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# $M$-PRECLOSED GRAPH AND $M^*$-PREOPEN MAPPING ON SPACES WITH MINIMAL STRUCTURES

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**Abstract.** We introduce the concepts of $M$-preclosed graph and $M^*$-preopen mapping on spaces with minimal structures and investigate some properties of $M^*$-preopen mapping. We also investigate the relationships between $M$-precontinuous mappings and several types of $m$-compactness.

## 1. Introduction

In [3], Popa and Noiri introduced the concept of minimal structure which is a generalization of a topology on a given nonempty set. They introduced the concept of $M$-continuous [4] mapping as a mapping defined between minimal structures. They showed that the $M$-continuous mappings have properties similar to those of continuous mappings between topological spaces. We introduced the concepts of $m$-preopen set and $M$-precontinuity on spaces with minimal structures in [1]. In this paper, we introduce the concepts of $M$-preclosed graph and $M^*$-preopen mapping on spaces with minimal structures and investigate some properties of $M^*$-preopen mapping. And we investigate the relationships between $M$-precontinuous mappings and several types of $m$-compactness. In particular, we show that Theorem 3.16: If $f : (X, m_X) \to (Y, m_Y)$ is an $M$-precontinuous and $M^*$-preopen mapping on two spaces with minimal structures $m_X$ and $m_Y$, and if $A$ is a nearly $m$-precompact set, then $f(A)$ is nearly $m$-compact.

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## 2. Preliminaries

A subfamily $m_X$ of the power set $P(X)$ of a nonempty set $X$ is called a *minimal structure* [3] on $X$ if $\emptyset \in m_X$ and $X \in m_X$. By $(X, m_X)$, we denote a nonempty set $X$ with a minimal structure $m_X$ on $X$. Simply we call $(X, m_X)$ a space with a minimal structure $m_X$ on $X$.

Let $(X, m_X)$ be a space with a minimal structure $m_X$ on $X$. For a subset $A$ of $X$, the closure of $A$ and the interior of $A$ [3] are defined as the following:
$mInt(A) = \cup\{U : U \subseteq A, U \in m_X\}$.
$mCl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}$.

**Theorem 2.1.** ([3]) Let $(X, m_X)$ be a space with a minimal structure $m_X$ on $X$ and $A \subseteq X$.
(1) $X = mInt(X)$ and $\emptyset = mCl(\emptyset)$.
(2) $mInt(A) \subseteq A$ and $A \subseteq mCl(A)$.
(3) If $A \in m_X$, then $mInt(A) = A$ and if $X - F \in m_X$, then $mCl(F) = F$.
(4) If $A \subseteq B$, then $mInt(A) \subseteq mInt(B)$ and $mCl(A) \subseteq mCl(B)$.
(5) $mInt(mInt(A)) = mInt(A)$ and $mCl(mCl(A)) = mCl(A)$.
(6) $mCl(X - A) = X - mInt(A)$ and $mInt(X - A) = X - mCl(A)$.

Let $(X, m_X)$ be a space with a minimal structure $m_X$ on $X$ and $A \subseteq X$. Then a set $A$ is called an *m-preopen set* [1] in $X$ if

$$A \subseteq mInt(mCl(A)).$$

A set $A$ is called an *m-preclosed set* if the complement of $A$ is $m$-preopen.

Any union of $m$-preopen sets is $m$-preopen [1].

The *m-pre-closure* and the *m-pre-interior* of $A$, denoted by $mpCl(A)$ and $mpInt(A)$, respectively, are defined as the following:

$$mpCl(A) = \cap\{F \subseteq X : A \subseteq F, F \text{ is } m\text{-preclosed in } X\}$$

$$mpInt(A) = \cup\{U \subseteq X : U \subseteq A, U \text{ is } m\text{-preopen in } X\}.$$

**Theorem 2.2.** *([1]) Let $(X, m_X)$ be a space with a minimal structure $m_X$ and $A \subseteq X$. Then*
*(1) $mpInt(A) \subseteq A$.*
*(2) If $A \subseteq B$, then $mpInt(A) \subseteq mpInt(B)$.*
*(3) $A$ is m-preopen iff $mpInt(A) = A$.*
*(4) $mpInt(mpInt(A)) = mpInt(A)$.*
*(5) $mpCl(X - A) = X - mpInt(A)$ and $mpInt(X - A) = X - mpCl(A)$.*

