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FACTORIZATION OF A HILBERT SPACE ON THE BIDISK

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Abstract. Let $S(z_1, z_2)$, $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ be power series with operator coefficients such that $S(z_1, z_2) = S_1(z_1, z_2)S_2(z_1, z_2)$. Assume that the multiplications by $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ are contractive transformations in $\mathbf{H}(\mathbb{D}^2, \mathcal{C})$. Then the factorizations of spaces $\mathcal{D}(\mathbb{D}, \tilde{S})$ and $\mathcal{D}(\mathbb{D}^2, S)$ are well-behaved.

1. Introduction

The multiplication of the transfer functions of linear systems induces a factorization of the state spaces of those linear systems [7,8]. In this paper, we study a factorization of the state space of a unitary linear system whose transfer function is of two variables inside bidisk.

Let $\mathcal{H}$ and $\mathcal{C}$ be Hilbert spaces. Then we define a linear system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathcal{C} \longrightarrow \mathcal{H} \oplus \mathcal{C} \tag{1.1}$$

that is a matrix of a continuous linear transformation where $\mathcal{H}$ is called a state space and $\mathcal{C}$ is called a coefficient space of a linear system. Also we define that the transfer function $S(z)$ of a given linear system is of the form

$$S(z) = D + zC(I - zA)^{-1}B.$$

A linear system is said to be observable if there is no nonzero element f of the state space such that $CA^n f = 0$ for every nonnegative integer n. A linear system is called to be in canonical form if it is observable and every element of the state space is power series with coefficients in $\mathcal{C}$.

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Let $\mathbb{D}$ be the open unit disk in the complex plain $\mathbb{C}$. Then the Hardy space $\mathbf{H}(\mathbb{D},\mathcal{C})$ is defined as

$$\mathbf{H}(\mathbb{D},\mathcal{C}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : a_n \in \mathcal{C}, z \in \mathbb{D}, \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{C}} < \infty\}$$

with the scalar product $\|f(z)\|^2_{\mathbf{H}(\mathbb{D},\mathcal{C})} = \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{C}}$. Moreover, the space $\mathbf{H}(\mathbb{D},\mathcal{C})$ is the state space of a canonical linear system whose transfer function is identically zero.

In this paper we study unitary linear systems of two variables. Therefore we define the Hardy of the bidisk. The Hardy space of the bidisk $\mathbf{H}(\mathbb{D}^2,\mathcal{C})$ is defined as

$$\mathbf{H}(\mathbb{D}^2,\mathcal{C}) = \{F(z_1,z_2) = \sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j : z_i, z_j \in \mathbb{D}, a_{ij} \in \mathcal{C}, \sum_{i,j=0}^{\infty} \|a_{ij}\|^2_{\mathcal{C}} < \infty\}$$

with the inner product

$$\|F(z_1,z_2)\|^2_{\mathbf{H}(\mathbb{D}^2,\mathcal{C})} = \sum_{i,j=0}^{\infty} \|a_{ij}\|^2_{\mathcal{C}}.$$

Assume that $F(z_1,z_2) = \sum_{i,j=0}^{\infty} a_{ij} z_1^i z_2^j$ is an element in $\mathbf{H}(\mathbb{D}^2,\mathcal{C})$ and

$$(1.2) \qquad f(z) = \begin{pmatrix} f_0(z) \\ f_1(z) \\ \vdots \end{pmatrix} \quad \text{and} \quad g(z) = \begin{pmatrix} g_0(z) \\ g_1(z) \\ \vdots \end{pmatrix}$$

where $f_j(z) = \sum_{i=0}^{\infty} a_{ij} z^i$ and $g_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j$. Then $f(z)$ and $g(z)$ belong to $\mathbf{H}(\mathbb{D}, l_2(\mathcal{C}))$, and $F(z_1,z_2)$ can be written by

$$F(z_1,z_2) = E(z_2)f(z_1) = E(z_1)g(z_2)$$

where $E(z) = (I_{\mathcal{C}}, zI_{\mathcal{C}}, z^2 I_{\mathcal{C}}, \ldots)$ and

$$\|F\|_{\mathbf{H}(\mathbb{D}^2,\mathcal{C})} = \|f\|_{\mathbf{H}(\mathbb{D},l_2(\mathcal{C}))} = \|g\|_{\mathbb{H}(\mathbb{D},l_2(\mathcal{C}))}.$$

Assume that $S(z_1,z_2)$ is a power series with operator coefficients such that the multiplication by $S(z_1,z_2)$ is a contractive transformation in $\mathbb{H}(\mathbb{D}^2,\mathcal{C})$.

