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A THEOREM OF G-INVARIANT MINIMAL HYPERSURFACES WITH CONSTANT SCALAR CURVATURES IN S^{n+1}

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Abstract. Let $G = O(k)\times O(k)\times O(q)$ and let M^n be a closed G-invariant minimal hypersurface with constant scalar curvature in S^{n+1}. Then we obtain a theorem: If M^n has 2 distinct principal curvatures at some point p, then the square norm of the second fundamental form of M^n, $S = n$.

Introduction

Let M^n be a closed minimally immersed hypersurface in the unit sphere S^{n+1}, and h its second fundamental form. Denote by R and S its scalar curvature and the square norm of h, respectively. It is well known that $S = n(n-1)-R$ from the structure equations of both M^n and S^{n+1}. In particular, S is constant if and only if M has constant scalar curvature. In 1968, J. Simons [6] observed that if $S \le n$ everywhere and S is constant, then $S \in \{0, n\}$. Clearly, M^n is an equatorial sphere if $S = 0$. And when $S = n$, M^n is indeed a product of spheres, due to the works of Chern, do Carmo, and Kobayashi [2] and Lawson [4].

We are concerned about the following conjecture posed by Chern [8].

Chern Conjecture. *For any $n \ge 3$, the set R_n of the real numbers each of which can be realized as the constant scalar curvature of a closed minimally immersed hypersurface in S^{n+1} is discrete.*

C. K. Peng and C. L. Terng [7] proved

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THEOREM [Peng and Terng, 1983]. *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S > n + 1/(12n)$.*

S. Chang [2] proved the following theorem by showing that $S = 3$ if $S \geq 3$ and M^3 has multiple principal curvatures at some point.

THEOREM [Chang, 1993]. *A closed minimally immersed hypersurface with constant scalar curvature in S^4 is either an equatorial 3-sphere, a product of spheres, or a Cartan's minimal hypersurface. In particular, $R_n = \{0, 3, 6\}$.*

H. Yang and Q. M. Cheng [10] proved

THEOREM [Yang and Cheng, 1998]. *Let M^n be a closed minimally immersed hypersurface with constant scalar curvature in S^{n+1}. If $S > n$, then $S \geq n + n/3$.*

Let $G \simeq O(k) \times O(k) \times O(q) \subset O(2k+q)$ and set $2k + q = n + 2$. Then W. Y. Hsiang [4] investigated G-*invariant, minimal hypersurfaces,* M^n in S^{n+1}, by studying their generating curves, M^n/G, in the orbit space S^{n+1}/G. He showed that there exit infinitely many closed minimal hypersurfaces in S^{n+1} for all $n \geq 2$, by proving the following theorem:

THEOREM [Hsiang, 1987]. *For each dimension $n \geq 2$, there exist infinitely many, mutually noncongruent closed G-invariant minimal hypersurfaces in S^{n+1}, where $G \simeq O(k) \times O(k) \times O(q)$ and $k = 2$ or 3.*

We studied a G-invariant minimal hypersurface M^n, in stead of minimal one, with constant scalar curvature in S^{n+1}. In this paper, we shall prove the following theorem:

THEOREM 3.2. *If M^n has 2 distinct principal curvatures at some point p, then $S = n$.*

1. Preliminaries

Let M^n be a manifold of dimension n immersed in a Riemannian manifold N^{n+1} of dimension $n+1$. Let $\overline{\nabla}$ and $\langle\, ,\rangle$ be the connection and metric tensor respectively of N^{n+1} and let $\bar{\mathcal{R}}$ be the curvature tensor with respect to the connection $\overline{\nabla}$ on N^{n+1}. Choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$

in N^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n. Denote the dual coframe by $\{\omega_A\}$. Here we will always use $i, j, k, \ldots,$ for indices running over $\{1, 2, \ldots, n\}$ and $A, B, C, \ldots,$ over $\{1, 2, \ldots, n+1\}$.

As usual, the *second fundamental form* h and the *mean curvature* H of M^n in N^{n+1} are respectively defined by

$$h(v, w) = \langle \overline{\nabla}_v w,\, e_{n+1}\rangle \quad \text{and} \quad H = \sum_i h(e_i,\, e_i).$$

M^n is said to be *minimal* if H vanishes identically. And the *scalar curvature* $\bar{R}$ of N^{n+1} is defined by

$$\bar{R} = \sum_{A,B} \langle \bar{\mathcal{R}}(e_A,\, e_B) e_B,\, e_A\rangle.$$

Then the structure equations of N^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

where $K_{ABCD} = \langle \bar{\mathcal{R}}(e_A,\, e_B) e_D,\, e_C\rangle$. When N^{n+1} is the unit sphere S^{n+1}, we have

$$K_{ABCD} = \delta_{AC}\, \delta_{BD} - \delta_{AD}\, \delta_{BC}.$$

Next, we restrict all tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n. Then

$$\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0.$$

By Cartan's lemma, we can write

$$\omega_{(n+1)i} = -\sum_j h_{ij}\, \omega_j.$$

Here,

$$h_{ij} = -\omega_{(n+1)i}(e_j) = -\langle \overline{\nabla}_{e_j} e_{n+1},\, e_i\rangle = \langle \overline{\nabla}_{e_j} e_i,\, e_{n+1}\rangle = h(e_j, e_i) = h(e_i, e_j).$$

Second, from

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - \frac{1}{2} \sum_{l,m} R_{ijlm}\, \omega_l \wedge \omega_m,$$

we find the curvature tensor of M^n is

$$R_{ijlm} = K_{ijlm} + h_{il}\,h_{jm} - h_{im}\,h_{jl}. \tag{1.1}$$

If M^n is a piece of minimally immersed hypersurface in the unit sphere S^{n+1} and R is the scalar curvature of M^n, then we have

$$R = n(n-1) - S, \tag{1.2}$$

where $S = \sum_{i,j} h_{ij}^2$ is the *square norm* of h.

