$ of $\delta_a$ in the sense of G. Birkhoff and R. C. James [9, page 93]. Kernel-range inequalities of type (1),
for all $x, y \in H$
i.e., the kernel approximants of the type recently proved by P. J. Maher [13] (for self-adjoint $a$
for further references). In this paper we look at the equation
\[ \delta \in X \]
p<\infty
a
for all
\begin{equation}
(2)
||a|| \leq ||a - [x^*, x]||
\end{equation}
for all $x \in J(\mathcal{X}) \cap \delta_a^{-1}(0)$. In the case in which $1 < p < \infty$, $\delta_a^{-1}(0) \subseteq \delta_a^{-1}(0)$ and
$a \in C_p$, it is proved that
\begin{equation}
||a||_p \leq \min\{||a - \delta_{x_1}(y)||_p, ||a - \delta_{x_2}(y)||_p\}
\end{equation}
for all $x = x_1 + ix_2$ and $y \in B(\mathcal{H})$ such that $\delta_{x_j}(y) \in C_p$ ($j = 1, 2$) if and only if
$x \in \delta_a^{-1}(0)$. We also prove that inequality (2) holds for essentially normal operators
$x \in B(\mathcal{H}) \cap \delta_a^{-1}(0)$ such that $||a||$ equals the essential norm $||a||_e$ of $a$.

2. Results

Evidently, $x \in \delta_a^{-1}(0) \iff a \in \delta_a^{-1}(0)$ for all $a, x \in B(\mathcal{X})$. Since $h \in H(\mathcal{X})$
do not (in general) imply that $h^2 \in H(\mathcal{X})$ [3, Example 1, Page 58], $J(\mathcal{X})$ is not (in
general) a subalgebra of $B(\mathcal{X})$. If however $J(\mathcal{X})$ is an algebra, then $h, k \in H(\mathcal{X})$
implies that $h^2$ and $hk + kh \in H(\mathcal{X})$ [3, Theorem 3, Page 59]. Recall that
\[ i(a_1a_2 - a_2a_1) \in H(\mathcal{X}) \]
whenever $a_1, a_2 \in H(\mathcal{X})$ [3, Lemma 4, Page 47]. Let $a = a_1 + ia_2$ and $b = b_1 + ib_2 \in J(\mathcal{X})$, and assume that $J(\mathcal{X})$ is an algebra. Then both
\[ ab + b^*a^* = \{(a_1b_1 + b_1a_1) - (a_2b_2 + b_2a_2)\} + i\{(a_2b_1 - b_1a_2) + (a_1b_2 - b_2a_1)\} \]
and
\[ i(ab - b^*a^*) = i\{(a_1b_1 - b_1a_1) - (a_2b_2 - b_2a_2)\} - \{(a_2b_1 + b_1a_2) + (a_1b_2 + b_2a_1)\} \]
are in $H(\mathcal{X})$. Hence

$$(ab)^* = \frac{1}{2} (ab + b^*a^*) + \frac{i}{2} (ab - b^*a^*) = b^*a^*.$$

**Theorem 2.1.** If $J(\mathcal{X})$ is an algebra and $\delta_\alpha^{-1}(0) \subseteq \delta_\alpha^{-1}(0)$ for some $a \in J(\mathcal{X})$, then $||a|| \leq ||a - [x^*, x]||$ for all $x \in J(\mathcal{X}) \cap \delta_\alpha^{-1}(0)$.

**Proof.** The hypotheses $J(\mathcal{X})$ is an algebra and $\delta_\alpha^{-1}(0) \subseteq \delta_\alpha^{-1}(0)$ imply that $\delta_x(a) = \delta_x'(a) = 0$ for every $x \in J(\mathcal{X}) \cap \delta_\alpha^{-1}(0)$. Hence, upon letting $x = x_1 + ix_2$, $\delta_{x_1}(a) = \delta_{x_2}(a) = 0$. Since $x_j \in H(\mathcal{X})$, $j = 1, 2$, it follows that

$$||a|| \leq \min\{||a - \delta_{x_1}(y)||, ||a - \delta_{x_2}(y)||\}$$

for all $y \in J(\mathcal{X})$ [12, Corollary 8]. Choose $y = 2ix_2$ (in $\delta_{x_1}(y)$); then $\delta_{x_1}(y) = [x^*, x]$ and $||a|| = ||a - [x^*, x]||$ for all $x \in J(\mathcal{X}) \cap \delta_\alpha^{-1}(0)$.