Let

$$(1.3) \qquad \tilde{S}(z) = \begin{pmatrix} s_0(z) & 0 & 0 & 0 & \cdots \\ s_1(z) & s_0(z) & 0 & 0 & \cdots \\ s_2(z) & s_1(z) & s_0(z) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

where $s_j(z) = \sum_{i=0}^{\infty} s_{ij} z^i$. Then $S(z_1, z_2)F(z_1, z_2) = E(z_2)\tilde{S}(z_1)f(z_1)$ for $F(z_1, z_2) \in \mathbb{H}(\mathbb{D}^2, \mathcal{C})$ where $F(z_1, z_2) = E(z_2)f(z_1)$ and the multiplication by $\tilde{S}(z)$ is a contractive transformation in $\mathbb{H}(\mathbb{D}, l_2(\mathcal{C}))$. See [1,2].

A general construction of unitary linear systems with given transfer function is given by Azizov [4]. The existence of a unitary canonical linear system with given transfer function has been given by de Branges [5,6]. His construction makes use of the complementation theory on the space $\mathbf{H}(\mathbb{D}, \mathcal{C})$. Also the reproducing Kernel function can be used to characterize the state space of a linear system [3,10]. Alpay and Bolotnikov [1] have shown that a canonical linear system is not uniquely determined by its transfer function if the state space is a Krein space.

The following complementation theorem given by de Branges [6] is the main tool for this paper.

Theorem 1.1. *If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, then there is a unique Hilbert space $\mathcal{Q}$ contained continuously and contractively in $\mathcal{H}$ such that the inequality*

$$\|c\|_{\mathcal{H}} \ \leq \ \|a\|_{\mathcal{P}} + \|b\|_{\mathcal{Q}}$$

holds whenever $c = a + b$ with $a \in \mathcal{P}$ and $b \in \mathcal{Q}$. In addition, every element c of $\mathcal{H}$ has a unique such decomposition for which equality holds. In this case, a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$.

2. The spaces $\mathcal{H}(\mathbb{D}, \tilde{S})$ and $\mathcal{D}(\mathbb{D}, \tilde{S})$

On the bidisk, the complementation theory can also be used to construct the state space of a unitary linear system with a given transfer function [9].

Theorem 2.1. *Let $S(z_1, z_2)$ be a power series with operator coefficients such that the multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Then there exists a Hilbert space $\mathcal{H}(\mathbb{D}, \tilde{S})$ which is the state space of a unitary linear system whose transfer function is $\tilde{S}(z)$.*

It is well known that the state space $\mathcal{H}(\mathbb{D}, \tilde{S})$ of a linear system is a subset of elements in $\mathbb{H}_2(\mathbb{D}, l_2(\mathcal{C}))$ which satisfies

$$m(f) = \sup_{g \in \mathbb{H}_2(\mathbb{D}, l_2(\mathcal{C}))} \{\|f + \tilde{S}g\|^2_{\mathbb{H}_2(\mathbb{D}, l_2(\mathcal{C}))} - \|g\|^2_{\mathbb{H}_2(\mathbb{D}, l_2(\mathcal{C}))}\} < \infty. \tag{2.1}$$

In this case, the scalar product is defined by $\|f\|^2_{\mathcal{H}(\mathbb{D},\tilde{S})} = m(f)$ and the linear system

$$(2.2) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\mathbb{D},\tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix} \to \begin{pmatrix} \mathcal{H}(\mathbb{D},\tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix}$$

defined by $A(f(z)) = \frac{[f(z)-f(0)]}{z}$, $Bc = \frac{[\tilde{S}(z)-\tilde{S}(0)]c}{z}$, $Cf(z) = f(0)$ and $Dc = \tilde{S}(0)c$ is unitary whose transfer function is $\tilde{S}(z)$.

Theorem 2.2. *Let $S(z_1, z_2)$ be a power series with operator coefficients such that the multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. If $\mathcal{H}(\mathbb{D}^2, S)$ is defined by*

$$\mathcal{H}(\mathbb{D}^2, S) = \{F(z_1, z_2) : F(z_1, z_2) = E(z_2)f(z_1), f \in \mathcal{H}(\mathbb{D}, \tilde{S})\}$$

with the scalar product $\|F\|_{\mathcal{H}(\mathbb{D}^2,S)} = \|f\|_{\mathcal{H}(\mathbb{D},\tilde{S})}$, then $\mathcal{H}(\mathbb{D}^2, S)$ is the state space of a unitary linear system whose transfer function is $S(z_1, z_2)$.

