

CLASSIFICATION OF SINGULAR SOLUTIONS FOR THE POISSON PROBLEM WITH VARIOUS BOUNDARY CONDITIONS

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Abstract. The precise form of singular functions, singular function representation and the extraction form for the stress intensity factor play an important role in the singular function methods to deal with the domain singularities for the Poisson problems with most common boundary conditions, e.g. Dirichlet or Mixed boundary condition [2, 4]. In this paper we give an elementary step to get the singular functions of the solution for Poisson problem with Neumann boundary condition or Robin boundary condition. We also give singular function representation and the extraction form for the stress intensity with a result showing the number of singular functions depending on the boundary conditions.

1. Introduction

The model problem is to find $u \in H_D^1(\Omega)$ such that

$$(1.1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} + au = 0 & \text{on } \partial\Omega \setminus \Gamma_D, \end{cases}$$

where f belongs to $L^2(\Omega)$, Δ stands for the Laplace operator, and Ω is a polygonal domain in R^2 . The singularity of the solution depends on

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the shape of the domain and the boundary condition. If $\partial\Omega \setminus \Gamma_D = \emptyset$, we have the Dirichlet boundary problem. We will consider the mixed case i.e. $\Gamma_D \neq \emptyset$ and $\partial\Omega \setminus \Gamma_D \neq \emptyset$. Here, we have Neumann condition if $a = 0$ and Robin boundary condition if $a > 0$.

The precise form of singular functions, singular function representation and the extraction form for the stress intensity factor play essential roles in the singular function methods to deal with the domain singularities for the Poisson problems with most common boundary conditions, e.g. Dirichlet or mixed boundary condition [2, 4, 6]. In this paper we give an elementary step to get the singular functions of the solution for Poisson problem with Neumann boundary condition or Robin boundary condition. We also give singular function representation and the extraction form for the stress intensity factor with results showing the number of singular functions depending the boundary conditions. Although the Robin boundary condition for Poisson problem is less interesting than inverse problem or problem of parabolic type, we expect to use the results to apply the singular function methods [2, 4] to more general type of problems.

We will consider the mixed cases with one singular corner, one of whose adjacent sides has Dirichlet boundary condition and the other has Neumann or Robin boundary condition. In this paper we will classify the singular functions for these cases and show the dependency of the number of singular solutions to the shape of the domain and the boundary conditions. We start this by introducing the elementary methods to obtain the singular solutions of Laplace's equation using polar coordinates.

In polar coordinates Laplace's equation takes the form

$$(1.2) \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

To apply the method of separation of variables to this problem we assume that

$$(1.3) \quad u(r, \theta) = R(r)\Theta(\theta),$$

and substitute the differential equation (1.2) for u . So we have

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

or

$$(1.4) \quad r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\Theta''}{\Theta} = k,$$

where k is a separation constant. Thus we obtain the two ordinary differential equations

$$(1.5) \quad r^2 R'' + rR' - kR = 0,$$

$$(1.6) \quad \Theta'' + k\Theta = 0.$$

It is easy to show boundary condition that the periodicity condition requires k to be real. We will consider in turn the cases in which k is negative, zero, and positive.

If $k < 0$, let $k = -\mu^2$, where $\mu > 0$. Then Eq. (1.6) becomes $\Theta'' - \mu^2\Theta = 0$, and consequently

$$(1.7) \quad \Theta(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

Thus $\Theta(\theta)$ can be periodic only if $c_1 = c_2 = 0$, and we conclude that k cannot be negative.

Similarly, if $k = 0$, then Eq. (1.3) is not a singular.

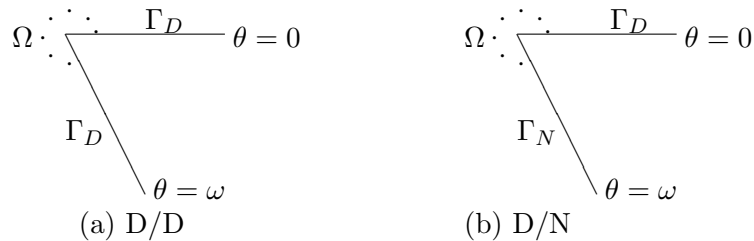


FIGURE 1. Dirichlet and Mixed(D/N) boundary conditions.

