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GENERALIZED DERIVATIONS OF BCI-ALGEBRAS

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Abstract. The notion of generalized derivations of a BCI-algebra is introduced, and some related properties are investigated. Also, the concept of a torsion free BCI-algebra is introduced and some properties are discussed.

1. Introduction

Several authors have studied derivations in rings and near rings after Posner [9] have given the definition of the derivation in ring theory. Bresar [2] introduced the generalized derivation in rings and then many mathematicians studied on this concept. Jun and Xin firstly discussed derivations in BCI-algebras [5]. As a continuation of the paper [5], in this paper, we introduce the concept of generalized derivations in BCI-algebras and torsion BCI-algebras, and investigate several properties.

2. preliminaries

A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCI-algebra* if it satisfies the following conditions:

(I) $((xy)(xz))(zy) = 0$,
(II) $(x(xy))y = 0$,
(III) $xx = 0$,
(IV) $xy = 0$ and $yx = 0$ imply $x = y$

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for all $x, y, z \in X$. In any BCI-algebra X one can define a partial ordering "$\leq$ " by putting $x \leq y$ if and only if $xy = 0$. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a *BCK-algebra*. A BCI-algebra X has the following properties for all $x, y, z \in X$,

(1) $x0 = x$,
(2) $(xy)z = (xz)y$,
(3) $x \leq y$ implies $xz \leq yz$ and $zy \leq zx$,
(4) $(xz)(yz) \leq xy$,
(5) $x(x(xy)) = xy$,
(6) $0(xy) = (0x)(oy)$,
(7) $x0 = 0$ implies $x = 0$.

For a BCI-algebra X, the *BCK-part* of X, denoted by X_+, is defined to be the set of all $x \in X$ such that $0 \leq x$, and the *BCI-G part* of X, denoted by $G(X)$, is defined to be the set of all $x \in X$ such that $0x = x$. Note that $G(X) \cap X_+ = \{0\}$ (see [3]). If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra*. In a p-semisimple BCI-algebra X, the following hold:

(8) $(xz)(yz) = xy$,
(9) $0(0x) = x$,
(10) $x(0y) = y(0x)$,
(11) $xy = 0$ implies $x = y$,
(12) $xa = xb$ implies $a = b$,
(13) $ax = bx$ implies $a = b$,
(14) $a(ax) = x$.

Let X be a p-semisimple BCI-algebra. If we define an addition "+" as $x + y = x(0y)$ for all $x, y \in X$, then $(X, +)$ is an abelian group with identity 0 and $x - y = xy$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $xy = x - y$. Then X is a p-semisimple BCI-algebra and $x + y = x(0y)$ for all $x, y \in X$ (see [7]). For a BCI-algebra X we denote $x \wedge y = y(yx)$, and $L_p(X) = \{a \in X \mid (\forall x \in X)(xa = 0 \Rightarrow x = a)\}$. We call the elements of $L_p(X)$ the *p-atoms* of X. Note that $L_p(X) = \{x \in X \mid 0(0x) = x\}$ which is the p-semisimple part of X, and X is a p-semisimple BCI-algebra if and only if $L_p(X) = X$ (see [4]). It is clear that $G(X) \subset L_p(X)$, and $x(xa) = a$ and $ax \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. For more details, refer to [1, 6, 8, 10, 11].

Definition 2.1. [5, Definition 3.5] Let X be a BCI-algebra. A *left-right derivation* (briefly, *(l,r)-derivation*) of X is defined to be a self-map d of X satisfying the identity $d(xy) = d(x)y \wedge xd(y)$ for all $x, y \in X$. If d satisfies the identity $d(xy) = xd(y) \wedge d(x)y$ for all $x, y \in X$, d is called a *right-left derivation* (briefly, *(r,l)-derivation*) of X. Moreover if d is both an (l,r)- and (r,l)-derivation, we say that d is a *derivation*.