Given a symmetric 2-tensor $T = \sum_{i,j} T_{ij}\,\omega_i\,\omega_j$ on M^n, we also define its covariant derivatives, denoted by ∇T, $\nabla^2 T$ and $\nabla^3 T$, etc. with components $T_{ij,k}$, $T_{ij,kl}$ and $T_{ij,klp}$, respectively, as follows:

$$\begin{aligned}
(1.3)\sum_k T_{ij,k}\,\omega_k &= d\,T_{ij} + \sum_s T_{sj}\,\omega_{si} + \sum_s T_{is}\,\omega_{sj},\\
\sum_l T_{ij,kl}\,\omega_l &= d\,T_{ij,k} + \sum_s T_{sj,k}\,\omega_{si} + \sum_s T_{is,k}\,\omega_{sj} + \sum_s T_{ij,s}\,\omega_{sk},\\
\sum_p T_{ij,klp}\,\omega_p &= d\,T_{ij,kl} + \sum_s T_{sj,kl}\,\omega_{si} + \sum_s T_{is,kl}\,\omega_{sj} + \sum_s T_{ij,sl}\,\omega_{sk} + \sum_s T_{ij,ks}\,\omega_{sl}.
\end{aligned}$$

In general, the resulting tensors are no longer symmetric, and the rule to switch sub-index obeys the Ricci formula as follows:

$$\begin{aligned}
(1.4)\quad T_{ij,kl} - T_{ij,lk} &= \sum_s T_{sj}\,R_{sikl} + \sum_s T_{is}\,R_{sjkl},\\
T_{ij,klp} - T_{ij,kpl} &= \sum_s T_{sj,k}\,R_{silp} + \sum_s T_{is,k}\,R_{sjlp} + \sum_s T_{ij,s}\,R_{sklp},\\
T_{ij,klpm} - T_{ij,klmp} &= \sum_s T_{sj,kl}\,R_{sipm}\\
&\quad + \sum_s T_{is,kl}\,R_{sjpm} + \sum_s T_{ij,sl}\,R_{skpm} + \sum_s T_{ij,ks}\,R_{slpm}.
\end{aligned}$$

For the sake of simplicity, we always omit the comma (,) between indices in the special case $T = \sum_{i,j} h_{ij}\,\omega_i\,\omega_j$ with $N^{n+1} = S^{n+1}$.

Since $\sum_{C,D} K_{(n+1)iCD}\,\omega_C \wedge \omega_D = 0$ on M^n when $N^{n+1} = S^{n+1}$, we find

$$d\left(\sum_j h_{ij}\,\omega_j\right) = \sum_{j,l} h_{jl}\,\omega_l \wedge \omega_{ji}.$$

Therefore,

$$\sum_{j,l} h_{ijl}\,\omega_l \wedge \omega_j = \sum_j \left(dh_{ij} + \sum_l h_{lj}\,\omega_{li} + \sum_l h_{il}\,\omega_{lj} \right) \wedge \omega_j = 0;$$

i.e., h_{ijl} is symmetric in all indices.

Moreover, in the case that M^n is minimal, we have

$$\begin{aligned}(1.5)\sum_l h_{ijll} &= \sum_l h_{lijl} = \sum_l \left\{ h_{lilj} + \sum_m (h_{mi}R_{mljl} + h_{lm}R_{mijl}) \right\} \\ &= (n-1)h_{ij} + \sum_{l,m} \{-h_{mi}h_{ml}h_{lj} + h_{lm}(\delta_{mj}\delta_{il} - \delta_{ml}\delta_{ij} + h_{mj}h_{il} - h_{ml}h_{ij})\} \\ &= nh_{ij} - \sum_{l,m} h_{lm}h_{ml}h_{ij} = (n-S)h_{ij}.\end{aligned}$$

It follows that

$$\frac{1}{2}\Delta S = (n-S)S + \sum_{i,j,l} h_{ijl}^2. \tag{1.6}$$

2. G-invariant Hypersurface in S^{n+1}

For $G \simeq O(k) \times O(k) \times O(q)$, $\mathbb{R}^{n+2}$ splits into the orthogonal direct sum of irreducible invariant subspaces, namely

$$\mathbb{R}^{n+2} \simeq \mathbb{R}^k \oplus \mathbb{R}^k \oplus \mathbb{R}^q = \{(X, Y, Z)\}$$

where X and Y are generic k-vectors and Z is a generic q-vector. Here if we set $x = |X|, y = |Y|$ and $z = |Z|$, then the orbit space $\mathbb{R}^{n+2}/G$ can be parametrized by $(x,\, y,\, z)$; $x,\, y,\, z \in \mathbb{R}_+$ and the orbital distance metric is given by $ds^2 = dx^2 + dy^2 + dz^2$. By restricting the above G-action to the unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, it is easy to see that

$$S^{n+1}/G \simeq \{(x, y, z) : x^2 + y^2 + z^2 = 1;\ x,\ y,\ z \geq 0\}$$

which is isometric to a spherical triangle of $S^2(1)$ with $\pi/2$ as its three angles. The orbit labeled by (x, y, z) is exactly $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$.

To investigate those *G-invariant minimal hypersurfaces*, M^n, in S^{n+1} we study their generating curves, $\gamma(s) = M^n/G$, in the orbit space S^{n+1}/G [3, 7].