Finally, if $k > 0$, we let $k = \mu^2$, where $\mu > 0$. Then (1.5) and (1.6) become

$$(1.8) \quad r^2 R'' + rR' - \mu^2 R = 0$$

and

$$(1.9) \quad \Theta'' + \mu^2\Theta = 0,$$

respectively. Equation (1.8) is an Euler equation and has the solution

$$(1.10) \quad R(r) = k_1 r^\mu + k_2 r^{-\mu},$$

while Eq. (1.9) has the solution

$$(1.11) \quad \Theta(\theta) = c_1 \sin \mu\theta + c_2 \cos \mu\theta.$$

Since the solution u of (1.1) belongs to $H^1(\Omega)$, the singular function we want is reduced to

$$(1.12) \quad R(r)\Theta(\theta) = r^\mu(c_1 \sin \mu\theta + c_2 \cos \mu\theta),$$

from (1.10) with $0 < \mu < 1$.

To control the singularity at the corner, we will use cut-off functions. To this, set

$$B(t_1; t_2) = \{(r, \theta) : t_1 < r < t_2 \text{ and } 0 < \theta < \omega\} \cap \Omega \quad \text{and} \quad B(t_1) = B(0; t_1).$$

A family of cut-off functions of r , $\eta_\rho(r)$, is then defined to be collections of smooth functions on $B(\infty)$ with the following property:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(\frac{1}{2}\rho R), \\ 0 & \text{in } \Omega \setminus \bar{B}(\rho R) \end{cases}$$

where ρ is a parameter in $(0, 2]$ and $R \in \mathbb{R}$ is a fixed number so that the $\eta_2 s$ has the same boundary condition as u . We summarize the necessary properties of the cut-off functions as follows;

$$(1.13) \quad \eta_\rho(\frac{1}{2}\rho R) = 1, \quad \eta'_\rho(\frac{1}{2}\rho R) = 0, \quad \eta_\rho(\rho R) = 0, \quad \eta'_\rho(\rho R) = 0.$$

If there are more than two corners with singularities, we assume that R is small enough so that the intersection of either $B_j(\rho R)$ and $B_i(2R)$ or $B_j(2R)$ and $B_i(\rho R)$ for $j \neq i$ is empty.

We recall the singular function depends on boundary condition. We get the extraction formula for the stress intensity factor for the Poisson problem with Robin boundary case. Although the Poisson problem itself with Robin boundary case is less considered, Robin boundary is important in many cases including the inverse problems. For future reference we classify the singularity for this boundary. In Section 2, we will classify the singular function representation of the solution for various boundary conditions using the above results. In Section 3, we give the extraction formula for the stress intensity factor for the Poisson problem with Robin boundary condition.

2. Classification of singular functions

The singularity is local property around the corner and depends on the inner angle and the boundary conditions on the edge adjacent the corner. We will compute every possible singular functions according to the inner angle ω and the boundary conditions in the following subsections. We will use the notations D/D , N/N , or D/N , etc. Obviously D/D and N/N means the Dirichlet and Neumann boundary conditions, respectively, whereas the mixed boundary condition D/N means that the boundary conditions changes from Dirichlet to Neumann boundary condition at the corner. We also use the notation R_a for the Robin boundary condition $\frac{\partial u}{\partial \nu} + au = 0$ with the constant a . So the notation D/R_a stands for the mixed boundary condition which changes from the Dirichlet condition to the Robin boundary condition $\frac{\partial u}{\partial \nu} + au = 0$ at the corner, when we move along the boundary near the corner with the inner region right hand side.