**Theorem 2.3.** *([1]) Let $(X, m_X)$ be a space with a minimal structure $m_X$ and $A \subseteq X$. Then*
*(1) $A \subseteq mpCl(A)$.*
*(2) If $A \subseteq B$, then $mpCl(A) \subseteq mpCl(B)$.*
*(3) $F$ is $m$-preclosed iff $mpCl(F) = F$.*
*(4) $mpCl(mpCl(A)) = mpCl(A)$.*

Let $f : (X, m_X) \to (Y, m_Y)$ be a mapping on spaces $(X, m_X)$ and $(Y, m_Y)$ with minimal structures $m_X$, $m_y$. Then $f$ is said to be
(1) *$M$-continuous* [4] if for $x \in X$ and each $m$-open set $V$ containing $f(x)$, there exists an $m$-open set $U$ containing $x$ such that $f(U) \subseteq V$;
(2) *$M$-precontinuous* [1] if for each point $x$ and each $m$-open set $V$ containing $f(x)$, there exists an $m$-preopen set $U$ containing $x$ such that $f(U) \subseteq V$.

**Theorem 2.4.** *([1]) Let $f : (X, m_X) \to (Y, m_Y)$ be a mapping on spaces $(X, m_X)$ and $(Y, m_Y)$ with minimal structures $m_X$, $m_y$. Then the following statements are equivalent:*
*(1) $f$ is $M$-precontinuous.*
*(2) $f^{-1}(V)$ is an $m$-preopen set for each $m$-open set $V$ in $Y$.*
*(3) $f(mpCl(A)) \subseteq mCl(f(A))$ for $A \subseteq X$.*
*(4) $mpCl(f^{-1}(B)) \subseteq f^{-1}(mCl(B))$ for $B \subseteq Y$.*
*(5) $f^{-1}(mInt(B)) \subseteq mpInt(f^{-1}(B))$ for $B \subseteq Y$.*

## 3. $M$-preclosed graph and $M^*$-preopen mapping

**Definition 3.1.** Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. Then $f$ has an *$M$-preclosed graph* if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $m$-preopen set $U$ containing $x$ and an $m$-open set $V$ containing $y$ such that $(U \times V) \cap G(f) = \emptyset$.

**Lemma 3.2.** *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. Then $f$ has an $M$-preclosed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $m$-preopen set $U$ containing $x$ and an $m$-open set $V$ containing $y$ such that $f(U) \cap V = \emptyset$.*

Let $(X, m_X)$ be a space with a minimal structure $m_X$. Then $X$ is said to be *$m$-$T_2$* [4] if for any distinct points $x$ and $y$ of $X$, there exist disjoint $m$-open sets $U$, $V$ such that $x \in U$ and $y \in V$.

**Theorem 3.3.** *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. If $f$ is $M$-precontinuous and $Y$ is $m$-$T_2$, then $G(f)$ is an $M$-preclosed graph.*

*Proof.* Let $(x, y) \in (X \times Y) - G(f)$; then $f(x) \neq y$. Since $Y$ is $m$-$T_2$, there are disjoint open sets $U$, $V$ such that $f(x) \in U$, $y \in V$. Then for $f(x) \in U$, by $m$-precontinuity, there exists an $m$-preopen set $G$ containing $x$ such that $f(G) \subseteq U$. Consequently, there exist an $m$-open set $V$ and $m$-preopen set $G$ containing $y, x$, respectively, such that $f(G) \cap V = \emptyset$. Therefore, $G(f)$ is $M$-preclosed graph. □

**Definition 3.4.** ([2]) Let $(X, m_X)$ be a space with a minimal structure $m_X$. Then $X$ is said to be *$m$-pre-$T_2$* if for any distinct points $x$ and $y$ of $X$, there exist disjoint $m$-preopen sets $U$, $V$ such that $x \in U$ and $y \in V$.

**Theorem 3.5.** *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. If $f$ is an injective and $M$-precontinuous function and if $Y$ is $m$-$T_2$, then $X$ is $m$-pre-$T_2$.*

*Proof.* Obvious. □

**Theorem 3.6.** *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. If $f$ is an injective $M$-precontinuous function with an $M$-preclosed graph, then $X$ is $m$-pre-$T_2$.*

*Proof.* Let $x_1$ and $x_2$ be any distinct points of $X$. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since the graph $G(f)$ is $m$-preclosed graph, there exist an $m$-preopen set $U$ containing $x_1$ and $V \in \tau$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since $f$ is $M$-precontinuous, $f^{-1}(V)$ is an $m$-preopen set containing $x_2$ such that $U \cap f^{-1}(V) = \emptyset$. Hence $X$ is $m$-pre-$T_2$. □

**Definition 3.7.** ([2]) A subset $A$ of a space $(X, m_X)$ with a minimal structure $m_X$ is said to be *$m$-precompact* (resp. *almost $m$-precompact*) relative to $A$ if every collection $\{U_i : i \in J\}$ of $m$-preopen subsets of $X$ such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset $J_0$ of $J$ such that $A \subseteq \cup\{U_j : j \in J_0\}$ (resp. $A \subseteq \cup\{mpCl(U_j) : j \in J_0\}$). A subset $A$ of a minimal structure $(X, m_X)$ is said to be *$m$-precompact* (resp. *almost $m$-precompact*) if $A$ is $m$-precompact (resp. almost $m$-precompact) as a subspace of $X$.

