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d-ALGEBRAS WITH COMPLICATED CONDITION

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Abstract. Any BCK-ideal of a d-algebra can be decomposed into the union of some sets. The notion of a complicated d-algebra is introduced and some related properties are obtained.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4,5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J. Neggers and H. S. Kim ([9]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras. After that some further aspects were studied([1,2,8,10]). In [3], P. J. Allen, H. S. Kim and J. Neggers developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type.

In this paper, we show that any BCK-ideal of a d-algebra can be decomposed into the union of some sets. We also introduce the notion of a complicated d-algebra and investigate some related properties.

2. Preliminaries

A *d-algebra* ([9]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms :

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

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(V) $(x * (x * y)) * y = 0$, for all $x, y, z \in X$.

For brevity we also call X a *d-algebra*. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

Definition 2.1. ([9]) Let X be a d-algebra and $x \in X$. Define $x * X := \{x * a | a \in X\}$. X is said to be *edge* if for any $x \in X$, $x * X = \{x, 0\}$.

Lemma 2.2. ([9]) *Let X be an edge d-algebra. Then*

(i) *$x * 0 = x$ for any $x \in X$,*
(ii) *the condition (V) holds.*

Definition 2.2. ([10]) Let X be a d-algebra and let $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* of X if it satisfies:

(D_0) $0 \in I$,
(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

A d-algebra X is called a *d^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$. A BCK-algebra is a d^*-algebra but the converse need not be true(See [9]).

Definition 2.3. ([9]) A d-algebra X is said to be *d-transitive* if $x * z = 0$ and $z * y = 0$, then $x * y = 0$.

Definition 2.4. ([2]) A d-algebra X is said to be *positive implicative* if for all $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

If X is a positive implicative d-algebra, then it is d-transitive, since " $\leq$ " is transitive, i.e., $x \leq y$ and $y \leq z$ imply $x \leq z$ for any $x, y, z \in X$.

Definition 2.5. ([2]) A d-algebra X is said to be *commutative* if for all $x, y \in X$, $x * (x * y) = y * (y * x)$. We denote $x \wedge y := y * (y * x)$.

3. Main Results

For any d-algebra X and $x, y \in X$, we denote

$$A(x, y) := \{z \in X | (z * x) * y = 0\}.$$

Theorem 3.1. *If I is a BCK-ideal of a d-algebra X, then $I = \cup_{x,y \in I} A(x, y)$.*

Proof. Let I be a BCK-ideal of a d-algebra X. If $z \in I$, then $(z * z) * 0 = 0 * 0 = 0$. Hence $z \in A(z, 0)$. It follows that

$$I \subseteq \cup_{z \in I} A(z, 0) \subseteq \cup_{x,y \in I} A(x, y).$$

Let $z \in \cup_{x,y \in I} A(x, y)$. Then there exist $a, b \in I$ such that $z \in A(a, b)$, so that $(z * a) * b = 0 \in I$. Since I is a BCK-ideal of X, we have $z \in I$. Thus $\cup_{x,y \in I} A(x, y) \subseteq I$, and consequently $I = \cup_{x,y \in I} A(x, y)$. □

Corollary 3.2. *If I is a BCK-ideal of a d-algebra X, then $I = \cup_{x \in I} A(x, 0)$.*

Proof. By Theorem 3.1, we have

$$\cup_{x \in I} A(x, 0) \subseteq \cup_{x,y \in I} A(x, y) = I.$$

If $x \in I$, then $x \in A(x, 0)$ since $(x * x) * 0 = 0 * 0 = 0$. Hence $I \subseteq \cup_{x \in I} A(x, 0)$. This competes the proof. □

We give an example satisfying Theorem 3.1 and Corollary 3.2. See the following example.

Example 3.3. (1) Let $X := \{0, a, b, c\}$ be a d-algebra ([2]) which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	c	b	0	c
c	c	b	b	0

Then $I := \{0, a\}$ is a BCK-ideal of X. Moreover, it is easy to check that $I = A(0, a) = A(a, 0) = A(a, a)$.
(2) Let $X := \{0, a, b, c\}$ be a d-algebra ([10]) which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $J := \{0, a\}$ is a BCK-ideal of X. Moreover, it is easy to check that $J = A(0, a) = A(a, 0) = A(a, a)$.

Theorem 3.4. *Let I be a non-empty subset of a d-algebra X such that $0 \in I$ and $I = \cup_{x,y \in I} A(x, y)$. Then I is a BCK-ideal of X.*

Proof. Let $a * b, b \in I = \cup_{x,y \in I} A(x, y)$. Since $(a * b) * (a * b) = 0$, we have $a \in A(b, a * b)$. Hence I is a BCK-ideal of X. □

Combining Theorems 3.1 and 3.4, we have the following corollary.

Corollary 3.5. *Let X be a d-algebra. Then I is a BCK-ideal of X if and only if $I = \cup_{x,y \in I} A(x, y)$.*

Definition 3.6. Let X be a d^*-algebra. $A(x, y) := \{z \in X | (z*x)*y = 0\}$ for any $x, y \in X$. X is said to be *complicated* if for any $x, y \in X$, the set $A(x, y)$ has the greatest element.

