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MATRIX RINGS AND ITS TOTAL RINGS OF FRACTIONS

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Abstract. Let R be a commutative ring with identity. Then we prove

$$\begin{aligned} M_n(R) =& GL_n(R) \\ & \cup \{A \in M_n(R) \mid \det A \neq 0 \text{ and } \det A \notin U(R)\} \\ & \cup Z(M_n(R)) \end{aligned}$$

where $U(R)$ denotes the set of all units of R. In particular, it will be proved that the full matrix ring $M_n(F)$ over a field F is the disjoint union of the general linear group $GL_n(F)$ of degree n over the field F and the set $Z(M_n(F))$ of all zero-divisors of $M_n(F)$. Using the result and universal mapping property we prove that $M_n(F)$ is its total ring of fractions.

0. Introduction

Unless we state explicitly, we shall not assume that our rings are commutative, but we shall always assume that every ring has an identity. Let R be a ring. An element $a \in R$ is called a *zero-divisor* of R if there exists a non-zero element $b \in R$ such that $ab = 0$. Let $Z(R)$ denote the set of all zero-divisors of R. Then $0 \in Z(R)$.

A commutative ring R is called an *integral domain* if $Z(R) = \{0\}$. For example, the ring $\mathbb{Z}$ of integers is an integral domain, but $\mathbb{Z}^2$ is not.

1. Localizations of Commutative Rings

Let A be a commutative ring and let S be a multiplicatively closed set in A. Define a map $f : A \longrightarrow S^{-1}A$ by $f(x) = x/1$, where $x \in A$. Then f is a ring

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homomorphism. This is called the *natural ring homomorphism*.

LEMMA 1.1. *Let A be an integral domain. Then the following are true.*

(1) *$A\backslash\{0\}$ is a multiplicatively closed set in A.*
(2) *$(A\backslash\{0\})^{-1}A$ is the field of fractions of A.*
(3) *If F is the field of fractions of A, then the natural ring homomorphism $f : A \longrightarrow F$ is injective.* □

However, the natural ring homomorphism $f : A \longrightarrow S^{-1}A$ is not necessarily injective.

Let A be a field and let F be the field of fractions of A. If s is a non-zero element of A, then

$$s^{-1}/1 = (s/1)^{-1} = 1/s \text{ in } F$$

Hence A is isomorphic to F.

Let A, B be commutative rings and let $f : A \longrightarrow B$ be a ring homomorphism. Then $f(A)$ is a subring of B. However, $f(A)$ is not always an ideal of B.

PROPOSITION 1.2. *Let A be a commutative ring and let F be a field. Let $g : A \longrightarrow F$ be a non-zero ring homomorphism and let $\mathfrak{p} = Ker(g)$. Then the following are true.*

(1) *$\mathfrak{p}$ is a prime ideal of A.*
(2) *The field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ can be embedded in the field F.* □

LEMMA 1.3. *Let A be a commutative ring and let I be an ideal of A. Let S be a multiplicatively closed set in A such that $S \cap I = \emptyset$. Then the following are true.*

(1) *If $\bar{S}$ is the image of S under the natural homomorphism $\pi : A \longrightarrow A/I$, then $\bar{S}$ is a multiplicatively closed set in A/I.*
(2)
$$\frac{S^{-1}A}{IS^{-1}A} \cong \bar{S}^{-1}(A/I).$$
□

If we use Lemma 1.3 (2) and Lemma 1.1, then we can get the following result.

PROPOSITION 1.4. *Let A be a commutative ring and let $\mathfrak{p}$ be a prime ideal of A. Then the following are true.*

(1) $\frac{(A\backslash\mathfrak{p})+\mathfrak{p}}{\mathfrak{p}} = A/\mathfrak{p}\backslash\{0+\mathfrak{p}\}$ *in* $A/\mathfrak{p}$.