LEMMA 2.1. *Let M^n be a G-invariant hypersurface in S^{n+1}. Then there is a local orthonormal frame field $e_1, \ldots, e_{n+1}$ on S^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n and $h_{ij} = 0$ if $i \neq j$.*

Proof. Let $(X_0, Y_0, Z_0) \in M^n \subset S^{n+1}$ with $x = |X_0|$, $y = |Y_0|$ and $z = |Z_0|$ and choose a local orthonormal frame field on a neighborhood of (X_0, Y_0, Z_0) as follows.

First, we choose vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ on a neighborhood U of (X_0, Y_0, Z_0) in the orbit $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ such that:

(1) $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}$ are lifts of orthonormal tangent vector fields $u_1, \ldots, u_{k-1}$ on a neighborhood of X_0 in $S^{k-1}(x)$ to $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ respectively,

(2) $\widetilde{v}_1, \ldots, \widetilde{v}_{k-1}$ are lifts of orthonormal tangent vector fields $v_1, \ldots, v_{k-1}$ on a neighborhood of Y_0 in $S^{k-1}(y)$ to $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ respectively,

(3) $\widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ are lifts of orthonormal tangent vector fields $w_1, \ldots, w_{q-1}$ on a neighborhood of Z_0 in $S^{q-1}(z)$ to $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ respectively.

Second, let $c(t) = (c_1(t), c_2(t), c_3(t))$ be a unit speed curve in S^{n+1}/G orthogonal to the curve $\gamma(s) = (x(s), y(s), z(s))$. For each $p = (X, Y, Z) \in U$, let $\widetilde{\gamma}(p, s)$ and $\widetilde{c}(p, t)$ be the horizontal lifts in S^{n+1} of $\gamma(s)$ and $c(t)$ through p respectively. Then we know

$$\widetilde{\gamma}(p, s) = \left(x(s)\frac{X}{x},\, y(s)\frac{Y}{y},\, z(s)\frac{Z}{z}\right) \text{ and } \widetilde{c}(p, t) = \left(c_1(t)\frac{X}{x},\, c_2(t)\frac{Y}{y},\, c_3(t)\frac{Z}{z}\right)$$

and so,

$$\widetilde{\gamma}'(p, s) = \left(x'(s)\frac{X}{x},\, y'(s)\frac{Y}{y},\, z'(s)\frac{Z}{z}\right) \text{ and } \widetilde{c}'(p, t) = \left(c_1'(t)\frac{X}{x},\, c_2'(t)\frac{Y}{y},\, c_3'(t)\frac{Z}{z}\right).$$

Third, we extend these vector fields over a neighborhood of (X_0, Y_0, Z_0) in S^{n+1} as follows:

(1) we translate $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ Euclidian parallel along $\widetilde{\gamma}$.

(2) next, we extend $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{c}'$ over a neighborhood of S^{n+1} properly.

Then these extended vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{c}'$ is a local orthonormal frame field in S^{n+1}. After restriction these vector fields to M^n, $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{k-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}'$ are tangent to M^n. For convenience, we write them as $e_1, \ldots, e_{n+1}$ in order.

Let $\bar{\alpha}_i(u) = (\alpha_i(u), Y, Z)$ be a curve in $S^{k-1}(x) \times S^{k-1}(y) \times S^{q-1}(z)$ through p such that $\bar{\alpha}_i'(0) = (\alpha_i'(0), 0, 0) = \widetilde{u}_i(p)$. Then,

$$\widetilde{\gamma}(\bar{\alpha}_i(u), s) = \left(x(s)\frac{\alpha_i(u)}{x}, y(s)\frac{Y}{y}, z(s)\frac{Z}{z}\right),$$

and

$$\widetilde{c}(\bar{\alpha}_i(u), t) = \left(c_1(t)\frac{\alpha_i(u)}{x}, c_2(t)\frac{Y}{y}, c_3(t)\frac{Z}{z}\right).$$

It implies that

$$\widetilde{\gamma}'(\bar{\alpha}_i(u), s) = \left(x'(s)\frac{\alpha_i(u)}{x}, y'(s)\frac{Y}{y}, z'(s)\frac{Z}{z}\right),$$

and

$$\widetilde{c}'(\bar{\alpha}_i(u), t) = \left(c_1'(t)\frac{\alpha_i(u)}{x}, c_2'(t)\frac{Y}{y}, c_3'(t)\frac{Z}{z}\right).$$

Let $\overline{\nabla}$ and $\overline{\overline{\nabla}}$ be the Riemannian connections on S^{n+1} and $\mathbb{R}^{n+2}$, respectively. Then since $\overline{\nabla} = \overline{\overline{\nabla}}^{\top}$, we have

$$(2.1)\quad \begin{cases} \overline{\nabla}_{\widetilde{u}_i(p)}\widetilde{\gamma}' = \left\{\dfrac{x'(0)}{x}(\alpha_i'(0), 0, 0)\right\}^{\top} = \left\{\dfrac{x'(0)}{x}\widetilde{u}_i(p)\right\}^{\top} = \dfrac{x'(0)}{x}\widetilde{u}_i(p), \\ \overline{\nabla}_{\widetilde{u}_i(p)}\widetilde{c}' = \left\{\dfrac{c_1'(0)}{x}(\alpha_i'(0), 0, 0)\right\}^{\top} = \left\{\dfrac{c_1'(0)}{x}\widetilde{u}_i(p)\right\}^{\top} = \dfrac{c_1'(0)}{x}\widetilde{u}_i(p) \end{cases}$$

and so,

$$(2.2)\qquad h_{ij} = \langle \overline{\nabla}_{\widetilde{u}_i(p)}\widetilde{u}_j, \widetilde{c}' \rangle = -\left\langle \widetilde{u}_j(p), \frac{c_1'(0)}{x}\widetilde{u}_i(p) \right\rangle = \frac{-c_1'(0)}{x}\delta_{ij}.$$