Proof. Let $F(z_1, z_2) = E(z_2)f(z_1)$ for some $f(z) \in \mathcal{H}(\mathbb{D}, \tilde{S})$ and $k(z) \in \mathcal{H}(\mathbb{D}, \mathcal{C})$. If $k(z) = \sum a_n z^n$, then $k(z) = E(z)c$ where

$$c = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \end{pmatrix} \in l_2(\mathcal{C}).$$

Define

$$(2.3) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\mathbb{D}^2, S) \\ \mathcal{H}(\mathbb{D},\mathcal{C}) \end{pmatrix} \to \begin{pmatrix} \mathcal{H}(\mathbb{D}^2, S) \\ \mathcal{H}(\mathbb{D},\mathcal{C}) \end{pmatrix}$$

by

$$A(F(z_1, z_2)) = E(z_2)\frac{[f(z_1) - f(0)]}{z_1},$$

$$B(k(z)) = E(z_2)\frac{[\tilde{S}(z_1) - \tilde{S}(0)]c}{z_1}$$

$$\text{and} \quad C(F(z_1, z_2) = E(z)f(0) \quad \text{and} \quad D(k(z)) = E(z)\tilde{S}(0)c.$$

Then using the theorem 1.1, we can easily show that the linear system (2.3) is a unitary linear system whose transfer function is $S(z_1, z_2)$. □

Assume that $S(z_1, z_2)$ is a power series with operator coefficients such that the multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Then the extension space $\mathcal{D}(\mathbb{D}, \tilde{S})$ consisting of pairs

$(f(z), g(z))$ where $f(z) \in \mathcal{H}(\mathbb{D}, \tilde{S})$ and $g(z) = \sum_{n=0}^{\infty} a_n z^n$ for which

$$f_n(z) = z^n f(z) - \tilde{S}(z)(a_0 z^{n-1} + \cdots + a_{n-1}) \in \mathcal{H}(\mathbb{D}, \tilde{S})$$

and

$$c^* a_{n-1} = \langle B^* f_{n-1}(z), c\rangle_{l_2(\mathcal{C})}$$

is a Hilbert space. Also we can easily show that $\|(f(z), g(z))\|_{\mathcal{D}(\mathbb{D},\tilde{S})} = \|f(z)\|_{\mathcal{H}(\mathbb{D},\tilde{S})}$.

Consequently, the extension space $\mathcal{D}(\mathbb{D}, \tilde{S})$ is a state space of a unitary linear system which is defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{D}(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix} \to \begin{pmatrix} \mathcal{D}(\mathbb{D}, \tilde{S}) \\ l_2(\mathcal{C}) \end{pmatrix} \tag{2.4}$$

where $A(f(z), g(z)) = (\frac{f(z)-f(0)}{z}, zg(z) - \tilde{S}^*(z)f(0))$, $Bc = (\frac{\tilde{S}(z)-\tilde{S}(0)}{z}c$, $[I_{l_2(\mathcal{C})} - \tilde{S}^*(z)\tilde{S}(0)]c)$, $Cf(z) = f(0)$ and $Dc = \tilde{S}(0)c$.

Theorem 2.3. *Let $S(z_1, z_2)$ be a power series with operator coefficients such that the multiplication by $S(z_1, z_2)$ is a contractive transformation in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Let $\mathcal{D}(\mathbb{D}^2, S)$ be a set of pairs $(F(z_1, z_2), G(z_1, z_2))$ such that $F(z_1, z_2) = E(z_2)f(z_1)$ and $G(z_1, z_2) = E(z_2)g(z_1)$ where $(f(z), g(z)) \in \mathcal{D}(\mathbb{D}, \tilde{S})$. Then $\mathcal{D}(\mathbb{D}^2, S)$ is a Hilbert space with the scalar product*

$$\|(F, G)\|_{\mathcal{D}(\mathbb{D}^2,S)} = \|(f, g)\|_{\mathcal{D}(\mathbb{D},\tilde{S})}$$

and the state space of a unitary linear system.