(2) *The field of fractions of the integral domain $A/\mathfrak{p}$ is isomorphic to $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.* □

Proposition 1.4 (2) can be proved alternatively as follows. Let $\pi : A \longrightarrow A/\mathfrak{p}$ be the natural homomorphism. Let $\varphi_0 : A/\mathfrak{p} \longrightarrow (A/\mathfrak{p})_0$ be the natural ring homomorphism. Then by Lemma 1.1, $\varphi_0 : A/\mathfrak{p} \longrightarrow (A/\mathfrak{p})_0$ is injective. Consider the composite map $g : A \xrightarrow{\pi} A/\mathfrak{p} \xrightarrow{\varphi_0} (A/\mathfrak{p})_0$. Then g is a ring homomorphism with $\mathrm{Ker}(g) = \mathfrak{p}$. Consider the following diagram

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & (A/\mathfrak{p})_0 \\ & A_{\mathfrak{p}} \arrow[ur] & \end{tikzcd}$$

Then by the universal mapping property(see [E95, p.60]) there exists a homomorphism $h : A_{\mathfrak{p}} \longrightarrow (A/\mathfrak{p})_0$ such that $h \circ f = g$. Further, h is an epimorphism with $\mathrm{Ker}(h) = \mathfrak{p}A_{\mathfrak{p}}$. Hence, by the first isomorphism theorem for rings, $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong (A/\mathfrak{p})_0$, as required.

COROLLARY 1.5. *Let A be a commutative ring. If $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is isomorphic to $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$.* □

2. Matrix Rings over Commutative Rings

Let R be a ring. An element $a \in R$ is called a *unit element* of R if there exists an element $b \in R$ such that $ab = 1$, $ba = 1$. If $a \in R$ is a unit element of R, then $a \neq 0$. Let $U(R)$ denote the set of all unit elements of R.

If F is a field, then $U(F) = F\backslash\{0\}$. For the following result, see [FIS03, Exercise 27 (a), p.231].

PROPOSITION 2.1. *Let F be a field and let $A \in M_n(F)$. Then*

$$det(adjA) = (detA)^{n-1}.$$

Proof. Let $A \in M_n(F)$. Then $A\ \mathrm{adj}A = (\det A)I$.

Assume that $\det A \neq 0$. Since $\det A \det(\mathrm{adj}A) = (\det A)^n$, we have

$$\det(\mathrm{adj}A) = (\det A)^{n-1}.$$

Assume that $\det A = 0$. If $A = 0$, then $\mathrm{adj}A = 0$ and hence

$$\det(\mathrm{adj}A) = 0 = (\det A)^{n-1}.$$

Assume that $A \neq 0$. Note that $A\ \mathrm{adj}A = (\det A)I = 0$. We can prove that $\mathrm{adj}A$ is singular. For otherwise, there exit nonsingular matrices P, Q such that $P(\mathrm{adj}A)Q = I$. Then

$$A = AP^{-1}IP = AP^{-1}(P(\mathrm{adj}A)Q)P = A(\mathrm{adj}\ A)QP = 0QP = 0.$$

This is a contradiction. Hence $\mathrm{adj}A$ is singular. Thus,

$$\det(\mathrm{adj}A) = 0 = (\det A)^{n-1}.$$

□

Let R be a commutative ring. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

be in $M_n(R)$. For each $i,\ j \in \{1, 2, \cdots, n\}$, let $\tilde{A}_{i,j}$ be the matrix which is obtained from A by deleting the i-th row and j-th column. Then

$$\tilde{A}_{i,j} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}.$$

For each $s \in \{1, 2, \cdots, i-1, i+1, \cdots, n\}$ and $t \in \{1, 2, \cdots, j-1, j+1, \cdots, n\}$, let A_{st} be the cofactor of a_{st} as an entry of the $(n-1) \times (n-1)$ matrix $\tilde{A}_{i,j}$. Then

$$\mathrm{adj}(\tilde{A}_{i,j}) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{i-1,1} & A_{i+1,1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{i-1,2} & A_{i+1,2} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{1,j-1} & A_{2,j-1} & \cdots & A_{i-1,j-1} & A_{i+1,j-1} & \cdots & A_{n,j-1} \\ A_{1,j+1} & A_{2,j+1} & \cdots & A_{i-1,j+1} & A_{i+1,j+1} & \cdots & A_{n,j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{i-1,n} & A_{i+1,n} & \cdots & A_{nn} \end{pmatrix}.$$