The following corollary is immediate from Theorem 2.1.

**Corollary 2.2.** If $a \in B(\mathcal{H})$ is such that $\delta_\alpha^{-1}(0) \subseteq \delta_\alpha^{-1}(0)$, then $||a|| \leq ||a - [x^*, x]||$ for all $x \in B(\mathcal{H}) \cap \delta_\alpha^{-1}(0)$.

An operator $a \in B(\mathcal{H})$ is essentially normal if $\pi(a)$ is normal, where $\pi : B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$ is the Calkin map. (Equivalently, $a$ is essentially normal if $\pi([a^*, a]) = 0$.) For essentially normal $x \in \delta_\alpha^{-1}(0)$, we have the following.

**Theorem 2.3.** If $x \in \delta_\alpha^{-1}(0) \cap B(\mathcal{H})$ is essentially normal, then $||\pi(a)|| \leq ||a - [x^*, x]||$.

**Proof.** If $x \in \delta_\alpha^{-1}(0)$, then $\pi(a) \in \delta_\alpha^{-1}(0)$. Since $B(\mathcal{H})/K(\mathcal{H})$ is a $C^*$-algebra, there exists a Hilbert space $\mathcal{H}_0$ and a *-isometric isomorphism $\psi : B(\mathcal{H})/K(\mathcal{H}) \to B(\mathcal{H}_0)$ such that $x_0 = \psi(\pi(x))$ is a normal element of $B(\mathcal{H}_0)$. Letting $a_0 = \psi(\pi(a))$, it follows that

$$||\pi(a)|| = ||a_0|| \leq ||a_0 - \delta_{x_0}(\psi(\pi(y)))|| = ||\pi(a - \delta_{x}(y))|| \leq ||a - \delta_{x}(y)||$$

for all $y \in B(\mathcal{H})$. Choose $y = -x^*$.

In general, $||\pi(a)|| \neq ||a||$. However, if $a \in B(\mathcal{H})$ is hyponormal (i.e., $|a^*|^2 \leq |a|^2$), or normaloid ($||a||$ equals the spectral radius of $a$) and without eigen-values of finite multiplicity, then $||\pi(a)|| = ||a||$ (see [8, Page 1730]): for such $a \in B(\mathcal{H})$, $||a|| = ||a - [x^*, x]||$.

A version of Theorems 2.1 has been proved by Maher [13, Theorems 4.1(a) and 4.2] for the von Neumann-Schatten $p$-classes ($\mathcal{C}_p, ||.||_p$); $1 \leq p < \infty$. Observe from the proof of Theorem 2.1 that if $\delta_\alpha^{-1}(0) \subseteq \delta_\alpha^{-1}(0)$, then $||a||_p \leq ||a - [x^*, x]||_p$ for all $a \in \mathcal{C}_p$ and $x \in \delta_\alpha^{-1}(0)$ such that $[x^*, x] \in \mathcal{C}_p$. The following theorem proves that the condition $x = x_1 + ix_2 \in \delta_\alpha^{-1}(0)$ is necessary for $||a||_p \leq \min\{||a -
\[
\delta_{x_1}(y)\|_p, \|a - \delta_{x_2}(y)\|_p \}
\] in the case in which \(1 < p < \infty\). But before that we introduce some terminology. If \((\mathcal{V}, \|\cdot\|)\) is a Banach space, then \(|\cdot|\) is said to be Gateaux–differentiable at a non-zero \(x \in \mathcal{V}\) if
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = \text{Re}D_a(y)
\]
equalsforall\(y \in \mathcal{V}\). Here \(t \in \mathbb{R}\) (= the set of reals), \(\text{Re}\) denotes the real part and \(D_a\) is the unique support functional in the dual space \(\mathcal{V}^*\) such that \(\|D_x\| = 1\) and \(\|D_a(x)\| = \|x\|\). The Gateaux–differentiability of \(|\cdot|\) at \(x\) implies that \(x\) is a smooth point of the sphere with radius \(\|x\|\). If \(a \in \mathcal{C}_p, 1 < p < \infty\), has the polar decomposition \(a = u|a|\), then \(|a|^{p-1}u^* \in \mathcal{C}_{p'}\), \(\frac{1}{p} + \frac{1}{p'} = 1\), and
\[
D_a(y) = \text{tr}(|a|^{p-1}u^*y)/|a|^{p-1}
\]
equalsforall\(y \in \mathcal{C}_p\) [1, Theorem 2.3]. (As usual, \(\text{tr}\) denotes the trace functional.) Recall from [10] that if \(a, b \in \mathcal{V}\) and \(a\) is a smooth point of \(\mathcal{V}\), then \(|a| \leq \|a + tb\|\) for all complex \(t\) if and only if \(D_a(b) = 0\).