Proof. From the definition of the scalar product, the space $\mathcal{D}(\mathbb{D}^2, S)$ is a Hilbert space. Let $k(z) \in \mathcal{H}(\mathbb{D}, \mathcal{C})$. If $k(z) = \sum a_n z^n$, then $k(z) = E(z)c$ where

$$c = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \end{pmatrix} \in l_2(\mathcal{C}).$$

Define

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\mathbb{D}^2, S) \\ \mathcal{H}(\mathbb{D}, \mathcal{C}) \end{pmatrix} \to \begin{pmatrix} \mathcal{H}(\mathbb{D}^2, S) \\ \mathcal{H}(\mathbb{D}, \mathcal{C}) \end{pmatrix} \tag{2.5}$$

by

$$\begin{aligned} & A(E(z_2)f(z_1), E(z_2)g(z_1)) \\ = \; & (E(z_2)\frac{f(z_1)-f(0)}{z_1}, E(z_2)[z_1 g(z_1) - \tilde{S}^*(z_1)f(0)]), \\ & B(k(z)) = (E(z_2)\frac{\tilde{S}(z_1)-\tilde{S}(0)}{z_1}c, E(z_2)[I_{l_2(\mathcal{C})} - \tilde{S}^*(z_1)\tilde{S}(0)]c), \\ & CE(z_2)f(z_1) = E(z)f(0) \text{ and } D(k(z)) = E(z)\tilde{S}(0)c \end{aligned}$$

for $(f(z), g(z)) \in \mathcal{D}(\mathbb{D}, \tilde{S})$. Then the liner system (2.5) is unitary. □

3. Factorization of spaces

Let $S(z_1, z_2)$ $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ be power series with operator coefficients such that $S(z_1, z_2) = S_1(z_1, z_2)S_2(z_1, z_2)$. Assume that the multiplication by $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ are contractive transformations in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Let $\tilde{S}(z)$, $\tilde{S}_1(z)$ and $\tilde{S}_2(z)$ be matrices of the form (1.3). Then $\tilde{S}(z) = \tilde{S}_1(z)\tilde{S}_2(z)$ and the multiplication by $\tilde{S}(z)$ are contractive transformations in $\mathbb{H}(\mathbb{D}, l_2(\mathcal{C}))$. From (2.1), we can easily show that if $f(z)$ and $p(z)$ are in $\mathcal{H}(\mathbb{D}, \tilde{S}_1)$ and $\mathcal{H}(\mathbb{D}, \tilde{S}_2)$ respectively, then $f(z) + \tilde{S}_1(z)p(z)$ is in $\mathcal{H}(\mathbb{D}, \tilde{S})$ and the inequality

$$\|f(z) + \tilde{S}_1(z)p(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S})} \le \|f(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_1)} + \|p(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_2)}$$

holds. Moreover, the space $\mathcal{H}(\mathbb{D}, \tilde{S}_1)$ is contained contractively in $\mathcal{H}(\mathbb{D}, \tilde{S})$ and $\tilde{S}_1(z)p(z)$ is in $\mathcal{H}(\mathbb{D}, \tilde{S})$ for $p(z) \in \mathcal{H}(\mathbb{D}, \tilde{S}_2)$.

We now use the complementation theory to factorize the spaces.

Theorem 3.1. *Let $S(z_1, z_2)$, $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ be power series with operator coefficients such that $S(z_1, z_2) = S_1(z_1, z_2)S_2(z_1, z_2)$. Assume that the multiplications by $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ are contractive transformations in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Then $\mathcal{H}(\mathbb{D}, \tilde{S}_1)$ is contained contractively in $\mathcal{H}(\mathbb{D}, \tilde{S})$. For every element $h(z)$ in $\mathcal{H}(\mathbb{D}, \tilde{S})$, there are unique elements $f(z) \in \mathcal{H}(\mathbb{D}, \tilde{S}_1)$ and $p(z) \in \mathcal{H}(\mathbb{D}, \tilde{S}_2)$ such that $h(z) = f(z) + \tilde{S}_1(z)p(z)$ and the identity*

$$\|h(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S})} = \|f(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_1)} + \|p(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_2)} \tag{3.1}$$

holds.