2.1. Dirichlet boundary condition : D/D

The first example is the Dirichlet boundary condition D/D , which is well known and most common example. Especially if $\partial\Omega \setminus \Gamma_D = \emptyset$ then the problem (1.1) is called Dirichlet boundary problem. Let ω be the internal angle of Ω satisfying $\pi < \omega < 2\pi$. Without the loss of generality, assume that the corresponding vertex is at the origin.

If we denote the singular function by $s(r, \theta)$;

$$s(r, \theta) = R(r)\Theta(\theta) = r^\mu(c_1 \sin \mu\theta + c_2 \cos \mu\theta),$$

then the boundary condition $\Theta(0) = 0, \Theta(\omega) = 0$ implies that $c_2 = 0$ and $\mu = \frac{\pi}{\omega}$. Therefore $\Theta(\theta)$ must be proportional to $\sin \mu\theta$. Thus we obtain the singular solution ([2, 5])

$$s(r, \theta) = r^{\frac{\pi}{\omega}} \sin \frac{\pi\theta}{\omega}.$$

2.2. Neumann boundary condition : N/N

If we have Neumann boundary condition (N/N) at the corner with inner angle $\omega > \pi$ we have a singularity. If $\Gamma_D = \emptyset$ and $a = 0$ then (1.1) is the Neumann boundary problem. For this case the form of the singular function

$$s(r, \theta) = R(r)\Theta(\theta) = r^\mu(c_1 \sin \mu\theta + c_2 \cos \mu\theta)$$

implies that $c_1 = 0$ and $\mu = \frac{\pi}{\omega}$ with the help of the boundary condition $\Theta'(0) = \Theta'(\omega) = 0$.

Thus we obtain the singular solution

$$s(r, \theta) = r^{\frac{\pi}{\omega}} \cos \frac{\pi\theta}{\omega}.$$

2.3. Mixed boundary condition : D/N or N/D

Now we assume the Mixed boundary conditions, i.e. D/N or N/D. If $\partial\Omega \setminus \Gamma_D \neq \emptyset$ with $a = 0$ and $\Gamma_D \neq \emptyset$, then the problem is said to have the Mixed boundary condition. Note we use the notation D/N or N/D to specify the type of boundary conditions which changes with adjacent to the vertex. Here we assume again that the corresponding vertex is at the origin.

First, we consider the case D/N, i.e. $\Theta(0) = 0, \Theta'(\omega) = 0$. If $\frac{\pi}{2} < \omega \leq \frac{3\pi}{2}$, there is a singular function of the form ([4])

$$s(r, \theta) = r^{\frac{\pi}{2\omega}} \sin \frac{\pi\theta}{2\omega}.$$

If $\frac{3\pi}{2} < \omega < 2\pi$, there are two singular functions of the form

$$s_1(r, \theta) = r^{\frac{\pi}{2\omega}} \sin \frac{\pi\theta}{2\omega} \quad \text{and} \quad s_3(r, \theta) = r^{\frac{3\pi}{2\omega}} \sin \frac{3\pi\theta}{2\omega}.$$

Secondly, we consider the case N/D, i.e. $\Theta'(0) = 0, \Theta(\omega) = 0$. If $\frac{\pi}{2} < \omega \leq \frac{3\pi}{2}$, there is a singular function of the form

$$s(r, \theta) = r^{\frac{\pi}{2\omega}} \cos \frac{\pi\theta}{2\omega}.$$

If $\frac{3\pi}{2} < \omega < 2\pi$, there are two singular functions of the form

$$s_1(r, \theta) = r^{\frac{\pi}{2\omega}} \cos \frac{\pi\theta}{2\omega} \quad \text{and} \quad s_3(r, \theta) = r^{\frac{3\pi}{2\omega}} \cos \frac{3\pi\theta}{2\omega}.$$

2.4. Robin boundary condition

Finally we assume the case $\partial\Omega \setminus \Gamma_D \neq \emptyset$ where $\frac{\partial u}{\partial \nu} + au = 0$ with $a > 0$, whose boundary condition is denoted by the notation R_a . The problem (1.1) is said to have the Robin boundary condition there.