A subset $A$ of a space $(X, m_X)$ with a minimal structure $m_X$ is said to be *$m$-compact* [4] (resp. *almost $m$-compact*, *nearly $m$-compact*) relative to $A$ if every collection $\{U_i : i \in J\}$ of $m$-open subsets of $X$ such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset $J_0$ of $J$ such that $A \subseteq \cup\{U_j : j \in J_0\}$ (resp. $A \subseteq \cup\{mCl(U_j) : j \in J_0\}$, $A \subseteq \cup\{mInt(mCl(U_j)) : j \in J_0\}$).

**Theorem 3.8.** *Let $f : (X, m_X) \to (Y, m_Y)$ be an $M$-precontinuous function on two spaces with minimal structures $m_X$ and $m_Y$. If $A$ is an $m$-precompact set, then $f(A)$ is $m$-compact.*

*Proof.* Obvious. □

**Theorem 3.9.** *Let $f : (X, m_X) \to (Y, m_Y)$ be an $M$-precontinuous function on two spaces with minimal structures $m_X$ and $m_Y$. If $A$ is an almost $m$-precompact set, then $f(A)$ is almost $m$-compact.*

*Proof.* Let $\{U_i : i \in J\}$ be an $m$-open cover of $f(A)$ in $Y$. Then since $f$ is an $M$-precontinuous function, $\{f^{-1}(U_i) : i \in J\}$ is an $m$-preopen cover of $A$ in $X$. By $m$-precompactness, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{j \in J_0} mpCl(f^{-1}(U_j))$. From $M$-precontinuity of $f$, we have
$f(A) \subseteq f(\cup_{j \in J_0} mpCl(f^{-1}(U_j))) \subseteq f(\cup_{j \in J_0} f^{-1}(mCl(U_j)))$
$\subseteq \cup_{j \in J_0} mCl(U_j)$.
Thus $f(A)$ is almost $m$-compact. □

**Definition 3.10.** Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. Then $f$ is said to be *$M^*$-preopen* if for each $m$-preopen set $U$ in $X$, $f(U)$ is $m$-open.

**Theorem 3.11.** *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$.*
*(1) $f$ is $M^*$-preopen.*
*(2) $f(mpInt(A)) \subseteq mInt(f(A))$ for $A \subseteq X$.*
*(3) $mpInt(f^{-1}(B)) \subseteq f^{-1}(mInt(B))$ for $B \subseteq Y$.*
*(4) For each $x \in X$ and each $m$-preopen set $U$ containing $x$, there is an $m$-open set $V$ such that $f(x) \in V \subseteq f(U)$.*
*Then* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

*Proof.* (1) $\Rightarrow$ (2) For $A \subseteq X$,

$$\begin{aligned} f(mpInt(A)) &= f(\cup\{B : B \subseteq A, B \text{ is } m\text{-preopen}\}) \\ &= \cup\{f(B) : f(B) \subseteq f(A), f(B) \text{ is } m\text{-open}\} \\ &\subseteq \cup\{U : U \subseteq f(A), U \text{ is } m\text{-open}\} \\ &= mInt(f(A)) \end{aligned}$$

Hence $f(mpInt(A)) \subseteq mInt(f(A))$.

(2) $\Rightarrow$ (3) For $B \subseteq Y$, from (2) it follows that

$$f(mpInt(f^{-1}(B))) \subseteq mInt(f(f^{-1}(B))) \subseteq mInt(B).$$

Hence we get (3).

Similarly, we get (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (4) For each $x \in X$ and each $m$-preopen set $U$ containing $x$, since $U = mpInt(U)$, by (2), $f(U) = mInt(f(U))$ is obtained. So from definition of interior operator on $m_Y$, there is an $m$-open set $V$ such that $f(x) \in V \subseteq f(U)$.