Similarly, we have

$$(2.3)\qquad \begin{cases} h_{(k-1+i)(k-1+j)} = \langle \overline{\nabla}_{\widetilde{v}_i(p)}\widetilde{v}_j, \widetilde{c}' \rangle = \dfrac{-c_2'(0)}{y}\delta_{ij}, \\ h_{(2k-2+i)(2k-2+j)} = \langle \overline{\nabla}_{\widetilde{w}_i(p)}\widetilde{w}_j, \widetilde{c}' \rangle = \dfrac{-c_3'(0)}{z}\delta_{ij}. \end{cases}$$

And, since $\nabla_{\gamma'(P)}\gamma' = (x''(0), y''(0), z''(0))^{\top}$ on S^{n+1}/G,

$$\begin{aligned}
(2.4)\ h_{nn} &= \langle\, \overline{\nabla}_{\widetilde{\gamma}'}\widetilde{\gamma}',\, \vec{c}'\,\rangle \\
&= \langle\, (x''(0)\frac{X}{x},\, y''(0)\frac{Y}{y},\, z''(0)\frac{Z}{z})^{\top},\, (c_1'(0)\frac{X}{x},\, c_2'(0)\frac{Y}{y},\, c_3'(0)\frac{Z}{z})\,\rangle \\
&= x''(0)\, c_1'(0) + y''(0)\, c_2'(0) + z''(0)\, c_3'(0) \\
&= \langle\, (x''(0), y''(0), z''(0)),\, \mathfrak{n}\,\rangle \\
&= \langle\, \nabla_{\gamma'}\gamma',\, \mathfrak{n}\,\rangle = \kappa_g(\gamma),
\end{aligned}$$

where $\mathfrak{n} = (c_1{}'(0), c_2{}'(0), c_3{}'(0))$ and $\kappa_g(\gamma)$ is the geodesic curvature. Recall that

$$\gamma(s) = (\sin r(s)\cos\theta(s), \sin r(s)\sin\theta(s), \cos r(s)).$$

Then, we have

$$\gamma'(s) = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta},$$

where $\partial/\partial r = (\cos r\cos\theta, \cos r\sin\theta, -\sin r)$ and $\partial/\partial\theta = \sin r(-\sin\theta, \cos\theta, 0)$. Thus, we see

$$|\frac{\partial}{\partial r}| = 1, \quad |\frac{\partial}{\partial\theta}|^2 = \sin^2 r \quad \text{and} \quad \langle\frac{\partial}{\partial r},\ \frac{\partial}{\partial\theta}\ \rangle = 0.$$

And we see

$$1 = |\gamma'(s)|^2 = \left(\frac{dr}{ds}\right)^2 + \left(\frac{d\theta}{ds}\right)^2 |\frac{\partial}{\partial\theta}|^2 = \left(\frac{dr}{ds}\right)^2 + \left(\frac{d\theta}{ds}\right)^2 \sin^2 r.$$

Hence, we obtain

$$\cos\alpha = \langle\gamma',\ \frac{\partial}{\partial r}\rangle/|\gamma'|\,|\frac{\partial}{\partial r}| = \frac{dr}{ds} \quad \text{and} \quad \sin\alpha = \frac{d\theta}{ds}\sin r,$$

where α is the angle between the curve γ and the radial direction $\partial/\partial r$.

Suppose S^{n+1}/G is orientated by the frame field $\{(\partial/\partial r),\ 1/\sin r\,(\partial/\partial\theta)\}$ and $U = (\partial/\partial r)\times 1/\sin r\,(\partial/\partial\theta)$. Then we have

$$\begin{aligned}
\mathfrak{n} = U\times T = U\times\gamma' &= U\times\left(\frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial\theta}\right) \\
&= \frac{1}{\sin r}\frac{dr}{ds}\frac{\partial}{\partial\theta} - \sin r\frac{d\theta}{ds}\frac{\partial}{\partial r} \\
&= \frac{dr}{ds}(-\sin\theta, \cos\theta, 0) - \sin r\frac{d\theta}{ds}(\cos r\cos\theta, \cos r\sin\theta, -\sin r) \\
&= (c_1{}'(0), c_2{}'(0), c_3{}'(0)).
\end{aligned}$$

Thus, we get

$$\begin{aligned}\kappa_g(\gamma) &= \langle \nabla_{\gamma'}\gamma', \mathfrak{n}\rangle \\ &= \langle \nabla_{\gamma'}\left(\frac{dr}{ds}\frac{\partial}{\partial r}+\frac{d\theta}{ds}\frac{\partial}{\partial\theta}\right), \left(\frac{1}{\sin r}\frac{dr}{ds}\frac{\partial}{\partial\theta}-\sin r\frac{d\theta}{ds}\frac{\partial}{\partial r}\right)\rangle \\ &= \frac{d\alpha}{ds}+\cos r\frac{d\theta}{ds}.\end{aligned}$$