**Theorem 2.4.** If \(\delta_a^{-1}(0) \subseteq \delta_a^{-1}(0)\) then
\[
|a|_p \leq \min\{\|a - \delta_{x_1}(y)|_p, \|a - \delta_{x_2}(y)\|_p\}
\]
equalsforall\(a \in \mathcal{C}_p\) and \(y \in B(\mathcal{H})\) such that \(\delta_{x_j}(y) \in \mathcal{C}_p, j = 1, 2\) and \(1 < p < \infty\), if and only if \(x = x_1 + ix_2 \in \delta_a^{-1}(0)\).

**Proof.** The ‘if part’ being evident (from \(\delta_a(x) = 0 \implies \delta_{x_1}(a) = \delta_{x_2}(a) = 0\), we prove the ‘only if’ part. Recall that \(\mathcal{C}_p, 1 < p < \infty\), is uniformly convex; hence operators \(a \in \mathcal{C}_p, a = \delta_a^{-1}(0)\). Let \(a\) have the polar decomposition \(a = u|a|\). Then, [10], the inequality of the statement of the theorem holds for all \(y \in B(\mathcal{H})\) such that \(\delta_{x_j}(y) \in \mathcal{C}_p, j = 1, 2\), if and only if the support functional \(D_a(\delta_{x_j}(y)) = \text{tr}(|a|^{p-1}u^*\delta_{x_j}(y))/|a|^{p-1}\) = 0. Set \(|a|^{p-1}u^* \equiv \tilde{a}\); then \(\tilde{a} \in \mathcal{C}_{p'}, 1/p + 1/p' = 1\). Choose \(y\) to be the rank one operator \(y = e \otimes f\) for some \(e, f \in \mathcal{H}\). Then \(\delta_{x_j}(y) \in \mathcal{C}_p\) and
\[
\text{tr}(\tilde{a}\delta_{x_j}(y)) = \text{tr}(\tilde{a}(x_jy - yx_j)) = \text{tr}((\tilde{a}x_j - x_j\tilde{a})y)
\]
\[
= \text{tr}(\delta_{x_j}(x_j)e \otimes f) = (\delta_{x_j}(x_j)e, f) = 0
\]
equalsforall\(e, f \in \mathcal{H}\). Hence \(\delta_a(x_j) = 0; j = 1, 2\). The operator \(x_j\) being self–adjoint
\[
u|a|^{p-1}x_j = x_j\nu|a|^{p-1} \implies |a|^{2(p-1)}x_j = |a|^{p-1}u^*x_ju|a|^{p-1} = x_j|a|^{2(p-1)}.
\]
Hence \([x_j, |a|] = 0\). Since \(\tilde{a}x_j = x_j\tilde{a}\) implies \([x_j, u]|_{\text{ran}|a|^{p-1}} = 0\), it follows that
\[
ax_j = u|x_j = ux_j|a| = x_ju|a| = x_ja.
\]
Hence \(\delta_a(x_1) + i\delta_a(x_2) = \delta_a(x) = 0\). □

A stronger result is possible in the case in which \(p = 2\).
Corollary 2.5. If \( a \in C_2 \), then
\[
\|a + \delta_{x_j}(y)\|^2_2 = \|a\|^2_2 + \|\delta_{x_j}(y)\|^2_2 = \|a^* + \delta_{x_j}(y)\|^2_2,
\]
for all \( y \in C_2 \) if and only if \( x = x_1 + ix_2 \in \delta_a^{-1}(0) \cap \delta_a^{-1}(0) \).