(4) $\Rightarrow$ (2) For $A \subseteq X$, if $y \in f(mpInt(A))$, then there is $x \in mpInt(A)$ such that $f(x) = y$. Since $mpInt(A)$ is $m$-preopen, by (4), there exists an $m$-open set $V$ such that $f(x) \in V \subseteq f(mpInt(A)) \subseteq f(A)$. Consequently, $y = f(x) \in mInt(f(A))$. $\square$

**Example 3.12.** Let $X = \{a, b, c\}$ and $m_X = \{\emptyset, \{a\}, \{b\}, X\}$. Consider the identity function $f : (X, m_X) \to (X, m_X)$ Then $f$ satisfies the condition (2) in Theorem 3.11, but it is not $M^*$-open because for $m$-preopen set $\{a, b\}$, $f(\{a, b\})$ is not $m$-open.

A minimal structure $m_X$ on a nonempty set $X$ is said to have property ($\mathcal{B}$) [4] if the union of any family of subsets belonging to $m_X$ belongs to $m_X$

**Lemma 3.13.** *([4]) Let $m_X$ be a minimal structure on a nonempty set $X$ satisfying ($\mathcal{B}$). For $A \subseteq X$, the following are equivalent:*
*(1) $A \in m_X$ if and only if $mInt(A) = A$.*
*(2) $A$ is $m$-closed if and only if $mCl(A) = A$.*

**Corollary 3.14.** Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures $m_X$ and $m_Y$. If $m_Y$ has property ($\mathcal{B}$), then the following are equivalent:
(1) $f$ is $M^*$-preopen.
(2) $f(mpInt(A)) \subseteq mInt(f(A))$ for $A \subseteq X$.
(3) $mpInt(f^{-1}(B)) \subseteq f^{-1}(mInt(B))$ for $B \subseteq Y$.
(4) For each $x \in X$ and each $m$-preopen set $U$ containing $x$, there is an $m$-open set $V$ such that $f(x) \in V \subseteq f(U)$.

**Definition 3.15.** A subset $A$ of a space $(X, m_X)$ with a minimal structure $m_X$ is said to be *nearly m-precompact* relative to $A$ if every collection $\{U_i : i \in J\}$ of $m$-open subsets of $X$ such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset $J_0$ of $J$ such that $A \subseteq \cup\{mpInt(mpCl(U_j)) : j \in J_0\}$. A subset $A$ of a minimal structure $(X, m_X)$ is said to be *nearly m-precompact* if $A$ is nearly $m$-precompact relative to $A$.

**Theorem 3.16.** *Let $f : (X, m_X) \to (Y, m_Y)$ be an $M$-precontinuous and $M^*$-preopen mapping on two spaces with minimal structures $m_X$ and $m_Y$. If $A$ is a nearly $m$-precompact set, then $f(A)$ is nearly $m$-compact.*

*Proof.* Let $\{U_i : i \in J\}$ be an $m$-open cover of $f(A)$ in $Y$. Then since $f$ is an $M$-precontinuous function, $\{f^{-1}(U_i) : i \in J\}$ is an $m$-preopen cover of $A$ in $X$. By $m$-precompactness, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{j\in J_0} mpInt(mpCl(f^{-1}(U_j)))$. From $M$-precontinuity and Theorem 3.11, it follows

$$\begin{aligned} f(\cup_{j\in J_0} mpInt(mpCl(f^{-1}(U_j)))) &\subseteq f(\cup_{j\in J_0} mpInt(f^{-1}(mCl(U_j)))) \\ &= \cup_{j\in J_0} f(mpInt(f^{-1}(mCl(U_j)))) \\ &\subseteq \cup_{j\in J_0} mInt(f(f^{-1}(mCl(U_j)))) \\ &\subseteq \cup_{j\in J_0} mInt(mCl(U_j)). \end{aligned}$$

This implies $f(A) \subseteq \cup_{j\in J_0} mInt(mCl(U_j))$ and hence $f(A)$ is nearly $m$-compact. □

## References


1. W. K. Min and Y. K. Kim; *m-Preopen Sets and M-Precontinuity On Spaces With Minimal Structures*, Advances in Fuzzy Sets and Systems, 4(3) (2009), 237-245.
2. ——————; *On Minimal Precontinuous Functions*, Journal of ChungCheong Mathematical Society, accepted.
3. V. Popa and T. Noiri; *On the definition of some generalized forms of continuity under minimal conditions*, Mem. Fac. Sci. Kochi. Univ. Ser. Math. 22(2001), 9-19.
4. V. Popa and T. Noiri; *On M-continuous functions*, Anal, Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., Fasc. II, 18(23) (2000), 31-41.

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