Therefore, from (2.2), (2.3) and (2.4) we obtain

$$\left\{\begin{aligned} h_{ii} &= -\frac{c_1'(0)}{x} = \cos r\,\frac{d\theta}{ds}+\frac{\tan\theta}{\sin r}\frac{dr}{ds}, \\ h_{(k-1+i)(k-1+i)} &= -\frac{c_2'(0)}{y} = \cos r\,\frac{d\theta}{ds}-\frac{\cot\theta}{\sin r}\frac{dr}{ds}, \\ h_{(2k-2+i)(2k-2+i)} &= -\frac{c_3'(0)}{z} = -\frac{\sin^2 r}{\cos r}\frac{d\theta}{ds}, \\ h_{nn} &= \kappa_g(\gamma) = \frac{d\alpha}{ds}+\cos r\frac{d\theta}{ds}, \\ h_{ij} &= 0, \quad \text{if } i\neq j.\end{aligned}\right. \tag{2.5}$$

The proof of Lemma 2.1 is complete. □

LEMMA 2.2. *Let M^n be a G-invariant hypersurface in S^{n+1} and let $\{e_A\}$ be the local orthonormal frame field on S^{n+1} in Lemma 2.1. Then,*

(1) all $h_{ijl}=0$ except when $\{i,j,l\}$ is a permutation of either $\{i,i,n\}$,

(2) all $h_{ijlm}=0$ except when $\{i,j,l,m\}$ is a permutation of either $\{i,i,j,j\}$.

Proof. (1) Since h_{ijl} is symmetric in all indices, it suffices to show that $h_{ijl}=0$ if $i\le j\le l$ and $\{i,j,l\}\neq\{i,i,n\}$.

(1.a) *Case* 1. $j\neq i:$ Lemma 2.1 implies that $h_{ij}=0$ and

$$h_{ijl} = e_l(h_{ij})+\sum_s h_{sj}\,\omega_{si}(e_l)+\sum_s h_{is}\,\omega_{sj}(e_l) = (h_{jj}-h_{ii})\,\omega_{ji}(e_l). \tag{2.6}$$

Since $h_{ii}=h_{jj}$ if $i,j\le k-1$, (2.6) implies $h_{ijl}=0$ for all l.

If $k\le i,j\le 2k-2$ or $2k-1\le i,j\le n-1$, then also $h_{ijl}=0$ for all l.

And, if $i\le k-1$ and $k\le j<n$, then for all l $(>i)$ we have

$$h_{ijl} = h_{lij} = e_j(h_{li})+(h_{ii}-h_{ll})\,\omega_{il}(e_j) = (h_{ii}-h_{ll})\langle\nabla_{e_j}e_i,\, e_l\rangle = 0, \tag{2.7}$$

since $\nabla_{e_j} e_i = 0$ by the Koszul formula. In the similar cases, we also have $h_{ijl} = 0$.

Moreover, if $j = l = n$, then $h_{inn} = h_{nni} = e_i(h_{nn}) = 0$ since h_{nn} is constant on each orbit from (2.5).

(1.*b*) *Case* 2. $j = i$ and $l \neq n$: Since h_{ii} is constant on each orbit,

$$(2.8) \qquad h_{ijl} = h_{iil} = e_l(h_{ii}) + \sum_s h_{si}\,\omega_{si}(e_l) + \sum_s h_{is}\,\omega_{si}(e_l) = e_l(h_{ii}) = 0.$$

Therefore, we see all $h_{ijl} = 0$ except when $\{i, j, l\}$ is a permutation of either $\{i, i, n\}$.

(2.*a*) *Case* 1. i, j, l, m are distinct : Without loss of generality, it suffices to show that $h_{ijln} = h_{ijnl} = 0$ and $h_{ijlm} = 0$ for all i, j, l, m such that i, j, l, $m < n$.

By using (1), we easily see that

$$(2.9) h_{ijln} = e_n(h_{ijl}) + \sum_s h_{sjl}\,\omega_{si}(e_n) + \sum_s h_{isl}\,\omega_{sj}(e_n) + \sum_s h_{ijs}\,\omega_{sl}(e_n) = 0,$$

since i, j, $l < n$ and i, j, l are distinct.

And, from (1.4) and Lemma 2.1 we also have

$$(2.10) \quad h_{ijnl} = h_{ijln} + \sum_s h_{sj}\,R_{sinl} + \sum_s h_{is}\,R_{sjnl} = h_{jj}\,R_{jinl} + h_{ii}\,R_{ijnl} = 0.$$

If i, j, l, $m < n$, then from (1) we can easily see

$$(2.11) \quad h_{ijlm} = e_m(h_{ijl}) + \sum_s \{h_{sjl}\,\omega_{sj}(e_m) + h_{isl}\,\omega_{sj}(e_m) + h_{ijs}\,\omega_{sl}(e_m)\} = 0.$$

(2.*b*) *Case* 2. $j \neq l$: Let us show that $h_{iijl} = h_{jlii} = h_{jjjl} = h_{ljjj} = 0$.

If $j \neq l$, then

$$(2.12) \qquad h_{iijl} - h_{iilj} = \sum_s h_{si}\,R_{sijl} + \sum_s h_{is}\,R_{sijl} = 2h_{ii}R_{iijl} = 0.$$

Hence, we may assume $l \neq n$. So, $e_l(h_{iij}) = 0$ since h_{iij} is constant on each orbit. Hence, we have

$$(2.13)\, h_{iijl} = e_l(h_{iij}) + \sum_s h_{sij}\,\omega_{si}(e_l) + \sum_s h_{isj}\,\omega_{si}(e_l) + \sum_s h_{iis}\,\omega_{sj}(e_l)$$
$$= 2h_{jij}\,\omega_{ji}(e_l) - h_{iin}\,\omega_{nj}(e_l) = 0,$$

since $h_{jji} = 0$ if $i \neq n$ and $\omega_{nj}(e_l) = \langle \nabla_{e_l} e_n,\, e_j \rangle = 0$ from (2.1).