Now, for each $j \in \{1, 2, \cdots, n\}$, construct an $n \times n$ matrix B_j over R as follows. The entries of j-th row and j-th column of B_j are all zero and the remaining entries of B_j are from $\mathrm{adj}(\tilde{A}_{i,j})$. More precisely,

$$B_j = \begin{cases} \begin{pmatrix} A_{11} & \cdots & A_{i-1,1} & 0 & A_{i+1,1} & \cdots & A_{n1} \\ A_{12} & \cdots & A_{i-1,2} & 0 & A_{i+1,2} & \cdots & A_{n2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1,j-1} & \cdots & A_{i-1,j-1} & 0 & A_{i+1,j-1} & \cdots & A_{n,j-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ A_{1,j+1} & \cdots & A_{i-1,j+1} & 0 & A_{i+1,j+1} & \cdots & A_{n,j+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{i-1,n} & 0 & A_{i+1,n} & \cdots & A_{nn} \end{pmatrix} & if\ j = i \\ \\ \begin{pmatrix} A_{11} & \cdots & A_{s-1,1} & 0 & A_{s,\ \ 1} & \cdots & A_{n1} \\ A_{12} & \cdots & A_{s-1,2} & 0 & A_{s,\ \ 2} & \cdots & A_{n2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1,j-1} & \cdots & A_{s-1,j-1} & 0 & A_{s,\ \ j-1} & \cdots & A_{n,j-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ A_{1,j+1} & \cdots & A_{s-1,j+1} & 0 & A_{s,\ \ j+1} & \cdots & A_{n,j+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{s-1,n} & 0 & A_{s,\ \ n} & \cdots & A_{nn} \end{pmatrix} & if\ j \neq i. \end{cases}$$

where

$$s \begin{cases} =j & if\ j < i \\ \geq i+2 & if\ j > i. \end{cases}$$

Then we have the following result.

Lemma 2.2. *Let R be a commutative ring. Let $A \in M_n(R)$. For each $j \in \{1, 2, \cdots, n\}$, let B_j be constructed as above. If adj$(A) = 0$, then for each $j \in \{1, 2, \cdots, n\}$, $AB_j = 0$.* $\square$

If $A \in M_2(R)$ and adj$A = 0$, then $A = 0$. Now, let $A \in M_3(R)$. Assume that adj$(A) = 0$. Then for every $i, j \in \{1, 2, 3\}$, $A_{ij} = 0$. Let $i, j \in \{1, 2, 3\}$. Then

$$\tilde{A}_{i,j}\text{adj}(\tilde{A}_{i,j}) = det(\tilde{A}_{ij})I = 0$$

since $(-1)^{i+j}det(\tilde{A}_{ij}) = 0$. If adj$(\tilde{A}_{i,j}) = 0$, then $\tilde{A}_{i,j} = 0$ since $\tilde{A}_{i,j} \in M_2(R)$.

If for every $i, j \in \{1, 2, 3\}$ adj$(\tilde{A}_{i,j}) = 0$, then for every $i, j \in \{1, 2, 3\}\tilde{A}_{i,j} = 0$ and hence $A = 0$. Hence, $A \in Z(M_3(R))$.

If there exist $i, j \in \{1, 2, 3\}$ such that adj$(\tilde{A}_{i,j}) \neq 0$, then $B_j \neq 0$. Hence by Lemma 2.2 $A \in Z(M_3(R))$. Therefore if R is a commutative ring, then

$$\begin{aligned} M_3(R) =& GL_3(R) \\ & \cup \{A \in M_3(R) \mid \text{det}A \neq 0 \text{ and det } A \notin U(R)\} \\ & \cup Z(M_3(R)) \end{aligned}$$

More generally we proceed as follows.