In Theorem 3.1, $f(z)$ is obtained from $h(z)$ under the adjoint of the inclusion of $\mathcal{H}(\mathbb{D}, \tilde{S}_1)$ in $\mathcal{H}(\mathbb{D}, \tilde{S})$ and $g(z)$ is obtained from $h(z)$ under the adjoint of the multiplication by $\tilde{S}_1(z)$ as a transformation of $\mathcal{H}(\mathbb{D}, \tilde{S}_2)$ into $\mathcal{H}(\mathbb{D}, \tilde{S})$. And the identity

$$\begin{aligned} (3.2) \quad & \|[\tilde{S}(z) - \tilde{S}(0)]c/z\|^2_{\mathcal{H}(\mathbb{D},\tilde{S})} \\ = \quad & \|[\tilde{S}_1(z) - \tilde{S}_1(0)]\tilde{S}_2(0)c/z\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_1)} + \|[\tilde{S}_2(z) - \tilde{S}_2(0)]c/z\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_2)} \end{aligned}$$

holds for every c in $l_2(\mathcal{C})$.

Theorem 3.2. *Let $S(z_1, z_2)$, $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ be power series with operator coefficients such that $S(z_1, z_2) = S_1(z_1, z_2)S_2(z_1, z_2)$. Assume that the multiplications by $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ are contractive transformations in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Then there is an isometry from $\mathcal{D}(\mathbb{D}, \tilde{S})$ to the Cartesian product space $\mathcal{D}(\mathbb{D}, \tilde{S}_1) \times \mathcal{D}(\mathbb{D}, \tilde{S}_2)$.*

Proof. Let $(h(z), k(z))$ and c be elements of $\mathcal{D}(\mathbb{D}, \tilde{S})$ and $l_2(\mathcal{C})$ respectively. Since $h(z) \in \mathcal{H}(\mathbb{D}, \tilde{S})$, by Theorem 3.1, there are unique elements $f(z) \in \mathcal{H}(\mathbb{D}, \tilde{S}_1)$ and $p(z) \in \mathcal{H}(\mathbb{D}, \tilde{S}_2)$ such that $h(z) = f(z) + \tilde{S}_1(z)p(z)$ and the identity (3.1) holds. If we choose $g(z)$ and $q(z)$ so that $(f(z), g(z))$ is in $\mathcal{D}(\mathbb{D}, \tilde{S}_1)$ and $(p(z), q(z))$ is in $\mathcal{D}(\mathbb{D}, \tilde{S}_2)$, then the identity

$$\begin{aligned} & \langle f(z) + \tilde{S}_1(z)p(z), [\tilde{S}(z) - \tilde{S}(0)]/z\rangle_{\mathcal{H}(\mathbb{D},\tilde{S})} \\ (3.3) \qquad = \; & \langle f(z), [\tilde{S}_1(z) - \tilde{S}_1(0)]\tilde{S}_2(0)c/z\rangle_{\mathcal{H}(\mathbb{D},\tilde{S}_1)} \\ & + \langle p(z), [\tilde{S}_2(z) - \tilde{S}_2(0)]c/z\rangle_{\mathcal{H}(\mathbb{D},\tilde{S}_2)} \end{aligned}$$

holds because of the identity (3.2). First claim that

$$k(z) = q(z) + \tilde{S}_2^*(z)g(z).$$

Let $S_2(z_1, z_2) = \sum_{i,j=0}^{\infty} s_{ij} z_1^i z_2^j$. Then $\tilde{S}_2(z)$ can be written by $\tilde{S}_2(z) = \sum_{n=0}^{\infty} S_n z^n$ where

$$(3.4) \qquad S_n = \begin{pmatrix} s_{n0} & 0 & 0 & 0 & \cdots \\ s_{n1} & s_{n0} & 0 & 0 & \cdots \\ s_{n2} & s_{n1} & s_{n0} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Since $[1 - \tilde{S}_2(z)\tilde{S}_2^*(0)]c \in \mathcal{H}(\mathbb{D}, \tilde{S}_2)$,

$$z^n[1 - \tilde{S}_2(z)\tilde{S}_2^*(0)]c - \tilde{S}_2(z)[a_0 z^{n-1} + \cdots + a_{n-1}]$$

is in $\mathcal{H}(\mathbb{D}, \tilde{S}_2)$ where $a_k = S^*_{k+1}c$. Write $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $q(z) = \sum_{n=0}^{\infty} q_n z^n$ and $k(z) = \sum_{n=0}^{\infty} k_n z^n$. Then