In this case, we may consider the several different cases: $R_a/R_b, R_a/D, N/R_a$, etc. We will consider the case R_a/D , when the appropriate boundary conditions for the singular function (1.12) is

$$(2.14) \quad \Theta'(0) + a\Theta(0) = 0 \quad \text{and} \quad \Theta(\omega) = 0.$$

In the next chapter, we will show the explicit form of singular function is given.

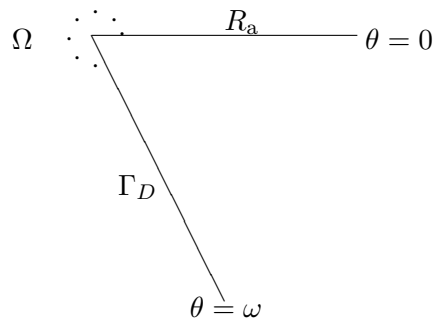


FIGURE 2. R_a/D boundary conditions.

3. Singularity for the R_a/D corner

To get the singular function corresponding to the case R_a/D , we have the following lemma.

Lemma 3.1. *Suppose*

$$(3.15) \quad u = R(r)\Theta(\theta) = r^\mu(c_1 \sin \mu\theta + c_2 \cos \mu\theta)$$

is the singular function satisfying the boundary condition R_a/D as in (2.14), then μ satisfies

$$(3.16) \quad \tan \mu\omega = \frac{\mu}{a}.$$

Proof. By substituting

$$(3.17) \quad \Theta(\theta) = c_1 \sin \mu\theta + c_2 \cos \mu\theta$$

into (2.14), we obtain

$$(3.18) \quad c_1\mu + ac_2 = 0 \quad \text{and} \quad c_1 \sin \mu\omega + c_2 \cos \mu\omega = 0.$$

Assuming that $c_1 = 1$, we have that $c_2 = -\mu/a$ and finally

$$\tan \mu\omega = \frac{\mu}{a}.$$

□

Note that for a precise singular corner, with given ω and a , we need to find the solution of (3.16) satisfying $0 < \mu < 1$, then we can get the singular function in the form;

$$s(r, \theta) = R(r)\Theta(\theta) = r^\mu(\sin \mu\theta - \frac{\mu}{a} \cos \mu\theta).$$

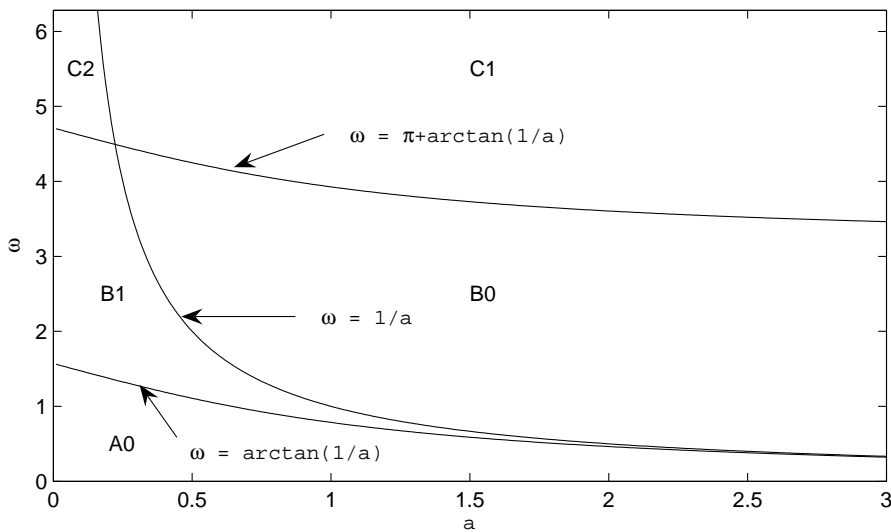


FIGURE 3. Numbers of singular functions at singular point with R_a/D condition with inner angle ω are given as : none on $A0 \cup B0$, one on $B1 \cup C1$, two on $C2$.