Lemma 2.3. *Let R be a commutative ring. Let $A \in M_n(R)$. If $detA = 0$, then $A \in Z(M_n(R))$.*

Proof. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

be in $M_n(R)$. For each $i, j \in \{1, 2, \cdots, n\}$, let M_{ij} be the minor of A. Denote elementary column operations by $\longrightarrow$. Then

$$A \longrightarrow \begin{pmatrix} a_{11}M_{11} & a_{12} & \cdots & a_{1n} \\ a_{21}M_{11} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}M_{11} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\longrightarrow \cdots$$

$$\longrightarrow \begin{pmatrix} a_{11}M_{11} & a_{12}M_{12} & \cdots & a_{1n}M_{1n} \\ a_{21}M_{11} & a_{22}M_{12} & \cdots & a_{2n}M_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}M_{11} & a_{n2}M_{12} & \cdots & a_{nn}M_{1n} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} a_{11}M_{11} & \cdots & a_{1,n-1}M_{1,n-1} & a_{11}M_{11} - \cdots + (-1)^{1+n}a_{1n}M_{1n} \\ a_{21}M_{11} & \cdots & a_{2,n-1}M_{1,n-1} & a_{21}M_{11} - \cdots + (-1)^{2+n}a_{2n}M_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1}M_{11} & \cdots & a_{n,n-1}M_{1,n-1} & a_{n1}M_{11} - \cdots + (-1)^{n+n}a_{nn}M_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}M_{11} & \cdots & a_{1,n-1}M_{1,n-1} & \det A \\ a_{21}M_{11} & \cdots & a_{2,n-1}M_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1}M_{11} & \cdots & a_{n,n-1}M_{1,n-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}M_{11} & \cdots & a_{1,n-1}M_{1,n-1} & 0 \\ a_{21}M_{11} & \cdots & a_{2,n-1}M_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1}M_{11} & \cdots & a_{n,n-1}M_{1,n-1} & 0 \end{pmatrix}.$$

So, there exists a non-singular matrix $P \in M_n(R)$ such that

$$AP = \begin{pmatrix} a_{11}M_{11} & \cdots & a_{1,n-1}M_{1,n-1} & 0 \\ a_{21}M_{11} & \cdots & a_{2,n-1}M_{1,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}M_{11} & \cdots & a_{n,n-1}M_{1,n-1} & 0 \end{pmatrix}.$$

Let E_{ij} be the matrix in which the only non-zero entry is a 1 in the ith row and jth column. Then

$$\begin{aligned} AP =& M_{11}(a_{11}E_{11} + a_{21}E_{21} + \cdots + a_{n1}E_{n1}) \\ &+ \cdots \\ &+ M_{1,n-1}(a_{1,n-1}E_{1,n-1} + a_{2,n-1}E_{2,n-1} + \cdots + a_{n,n-1}E_{n,n-1}). \end{aligned}$$

Multiplying both sides of this equation by E_{nn}, we have $APE_{nn} = 0$. Since $PE_{nn} \neq 0$, it follows that $A \in Z(M_n(R))$. $\square$

If R is a commutative ring, then $U(M_n(R))$ forms a group under the matrix multiplication. This group is called the *general linear group* of degree n over R and is denoted by $GL_n(R)$. Since $A \in U(M_n(R))$ if and only if $\det A \in U(R)$, we have

$$GL_n(R) = \{A \in M_n(R) \mid \det A \in U(R)\}.$$

THEOREM 2.4. *Let R be a commutative ring. Then*

$$\begin{aligned} M_n(R) =& GL_n(R) \\ &\cup \{A \in M_n(R) \mid detA \neq 0 \text{ and } det\ A \notin U(R)\} \\ &\cup Z(M_n(R)) \end{aligned}$$

Proof. Let $A \in M_n(R)$. Then either $\det A \in U(R)$ or $\det A \notin U(R)$. If $\det A \in U(R)$, then $A \in GL_n(R)$. Assume that $\det A \notin U(R)$. If $\det A \neq 0$, then $A \in \{A \in M_n(R) \mid \det A \neq 0 \text{ and } \det A \notin U(R)\}$. If $\det A = 0$, then by Lemma 2.3 $A \in Z(M_n(R))$. $\square$