And since $j \neq l$, from (1.4), Lemma 2.1 and (2.13) we also have

$$(2.14) \qquad h_{jlii} = h_{ijli} = h_{ijil} + \sum_s h_{sj}\,R_{sili} + \sum_s h_{is}\,R_{sjli}$$
$$= h_{iijl} + h_{jj}\,R_{jili} + h_{ii}\,R_{ijli} = 0.$$

Moreover, if $j \neq n$, then

$$(2.15) \qquad h_{jjjl} = e_l(h_{jjj}) + \sum_s h_{sjj}\,\omega_{sj}(e_l) + \sum_s h_{jsj}\,\omega_{sj}(e_l)$$
$$+ \sum_s h_{jjs}\,\omega_{sj}(e_l) = 3h_{jjn}\,\omega_{nj}(e_l) = 0,$$

since $\omega_{nj}(e_l) = \langle \nabla_{e_l} e_n, e_j \rangle = 0$ from (2.1). And so,

$$(2.16) \quad h_{jjlj} = h_{jjjl} + \sum_s h_{sj}\,R_{sjlj} + \sum_s h_{js}\,R_{sjlj} = h_{jjjl} + 2h_{jj}\,R_{jjlj}$$
$$= 0.$$

Hence, we have that if $j \neq n$, then $h_{jjjl} = h_{ljjj} = 0$.

If $j = n,\ l \neq n$, then from (1),

$$(2.17) \quad h_{lnnn} = h_{nnnl} = e_l(h_{nnn}) + \sum_s h_{snn}\,\omega_{sn}(e_l) + \sum_s h_{nsn}\,\omega_{sn}(e_l)$$
$$+ \sum_s h_{nns}\,\omega_{sn}(e_l) = e_l(h_{nnn}) = 0.$$

since $e_l(h_{nnn}) = 0$ since $h_{nnn} = e_n(h_{nn})$ is also constant on each orbit from (2.5). It completes the proof of Lemma 2.2. □

Under such frame field in Lemma 2.1, we have

$$(2.18) \qquad e_k(h_{ii}) = h_{iik} - \sum_s h_{si}\omega_{si}(e_k) - \sum_s h_{is}\omega_{si}(e_k) = h_{iik}.$$

Hence, in the case M^n is minimal, by differentiating $\sum_m h_{mm} = 0$ we have

$$\sum_m h_{mmij} = 0. \tag{2.19}$$

In the case S is constant, by differentiating $\sum_{i,j} h_{ij}^2 = S$ twice, we have

$$\sum_{i,j}(h_{ij}h_{ijkl} + \sum_{i,j} h_{ijk}\, h_{ijl}) = 0. \tag{2.20}$$

3. G-invariant Minimal Hypersurface in S^{n+1}

Throughout this section, we assume that $G \simeq O(k) \times O(k) \times O(q)$ and M^n is a closed G-invariant minimal hypersurface with constant scalar curvature in S^{n+1}. Let $\{e_A\}$ be the local orthonormal frame field on S^{n+1} in Lemma 2.1. For convenience, we rewrite

$$\begin{cases} h_{11} = \cdots = h_{(k-1)(k-1)} = h_{11} = \lambda_1, \\ h_{kk} = \cdots = h_{(2k-2)(2k-2)} = h_{22} = \lambda_2, \\ h_{(2k-1)(2k-1)} = \cdots = h_{(n-1)(n-1)} = h_{33} = \lambda_3. \end{cases} \tag{3.1}$$

Then

$$\begin{cases} \sum_i h_{ii} = (k-1)h_{11} + (k-1)h_{22} + (q-1)h_{33} + h_{nn} = 0, \\ \sum_i h_{ii}^2 = (k-1)h_{11}^2 + (k-1)h_{22}^2 + (q-1)h_{33}^2 + h_{nn}^2 = S. \end{cases} \tag{3.2}$$

By differentiating the both sides of (3.2) with respect to e_n respectively, we have

$$\begin{cases} (k-1)h_{11n} + (k-1)h_{22n} + (q-1)h_{33n} + h_{nnn} = 0, \\ (k-1)h_{11}h_{11n} + (k-1)h_{22}h_{22n} + (q-1)h_{33}h_{33n} + h_{nn}h_{nnn} = 0. \end{cases} \tag{3.3}$$

By differentiating (3.3) with respect to e_n respectively, we have

$$\begin{cases} (k-1)h_{11nn} + (k-1)h_{22nn} + (q-1)h_{33nn} + h_{nnnn} = 0, \\ (k-1)h_{11}h_{11nn} + (k-1)h_{22}h_{22nn} + (q-1)h_{33}h_{33nn} + h_{nn}h_{nnnn} \\ \quad +(k-1)h_{11n}^2 + (k-1)h_{22n}^2 + (q-1)h_{33n}^2 + h_{nnn}^2 = 0, \end{cases} \tag{3.4}$$

since

$$e_n(h_{iin}) = h_{iinn} - \sum_s \{h_{sin}\,\omega_{si}(e_n) + h_{isn}\,\omega_{si}(e_n) + h_{iis}\,\omega_{sn}(e_n)\} = h_{iinn}.$$

From (1.5), we also have

$$h_{ii11} + h_{ii22} + \cdots + h_{iinn} = (n - S)h_{ii}. \tag{3.5}$$

Since S is constant, from (1.6) and Lemma 2.2 we have

$$3(k-1)h_{11n}^2 + 3(k-1)h_{22n}^2 + 3(q-1)h_{33n}^2 + h_{nnn}^2 = S(S-n). \tag{3.6}$$