By using the above lemma we show the number of singular functions as in the following theorem. We show the possible combination of the values of ω and a for R_a/D boundary in Figure 3.

Theorem 3.1. *The number of singular functions of the system (1.1) at the given singular corner with R_a/D condition is given by $m - l$, where*

$$\begin{aligned} l &= 0 & \text{if } \omega a < 1, \\ l &= 1 & \text{if } \omega a \geq 1, \\ m &= 0 & \text{if } 0 < \omega \leq \tan^{-1} \frac{1}{a}, \\ m &= 1 & \text{if } \tan^{-1} \frac{1}{a} < \omega \leq \tan^{-1} \frac{1}{a} + \pi, \\ m &= 2 & \text{if } \tan^{-1} \frac{1}{a} + \pi < \omega \leq 2\pi. \end{aligned}$$

Proof. By recalling the condition $0 < a$ and $0 < \omega < 2\pi$, we observe that the maximum number of solutions $0 < \mu < 1$ of (3.16) is at most 2. Then we can figure out the theorem by observing the graphs of tangent. \square

For the case as in set C2 of the Figure 3, it is possible that two singular functions exist at a corner. For this case we use subindex i (or j) in singular functions,

$$s_i(r, \theta) = r^{\mu_i} \left(\sin \mu_i \theta - \frac{\mu_i}{a} \cos \mu_i \theta \right)$$

and its dual singular functions

$$s_{-i}(r, \theta) = r^{-\mu_i} \left(\sin \mu_i \theta - \frac{\mu_i}{a} \cos \mu_i \theta \right).$$

Now we will give the singular representation for the solution of (1.1) for the robin boundary condition, specially, R_a/D condition. For simplicity of presentation, assume that there is only one reentrant corner at the origin.

As in [2, 5] we conclude that the solution of problem (1.1) has the following singular function representation:

$$(3.19) \quad u = w + \sum_{l \in L} \lambda_l \eta_\rho(r) s_l(r, \theta)$$

where $w \in H^2(\Omega) \cap H_D^1(\Omega)$ is the regular part of the solution and $\lambda_l \in \mathbb{R}$ are the stress intensity factors. Note $|L| = 0$ for $A0 \cup B0$, $|L| = 1$ for $B1 \cup C1$, and $|L| = 2$ for C2 in Figure 3 by Theorem 3.1. So, there is no singular function for $A0 \cup B0$ and only one for $B1 \cup C1$ whereas two for C2, in which case we use the subindex i or j . To see the extraction formula for the stress intensity factor λ_l we need the following lemma.

Lemma 3.2. For $\rho \in (0, 1]$, we have

$$(3.20) \quad (\Delta(\eta_\rho s_i), \eta_2 s_{-j}) = -\alpha_i \delta_{ij}$$

with $\alpha_i = \mu_i(\omega + \mu_i^2 \omega / a^2 - 1/a) \neq 0$.

Proof. Let $B := B(\frac{\rho R}{2}; \rho R)$. Noting the facts $\Delta(\eta_\rho s_i) = \Delta(s_i) = 0$ on $B(\frac{\rho R}{2})$ and $\eta_2 = 1$ on the support of $\Delta(\eta_\rho s_i)$, we have

$$\begin{aligned} & (\Delta(\eta_\rho s_i), \eta_2 s_{-j}) \\ &= (s_i \Delta \eta_\rho + 2 \nabla \eta_\rho \cdot \nabla s_i, s_{-j}) \\ &= (s_i \Delta \eta_\rho, s_{-j}) + (2 \nabla \eta_\rho \cdot \nabla s_i, s_{-j}) \\ &= \int_B \Delta \eta (\sin \mu_i \theta - \frac{\mu_i}{a} \cos \mu_i \theta) (\sin \mu_j \theta - \frac{\mu_j}{a} \cos \mu_j \theta) dx \\ &+ \int_B 2 \mu_i r^{-1} (\sin \mu_i \theta - \frac{\mu_i}{a} \cos \mu_i \theta) (\sin \mu_j \theta - \frac{\mu_j}{a} \cos \mu_j \theta) \eta_r dx \\ &= \int_{\frac{R}{2}}^R r \eta_{rr} + (1 + 2 \mu_i) \eta_r dr \\ &\cdot \int_0^\omega (\sin \mu_i \theta - \tan \mu_i \omega \cos \mu_i \theta) (\sin \mu_j \theta - \tan \mu_j \omega \cos \mu_j \theta) d\theta \\ &:= F_1(r) F_2(\theta). \end{aligned}$$