3. The Total Rings of Fractions of Matrix Rings

LEMMA 3.1. *If R is a ring, then*

$$U(R) \cap Z(R) = \emptyset.$$

Proof. Suppose that $U(R) \cap Z(R) \neq \emptyset$. Take an element $a \in U(R) \cap Z(R)$. Then $a \in U(R)$, so there exists an element $b \in R$ such that $ab = 1$, $ba = 1$. $a \in Z(R)$, so there exists a non-zero element $c \in R$ such that $ac = 0$. Then

$$c = 1c = (ba)c = b(ac) = b0 = 0.$$

This contradiction shows that $U(R) \cap Z(R) = \emptyset$. $\square$

DEFINITION. Let R be a ring (not necessarily commutative). A multiplicatively closed subset S of R is said to be *saturated* if whenever $xy \in S$, where x, $y \in R$, then $x \in S$ and $y \in S$.

Note that this definition in non-commutative case coincides completely with that in commutative case.

THEOREM 3.2. *Let R be a unique factorization domain and let*

$$S = U(R) \cup \{a \in R \mid a \text{ is a product of principal primes of } R\}.$$

Then

(1) *S is a saturated multiplicatively closed set in R.*
(2) *$M_n(R) = \{A \in M_n(R) \mid \det A \in S\} \cup Z(M_n(R))$.*
(3) *$M_n(R)_S = U(M_n(R)_S) \,\dot{\cup}\, Z(M_n(R)_S)$.*
(4) *$U(M_n(R)_S)$ is a subgroup of $GL_n(R_S)$.*
(5) *$M_n(R)_S$ is R_S-isomorphic to $M_n(R_S)$.*

Proof. (1) See [K74]. (2) By Theorem 2.4. (3) Let A/s be any element of $M_n(R)_S$. Assume $\det A \in S$. Then

$$(A/s)(s\ \mathrm{adj}A/\det A)) = I/1,$$

$$(s\ \mathrm{adj}A/\det A))(A/s) = I/1.$$

So, $(A/s)^{-1} = s\,\mathrm{adj}A/\det A) \in M_n(R)_S$. Hence $A/s \in U(M_n(R)_S)$. Or, assume $\det A \notin S$. Then by (2) $A \in Z(M_n(A))$. Hence $A/s \in Z(M_n(A)_S)$. Therefore it follows from Lemma 3.1 that (3) holds.

(4) Define a map $\varphi : M_n(R)_S \longrightarrow M_n(R_S)$ by

$$\varphi(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} /s) = \begin{pmatrix} a_{11}/s & a_{12}/s & \cdots & a_{1n}/s \\ a_{21}/s & a_{22}/s & \cdots & a_{2n}/s \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}/s & a_{n2}/s & \cdots & a_{nn}/s \end{pmatrix}$$

Then φ is a well-defined injective ring homomorphism. Hence, $U(M_n(R)_S)$ is a subgroup of $GL_n(R_S)$.

(5)

$$\begin{aligned} M_n(R)_S &\cong M_n(R) \otimes_R R_S \\ &\cong R^{n^2} \otimes_R R_S \\ &\cong (R \oplus R \oplus \cdots \oplus R) \otimes_R R_S \\ &\cong (R \otimes_R R_S) \oplus (R \otimes_R R_S) \oplus \cdots \oplus (R \otimes_R R_S) \\ &\cong (R \otimes_R R_S)^{n^2} \\ &\cong {R_S}^{n^2} \\ &\cong M_n(R_S). \end{aligned}$$

□

The following result comes from Theorem 2.4 and Lemma 3.1. However, we give an alternative proof.

Theorem 3.3. *Let F be a field. Then $M_n(F) = GL_n(F) \,\dot\cup\, Z(M_n(F))$.*

Proof. It is clear that the result holds for $n = 1$.

Let $n \geq 2$. Assume $A = 0$. Then $AI = A = 0$. Hence $A \in Z(M_n(F))$.