Here, if $i \neq n$, from (1.3) we know

$$h_{iin} = h_{ini} = e_i(h_{in}) + \sum_s h_{sn}\omega_{si}(e_i) + h_{is}\omega_{sn}(e_i) = (h_{nn} - h_{ii})\,\omega_{ni}(e_i) \tag{3.7}$$

and

$$\begin{aligned} h_{iiii} &= e_i(h_{iii}) + \sum_s \{h_{sii}\omega_{si}(e_i) + h_{isi}\omega_{si}(e_i) + h_{iis}\omega_{si}(e_i)\} \\ &= 3h_{iin}\,\omega_{ni}(e_i). \end{aligned} \tag{3.8}$$

Moreover, if $i,\ j \neq n$ and $i \neq j$, then

$$\begin{aligned} h_{iijj} &= e_j(h_{iij}) + \sum_s \{h_{sij}\omega_{si}(e_j) + h_{isj}\omega_{si}(e_j) + h_{iis}\omega_{sj}(e_j)\} \\ &= h_{iin}\,\omega_{nj}(e_j). \end{aligned} \tag{3.9}$$

Now, to prove Theorem 3.2 we need the following lemma.

LEMMA 3.1. *With notation as above,*
(1) If $h_{11} = h_{nn} = \lambda$ at some point p, then

$$(1-2\lambda^2)S + (k-1)\lambda_2^3\lambda + (q-1)\lambda_3^3\lambda + k\lambda^4 + n\lambda^2 = 0.$$

(2) If $h_{22} = h_{nn} = \lambda$ at some point p, then

$$(1-2\lambda^2)S + (k-1)\lambda_1^3\lambda + (q-1)\lambda_3^3\lambda + k\lambda^4 + n\lambda^2 = 0.$$

(3) If $h_{33} = h_{nn} = \lambda$ at some point p, then

$$(1-2\lambda^2)S + (k-1)\lambda_1^3\lambda + (k-1)\lambda_2^3\lambda + q\lambda^4 + n\lambda^2 = 0.$$

Proof. (1) Suppose $h_{11} = h_{nn} = \lambda$ at some point p. From (3.7), we have

$$h_{11n}(p) = 0. \tag{3.10}$$

Using (3.8) and (3.10), we have at p

$$h_{1111} = h_{1122} = \cdots = h_{11(n-1)(n-1)} = 0. \tag{3.11}$$

Hence, (3.5) and (3.11) imply

$$h_{11nn} = (n - S)h_{nn} \tag{3.12}$$

and (1.4) implies

$$h_{nn11} = h_{11nn} + (h_{nn} - h_{11})(1 + h_{nn}h_{11}) = h_{11nn}. \tag{3.13}$$

Since $\sum_{i,j} h_{ij1}^2 = 0$ at p, from (2.17) we have

$$(k-1)(h_{11}h_{1111} + h_{22}\, h_{2211}) + (q-1)h_{33}\, h_{3311} + h_{nn}\, h_{nn11} = 0. \tag{3.14}$$

Then, by using (1.4) and (3.11) we know

$$h_{2211} = (\lambda_2 - \lambda)(1 + \lambda_2\, \lambda) \quad \text{and} \quad h_{3311} = (\lambda_3 - \lambda)(1 + \lambda_3\, \lambda). \tag{3.15}$$

Hence, (3.14) and (3.15) imply

$$(k-1)\lambda_2(\lambda_2 - \lambda)(1+\lambda_2\, \lambda) + (q-1)\lambda_3\, (\lambda_3 - \lambda)(1+\lambda_3\, \lambda) + \lambda^2\, (n-S) = 0. \tag{3.16}$$

Here, since

$$k\lambda + (k-1)\lambda_2 + (q-1)\lambda_3 = 0 \quad \text{and} \quad k\lambda^2 + (k-1)\lambda_2^2 + (q-1)\lambda_3^2 = S, \tag{3.17}$$

(3.16) becomes

$$(1 - 2\lambda^2)S + (k-1)\lambda_2^3\lambda + (q-1)\lambda_3^3\lambda + k\lambda^4 + n\lambda^2 = 0. \tag{3.18}$$

(2) and (3) These proofs use exactly the same argument; one just replaces h_{11} by h_{22} and h_{33} throughout, respectively. It completes the proof of Lemma 3.1. □

THEOREM 3.2. *If M^n has 2 distinct principal curvatures at some point p, then $S = n$.*

Proof. Suppose M^n has 2 distinct principal curvatures at p. Consider now the four cases in the proof of that theorem for some $\lambda \neq 0$.

Case 1. $h_{22} = h_{33} = h_{nn} = \lambda$ $(\neq h_{11})$ *at the point p.* Then (3.2) becomes

$$\begin{cases} (k-1)h_{11} + (k-1)h_{22} + (q-1)h_{33} + h_{nn} = (k-1)\lambda_1 + (k+q-1)\lambda = 0, \\ S = (k-1)h_{11}^2 + (k-1)h_{22}^2 + (q-1)h_{33}^2 + h_{nn}^2 = (k-1)\lambda_1^2 + (k+q-1)\lambda^2. \end{cases}$$

From the above,

$$\lambda_1 = -\frac{k+q-1}{k-1}\lambda. \tag{3.19}$$

And so, since $2k + q - 2 = n$

$$S = \left\{(k-1)\frac{(k+q-1)^2}{(k-1)^2} + (k+q-1)\right\}\lambda^2 = \frac{n(k+q-1)}{k-1}\lambda^2. \tag{3.20}$$