Now, we have $F_1(r) = -2 \mu_i$, by the definition of the cut-off function, or (1.13), and

$$\begin{aligned} F_2(\theta) &= \int_0^\omega (\sin \mu_i \theta - \tan \mu_i \omega \cos \mu_i \theta) (\sin \mu_j \theta - \tan \mu_j \omega \cos \mu_j \theta) d\theta \\ &= \int_0^\omega \sin \mu_i \theta \sin \mu_j \theta d\theta + (\tan \mu_i \omega \tan \mu_j \omega) \int_0^\omega \cos \mu_i \theta \cos \mu_j \theta d\theta \\ &- \tan \mu_i \omega \int_0^\omega \cos \mu_i \theta \sin \mu_j \theta d\theta - \tan \mu_j \omega \int_0^\omega \sin \mu_i \theta \cos \mu_j \theta d\theta. \end{aligned}$$

Finally, we have, if $i = j$,

$$\begin{aligned} F_2(\theta) &= \int_0^\omega (\sin \mu_i \theta - \tan \mu_i \omega \cos \mu_i \theta) (\sin \mu_j \theta - \tan \mu_j \omega \cos \mu_j \theta) d\theta \\ &= \frac{1}{2} (\omega + \frac{\mu_i^2 \omega}{a^2} - \frac{1}{a}) \end{aligned}$$

and, if $i \neq j$,

$$F_2(\theta) = \int_0^\omega (\sin \mu_i \theta - \tan \mu_i \omega \cos \mu_i \theta) (\sin \mu_j \theta - \tan \mu_j \omega \cos \mu_j \theta) d\theta = 0.$$

Now it is enough to show the following inequality;

$$G(\omega, \mu, a) := \frac{1}{2} (\omega + \frac{\mu^2 \omega}{a^2} - \frac{1}{a}) > 0$$

for $0 < \mu < 1$ satisfying (3.16) with given $0 < \omega < 2\pi$ and $0 < a$. By substituting

$$\frac{1}{a} = \frac{\tan \mu \omega}{\mu}$$

and elementary computations, we observe

$$G(\omega, \mu, a) = \frac{\omega}{2 \cos^2 \mu \omega} (1 - \frac{\sin \mu \omega \cos \mu \omega}{\mu \omega}) \geq \frac{\omega}{2} (1 - \frac{\sin \mu \omega \cos \mu \omega}{\mu \omega}) > 0$$

and have the lemma. \square

Now we use the approach in [2] to get the extraction formula for the stress intensity factor λ_l . First we note that the singular function representation of the solution of problem (1.1) has the form

$$u = w + \sum_l \lambda_l \eta_\rho s_l,$$

where $w \in H_D^1(\Omega) \cap H^2(\Omega)$ satisfies

$$(3.21) \quad -\Delta w - \sum_l \lambda_l \Delta(\eta_\rho s_l) = f \text{ in } \Omega.$$

Then we multiply $\eta_2 s_k$ and integrate each side and have the following theorem using Lemma 3.2 and Theorem 1.5.3.6 in [5].

Theorem 3.2. *The stress intensity factors λ_l can be expressed in terms of w by the following extraction formula:*

$$(3.22) \quad \lambda_l = \frac{1}{\alpha_l} (f, \eta_2 s_{-l}) + \frac{1}{\alpha_l} (w, \Delta(\eta_2 s_{-l})).$$

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