$$\begin{array}{rl} & [z^n f(z) - \tilde{S}_1(z)(g_0 z^{n-1} + \cdots + g_{n-1}] \\ + & \tilde{S}_1(z)[z^{n-1}(1 - \tilde{S}_2(z)S_2^*(0))g_0 - \tilde{S}_2(z)(S_1^* g_0 z^{n-2} + \cdots + S^*_{n-1}g_0) \\ + & \cdots \\ + & (1 - \tilde{S}_2(z)S_2^*(0))g_{n-1}] \\ + & \tilde{S}_1(z)[z^n p(z) - \tilde{S}_2(z)(q_0 z^{n-1} + \cdots + q_{n-1})] \\ = & [z^n(f(z) + \tilde{S}_1(z)p(z)) - \tilde{S}(z)(k_0 z^{n-1} + \cdots + k_{n-1})] \end{array}$$

is in $\mathcal{H}(\mathbb{D}, \tilde{S})$ where $k_i = q_i + \sum_{j=1}^{i} S_j^* g_{i-j}$. This implies $k(z) = q(z) + \tilde{S}_2^*(z)g(z)$. Therefore, the identity (3.3) implies that $k(0) = q(0) + \tilde{S}_2^*(0)g(0)$. If we now define $T : \mathcal{D}(\mathbb{D}, \tilde{S}) \longrightarrow \mathcal{D}(\mathbb{D}, \tilde{S}_1) \times \mathcal{D}(\mathbb{D}, \tilde{S}_2)$ by

$$T(h(z), k(z)) = ((f(z), g(z)), (p(z), q(z)),$$

then the identity

$$\begin{aligned} \|T(h(z), k(z))\|^2_{\mathcal{D}(\mathbb{D},\tilde{S})} &= \|(f(z), g(z))\|^2_{\mathcal{D}(\mathbb{D},\tilde{S}_1)} + \|(p(z), q(z))\|^2_{\mathcal{D}(\mathbb{D},\tilde{S}_2)} \\ &= \|f(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_1)} + \|p(z)\|^2_{\mathcal{H}(\mathbb{D},\tilde{S}_2)} \\ &= \|(h(z), k(z))\|^2_{\mathcal{D}(\mathbb{D},\tilde{S})} \end{aligned}$$

implies that T is isometric. This completes the proof. □

If we apply the same argument to the space $\mathcal{D}(\mathbb{D}^2, \tilde{S})$, we can easily obtain the following Theorem.

Theorem 3.3. *Let $S(z_1, z_2)$ $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ be power series with operator coefficients such that $S(z_1, z_2) = S_1(z_1, z_2)S_2(z_1, z_2)$. Assume that the multiplications by $S_1(z_1, z_2)$ and $S_2(z_1, z_2)$ are contractive transformations in $\mathbb{H}(\mathbb{D}^2, \mathcal{C})$. Then there is an isometry from $\mathcal{D}(\mathbb{D}^2, S)$ to the Cartesian product space $\mathcal{D}(\mathbb{D}^2, S_1) \times \mathcal{D}(\mathbb{D}^2, S_2)$.*

References

[1] D. Alpay and V. Bolotnikov, *On the tangential interpolation problem for matrix-valued H_2-functions of two variables*, Proc. Amer. Math. Soc. **127**(1999), 25-50

[2] D. Alpay, V. Bolotnikov, A. Dijksma and C. Sadosky, *Hilbert spaces contractively included in the Hardy space of the bidisk*, Positivity **5**(2001), 1385-1292.

[3] D. Alpay, A. Dijksma, J. Rovnyak and H. de Snoo, *Schur functions,operator colligations and reproducing kernel Pontryagin spaces*, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1997.

[4] T. Azizov, *Extension of J-isometric and J-symmetric operators*, Functional Anal. Appl. **18**(1984), 46-48.

[5] L. de Branges, *Krein spaces of analytic functions*, J. Functional Analysis **181**(1988), 219-259.

[6] L. de Branges, *Complementation in Krein spaces*, Trans. Amer. Math. Soc. **305**(1988), 277-291.

[7] L. de Branges, *Factorization in Krein spaces*, J. Functional Analysis **124**(1994), 228–262.

[8] L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (1966), 295-392.

[9] M. Yang and B. I. Hong *A unitary linear system on the bidisk*, Honam Math. Jour. **29**(2007), 511-521

[10] M. Yang, *Reproducing Kernel Krein spaces*, Korean J. Compu. & Appl. Math. **8** (2001), 659-668

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