Moreover, by substituting (3.19) and (3.20) for (3) of Lemma 3.1

$$\begin{aligned} 0 &= (1-2\lambda^2)S + (k-1)(\lambda_1^3 + \lambda_2^3)\lambda + q\lambda^4 + n\lambda^2 \\ &= (1-2\lambda^2)\frac{n(k+q-1)}{k-1}\lambda^2 + (k-1)\left(1 - \frac{(k+q-1)^3}{(k-1)^3}\right)\lambda^4 + q\lambda^4 + n\lambda^2. \end{aligned}$$

And so,

$$\begin{aligned} 0 &= (1-2\lambda^2)n(k+q-1)(k-1) + \{(k-1)^3 - (k+q-1)^3\}\lambda^2 + (q\lambda^2+n)(k-1)^2 \\ &= (k+q-1)\{-2n(k-1) + (k-1)^2 - (k+q-1)^2\}\lambda^2 + n^2(k-1) \\ &= -(k+q-1)n^2\lambda^2 + n^2(k-1) \end{aligned}$$

since $2k + q - 2 = n$ and $k + q - 1 = n - k + 1$. Hence,

$$\lambda^2 = \frac{k-1}{k+q-1} \tag{3.21}$$

and

$$S = \frac{n(k+q-1)}{k-1}\lambda^2 = n. \tag{3.22}$$

i.e.,

$$M^n = S^{k-1}\left(\sqrt{\frac{k-1}{n}}\right) \times S^{k+q-1}\left(\sqrt{\frac{k+q-1}{n}}\right).$$

Case 2. $h_{11} = h_{22} = h_{nn} = \lambda$ ($\neq h_{33}$) *at the point* p. Then (3.2) becomes

$$\begin{cases} (2k-1)\lambda + (q-1)\lambda_3 = 0, \\ S = (2k-1)\lambda^2 + (q-1)\lambda_3^2. \end{cases}$$

From the above,

$$\lambda_3 = -\frac{2k-1}{q-1}\lambda. \tag{3.23}$$

And so

$$S = \frac{n(2k-1)}{q-1}\lambda^2. \tag{3.24}$$

Moreover, by substituting (3.23) and (3.24) for (1) of Lemma 3.1

$$\begin{aligned} 0 &= (1-2\lambda^2)S + (k-1)\lambda_2^3\lambda + (q-1)\lambda_3^3\lambda + k\lambda^4 + n\lambda^2 \\ &= (1-2\lambda^2)\frac{n(2k-1)}{q-1}\lambda^2 + (2k-1)\lambda^4 - (q-1)\frac{(2k-1)^3}{(q-1)^3}\lambda^4 + n\lambda^2. \end{aligned}$$

And so,

$$(2k-1)\left\{-2n(q-1) + (q-1)^2 - (n-q+1)^2\right\}\lambda^2 = -n^2(q-1). \tag{3.25}$$

Hence,

$$\lambda^2 = \frac{q-1}{2k-1}, \tag{3.26}$$

and

$$S = \frac{n(2k-1)}{q-1}\lambda^2 = n. \tag{3.27}$$

i.e.,

$$M^n = S^{2k-1}\left(\sqrt{\frac{2k-1}{n}}\right) \times S^{q-1}\left(\sqrt{\frac{q-1}{n}}\right).$$

Case 3. $h_{11} = h_{22} = \lambda_1$, $h_{33} = h_{nn} = \lambda$ *at the point* p. Then (3.2) becomes

$$\begin{cases} (2k-2)\lambda_1 + q\lambda = 0, \\ S = (2k-2)\lambda_1^2 + q\lambda^2. \end{cases}$$

From the above,

$$\lambda_1 = -\frac{q}{2k-2}\lambda. \tag{3.28}$$

And so

$$S = \frac{nq}{(n-q)}\lambda^2 \tag{3.29}$$

since $2k-2 = n-q$. By substituting (3.28) and (3.29) for (3) of Lemma 3.1,

$$\begin{aligned} 0 &= (1-2\lambda^2)S + (k-1)(\lambda_1^3\lambda + \lambda_2^3\lambda) + q\lambda^4 + n\lambda^2 \\ &= (1-2\lambda^2)\frac{nq}{(2k-2)}\lambda^2 - 2(k-1)\frac{q^3}{(2k-2)^3}\lambda^4 + q\lambda^4 + n\lambda^2. \end{aligned}$$

And so,

$$\left\{-2nq(n-q) - q^3 + q(n-q)^2\right\}\lambda^2 = -n^2(n-q). \tag{3.30}$$

Hence, we have

$$\lambda^2 = \frac{n-q}{q}, \tag{3.31}$$

and

$$S = \frac{nq}{(n-q)}\lambda^2 = n. \tag{3.32}$$

i.e.,

$$M^n = S^{2k-2}\left(\sqrt{\frac{2k-2}{n}}\right) \times S^q\left(\sqrt{\frac{q}{n}}\right).$$

But, it is not G-invariant.

Case 4. $h_{11} = h_{22} = h_{33} = \lambda\ (\neq h_{nn})$ *at the point* p. Then from (2.5), we have at p

$$\cos r\,\frac{d\theta}{ds} + \frac{\tan\theta}{\sin r}\frac{dr}{ds} = \cos r\,\frac{d\theta}{ds} - \frac{\cot\theta}{\sin r}\frac{dr}{ds} = -\frac{\sin^2 r}{\cos r}\frac{d\theta}{ds}.$$

It implies that

$$\frac{dr}{ds} = 0 \quad\text{and}\quad \frac{d\theta}{ds} = 0,$$

which means that $h_{11} = h_{22} = h_{33} = h_{nn} = \lambda = 0$ at p. It is contrary to the hypothesis. We complete the proof of our theorem. □

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