Proof. \( C_2 \) has a Hilbert space structure with inner product \((s, t) = \text{tr}(t^*s)\). Since
\[
\|a + \delta_{x_j}(y)\|^2_2 = \|a\|^2_2 + \|\delta_{x_j}(y)\|^2_2 + 2\text{Re}(\delta_{x_j}(y), a),
\]
\[
\|a^* + \delta_{x_j}(y)\|^2_2 = \|a\|^2_2 + \|\delta_{x_j}(y)\|^2_2 + 2\text{Re}(a, \delta_{x_j}^*(y)) = \|a\|^2_2 + \|\delta_{x_j}(y)\|^2_2 + 2\text{Re}(a, \delta_{x_j}(y)),
\]
it follows that if \( x \in \delta_a^{-1}(0) \cap \delta_a^{-1}(0) \), then \( \|a + \delta_{x_j}(y)\|^2_2 = \|a\|^2_2 + \|\delta_{x_j}(y)\|^2_2 = \|a^* + \delta_{x_j}(y)\|^2_2 \) for all \( y \in C_2 \). Conversely, if this equality is satisfied, then the argument of the proof Theorem 2.4 (with \( p = 2 \)) applied to the inequalities \( \|a\|_2 \leq \|a - \delta_{x_j}(y)\|_2 \) and \( \|a^*\|_2 \leq \|a^* - \delta_{x_j}(y)\|_2 \) implies that \( x_j \), and so also \( x \in \delta_a^{-1}(0) \cap \delta_a^{-1}(0) \). \( \square \)

The elementary operator \( \Delta_a(x) = axa - x \). We close this note with a remark on the elementary operator \( \Delta_a \). If \( a \in B(\mathcal{X}) \) is a contraction, then \( L_aR_a \) is a contraction. Hence
\[
V(B(B(\mathcal{X}))), L_aR_a) \subseteq \{ \lambda \in C : |\lambda| \leq 1 \}
\]
and
\[
V(B(B(\mathcal{X}))), \Delta_a) = V(B(B(\mathcal{X}))), L_aR_a - I) \subseteq \{ \lambda \in C : |\lambda + 1| \leq 1 \}
\]
[5, Proposition 4, Page 52]. (Here \( C \) denotes the complex plane.) This implies that the operator \( \Delta_a \) is dissipative [3, Page 30], and hence
\[
||x|| \leq ||x - \Delta_a(y)||
\]
for all \( x \in \Delta_a^{-1}(0) \) and \( y \in B(\mathcal{X}) \) [12, Theorem 7]. Although \( \Delta_a \) may not be normal even for normal \( a \in B(\mathcal{X}) \), see [7, Example 2.1], a number of kernel–range orthogonality results for the elementary operator \( \Delta_a \in B(B(\mathcal{H})) \) and \( \Delta_a \in B(C_p) \) are to be found in the extant literature; see for example [6], [7], [11], [14]. Seemingly, self-commutator approximant inequalities of the type (2) are not possible for \( \Delta_a \). However, one does have the following interesting result.

Theorem 2.6. Assume that \( \Delta_a^{-1}(0) \subseteq \Delta_a^{-1}(0) \). If \( a \in B(\mathcal{H}) \) (resp., \( a \in C_p \)), then
\[
||a|| \leq ||a - ||x, x^*|||| \text{ for all } x \in B(\mathcal{H}) \cap \Delta_a^{-1}(0) \text{ (resp., } ||a||_p \leq ||a - ||x, x^*||||_p \text{ for all } x \in C_p \cap \Delta_a^{-1}(0) \)).
\]
Proof. If $x \in \Delta_{a}^{-1}(0)$, then $axa = x$ and $a^*xa^* = x$ ($\iff ax^*a = x^*$). Since  

$$ax^* = (ax^*)axa = (ax^*)xa = x^*xa,$$

$[a, |x|] = 0$. Hence $\delta_{|x|}(a) = 0$, which, since $|x| \geq 0$, implies that $||a|| \leq ||a - \delta_{|x|}(y)||$ for all $y \in B(H)$ (resp., $||a||_p \leq ||a - \delta_{|x|}(y)||_p$ for all $y \in B(H)$ such that $\delta_{|x|}(y) \in C_p$). Choose $y = |x^*|$. Since $x \in C_p$ implies $|x| \in C_p$, the proof is complete.  

\[\square\]

References