Note that $A(x, y)$ is a non-empty set, since $0, x, y \in A(x, y)$, where X is a d^*-algebra. The greatest element of $A(x, y)$ is denoted by $x + y$.

Example 3.7. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	1	2	1	0

Then X is a complicated d^*-algebra which is not a BCK-algebra, since $(3 * (3 * 0)) * 0 = (3 * 1) * 0 = 2 * 0 = 2 \neq 0$. But it is neither positive implicative nor commutative.

Example 3.8. (1) Let $X := \{0, 1, 2, 3, 4\}$ be a d-algebra ([2]) which is not a BCK-algebra with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	3	2	1
2	2	4	0	2	1
3	3	4	3	0	1
4	4	4	3	2	0

Then X is not complicated, because $A(2, 3) = \{z \in X | (z * 2) * 3 = 0\} = \{0, 1, 2, 3, 4\}$ has no greatest element. Moreover, it is neither positive implicative nor commutative.

(2) Let $X := \{0, a, b, c\}$ be a set with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then X is a positive implicative d-algebra. But it is not complicated, because $A(a, c) = \{0, a, c\}$ has no greatest element.

Lemma 3.9. *Let X be a positive implicative d-algebra. If $x \leq y$, then $x * z \leq y * z$ for any $x, y, z \in X$.*

Proof. Let $x, y \in X$ with $x \leq y$. Then $x * y = 0$. Since X is positive implicative, we have $(x * z) * (y * z) = (x * y) * z = 0 * z = 0$. Hence $x * z \leq y * z$. This completes the proof. □

Proposition 3.10. *Let X be a complicated d^*-algebra. Then for any $x, y, z \in X$, the following hold:*

(i) $x \leq x + y, y \leq x + y$,
(ii) *if X is an edge d^*-algebra, then $x + 0 = x = 0 + x$.*

Proof. (i) and (ii) are straightforward. □

Theorem 3.11. *Let X be a positive implicative complicated d^*-algebra and let $a, b \in X$. Then the set*

$$\mathcal{H}(a, b) := \{x \in X | a \leq b + x\}$$

has the least element, and it is $a * b$.

Proof. The inequality $a * b \leq a * b$ implies that $a \leq b + (a * b)$ and so $a * b \in \mathcal{H}(a, b)$. Let $z \in \mathcal{H}(a, b)$. Then $a \leq b + z$, which implies from Lemma 3.9 and Definition 3.6 that $a * b \leq (b + z) * b \leq z$. Since X is d-transitive, we have $a * b \leq z$. Thus $a * b$ is the least element of $\mathcal{H}(a, b)$. □

We provide some characterizations of ideals in a complicated d^*-algebra.

Proposition 3.12. *Let A be a non-empty subset of a complicated d^*-algebra X. If A is a BCK-ideal of X, then it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \implies y \in A)$.
(ii) $(\forall x, y \in A)(\exists z \in A)\ (x \leq z, y \leq z)$.

Proof. Assume that A is a BCK-ideal of X. Let $x \in A, y \in X$ with $y \leq x$. Then $y * x = 0$. From the definition of BCK-ideal of X, we have $y \in A$. (i) is valid.

Let $x, y \in A$. Since $(x + y) * x \leq y$ and $y \in A$, it follows from (i) that $(x + y) * x \in A$ so that $x + y \in A$ because A is a BCK-ideal of X. If we take $z := x + y$, then $x \leq z$ and $y \leq z$ by Proposition 3.10 (i). This completes the proof. □

In Proposition 3.12, the condition, "complicated", is very necessary. See the following example.

Example 3.13. Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	4	3	0	0
4	3	4	3	2	0

It is easy to show that X is a d^*-algebra which is not a BCK-algebra because $4 * 0 = 3$. Moreover, X is not complicated, since $A(4, 3) = \{z \in X | (z * 4) * 3 = 0\} = \{0, 1, 2, 3, 4\}$ has no greatest element. It is easy to see that $\{0, 1, 2\}$ is a BCK-ideal of X, but there is no element $z \in A$ such that $x \leq z, y \leq z$, proving that the condition, "complicated", is necessary in Proposition 3.12. .

Theorem 3.14. *Let A be a non-empty subset of a positive implicative complicated d^*-algebra X. Then A is a BCK-ideal of X if and only if it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \leq x \implies y \in A)$.
(ii) $(\forall x, y \in A)(x, y \in A \implies x + y \in A)$.

Proof. The necessity follows immediately from Propositions 3.10 and 3.12.

Conversely, let A be a non-empty subset of X satisfying conditions (i) and (ii). Obviously $0 \in A$ by (i) and (II). Let $x, y \in X$ satisfying $y \in A$ and $x * y \in A$. Then $y + (x * y) \in A$ by (ii). Since $x \leq y + (x * y)$ by Theorem 3.11, it follows from (i) that $x \in A$. Thus A is a BCK-ideal of X. □

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