Assume $A \neq 0$. Let E_{ij} be the matrix in which the only non-zero entry is a 1 in the ith row and jth column. Then by [K96, Theorem 1.20, p.65], A is equivalent to $E_{11} + E_{22} + \cdots + E_{rr}$ for some $r \in \{1, 2, 3, \cdots, n\}$. There exist nonsingular matrices P, Q such that

$$PAQ = E_{11} + E_{22} + \cdots + E_{rr}. \tag{*}$$

If $r = n$, then $PAQ = E_{11} + E_{22} + \cdots + E_{nn} = I$ and hence det $A \neq 0$. Thus, $A \in GL_n(F)$. Assume $r < n$. Multiplying $E_{r+1,1}$ on both sides of the equation

(*), we have

$$\begin{aligned} PAQE_{r+1,1} &= E_{11}E_{r+1,1} + E_{22}E_{r+1,1} + \cdots + E_{rr}E_{r+1,1} \\ &= \delta_{1,r+1}E_{11} + \delta_{2,r+1}E_{21} + \cdots + \delta_{r,r+1}E_{r1} \\ &= 0 \end{aligned}$$

Hence $AQE_{r+1,1} = 0$ and $QE_{r+1,1} \neq 0$. Thus $A \in Z(M_n(F))$. Therefore $M_n(F) = GL_n(F) \cup Z(M_n(F))$. It follows from Lemma 3.1 that

$$M_n(F) = GL_n(F) \mathbin{\dot\cup} Z(M_n(F)).$$

□

THEOREM 3.4. *If F is a field, then the matrix ring $M_n(F)$ is its total ring of fractions.*

Proof. Assume that F is a field. Then

$$GL_n(F) = \{A \in M_n(F) \mid det(A) \neq 0\} = U(M_n(F)).$$

Hence it is clear that $G_n(F)$ is a saturated multiplicatively closed subset of $M_n(F)$. Let $S_0 = M_n(F) \backslash Z(M_n(F))$. Then by Theorem 3.3, $S_0 = GL_n(F)$ and hence S_0 is a saturated multiplicatively closed subset of $M_n(F)$. Thus $M_n(F)_{S_0}$ is the total ring of fractions of $M_n(F)$ ([AM69, Chapter 3, Exercise 9, p.44] and [H88].) Let $id : M_n(F) \to M_n(F)$ be the identity ring homomorphism. Since

$$S_0 = GL_n(F) = U(M_n(F)),$$

we can see that $id(A)$ is a unit in $M_n(F)$ for all $A \in S_0$. Consider the following diagram

$$\begin{tikzcd} M_n(F) \arrow[rr, "\mathrm{id}"] \arrow[dr, "f"'] & & M_n(F) \\ & M_n(F)_{S_0} \arrow[ur] & \end{tikzcd}$$

Then by the universal mapping property (see [E95, p.60]) there exists a homomorphism $h : M_n(F)_{S_0} \to M_n(F)$ such that $h \circ f = \mathrm{id}$. From this equation, it follows that h is an epimorphism. Further, for any $A/B \in M_n(F)_{S_0}$,

$$h(A/B) = h(A/I \cdot I/B) = h(f(A))h(f(B^{-1})) = id(A)id(B^{-1}) = AB^{-1}.$$

From this equation we can see that h is injective. Hence h is bijective. Thus h is an isomorphism. This shows that $M_n(F) \cong M_n(F)_{S_0}$. Therefore $M_n(F)$ is its total ring of fractions. $\square$

Let $\mathbb{C}$ be the complex number field. Then it is clear that $M_n(\mathbb{C})$ is isomorphic to $M_n(\mathbb{C})_{GL_n(\mathbb{C})}$ because $GL_n(\mathbb{C}) = U(M_n(\mathbb{C}))$. If we let $S_0 = M_n(\mathbb{C})\backslash Z(M_n(\mathbb{C}))$, then it follows from Theorem 3.3 that

$$M_n(\mathbb{C}) \cong M_n(\mathbb{C})_{S_0}.$$

Hence $M_n(\mathbb{C})$ is its total ring of fractions. In fact, the proof of Theorem 3.4 is standard.

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