3. Generalized Derivations

Definition 3.1. Let X be a BCI-algebra. A mapping $D : X \to X$ is called a *generalized (l,r)-derivation* if there exist an (l,r)-derivation $d : X \to X$ such that $D(xy) = D(x)y \wedge xd(y)$ for all $x, y \in X$. If there exist an (r,l)-derivation $d : X \to X$ such that $D(xy) = xD(y) \wedge d(x)y$ for all $x, y \in X$, the mapping $D : X \to X$ is called a *generalized (r,l)-derivation.* Moreover if D is both a generalized (l,r)- and (r,l)-derivation, we say that D is a *generalized derivation.*

Example 3.2. Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table.

	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a map $d : X \to X$ by $d(x) = \begin{cases} b & \text{if } x = 0 \\ 0 & \text{if } x = a \\ a & \text{if } x = b \end{cases}$. Then d is an (l,r)-derivation of X. But d is not an (r,l)-derivation of X since $d(ab) \neq ad(b) \wedge d(a)b$. Now we define a map $D : X \to X$ by

$$D(x) = \begin{cases} a & \text{if } x = 0, \\ b & \text{if } x = a, \\ 0 & \text{if } x = b. \end{cases}$$

It is easy to verify that D satisfies the equality $D(xy) = D(x)y \wedge xd(y)$ for all $x, y \in X$. Hence D is a generalized (l,r)-derivation of X. Also, let $D : X \to X$ satisfy $D(xy) = D(x)y$ for all $x, y \in X$. Note that X satisfies $x \wedge y = x$ for all $x, y \in X$. Hence, for every (l,r)-derivation d of

X we have $D(xy) = D(x)y = D(x)y \wedge xd(y)$ for all $x, y \in X$. Therefore D is a generalized (l, r)-derivation of X.

Example 3.3. Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define a map $d : X \to X$ by $d(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$. Then d is a derivation of X. Now we define a map $D : X \to X$ by

$$D(x) = \begin{cases} b & \text{if } x = 0, a \\ 0 & \text{if } x = b. \end{cases}$$

It is easy to verify that D is a generalized derivation of X.

Proposition 3.4. *Let D be a self-map of a BCI-algebra X. Then*

(i) *If D is a generalized (l, r)-derivation of X, then $D(x) = D(x) \wedge x$ for all $x \in X$.*

(ii) *If D is a generalized (r, l)-derivation of X, then $D(0) = 0$ if and only if $D(x) = x \wedge d(x)$ for all $x \in X$ and for some (r, l)-derivation d of X.*

Proof. (i) If D is a generalized (l, r)-derivation, then there exist an (l, r)-derivation d such that $D(xy) = D(x)y \wedge xd(y)$ for all $x, y \in X$. Hence we get

$$\begin{aligned} D(x) &= D(x0) = D(x)0 \wedge xd(0) = D(x) \wedge xd(0) \\ &= (xd(0))((xd(0))D(x)) = (xd(0))((xD(x))d(0)) \\ &\leq x(xD(x)) = D(x) \wedge x. \end{aligned}$$

But $D(x) \wedge x \leq D(x)$ is trivial and so (i) holds.

(ii) Suppose that D is a generalized (r, l)-derivation of X. Then there exist an (r, l)-derivation d such that $D(xy) = xD(y) \wedge d(x)y$ for all $x, y \in X$. If $D(0) = 0$, then we have $D(0) = D(x0) = xD(0) \wedge d(x)0 = x \wedge d(x)$. Conversely, if $D(x) = x \wedge d(x)$, then $D(0) = 0 \wedge d(0) = d(0)(d(0)0) = d(0)d(0) = 0$. This completes the proof. □

Proposition 3.5. *Let D be a generalized (l, r)-derivation of a BCI-algebra X. Then*

(i) $D(0) \in L_p(X)$.
(ii) $(\forall a \in L_p(X))$ $(D(a) = D(0) + a \in L_p(X))$.
(iii) $(\forall a \in L_p(X))$ $(\forall x \in X)$ $(D(ax) = D(a)x)$.
(iv) $(\forall a \in L_p(X))$ $(\forall x \in X)$ $(D(a + x) = D(a) + x)$.
(v) $(\forall a, b \in L_p(X))$ $(D(a + b) = D(a) + b = a + D(b))$.

Proof. (i) Using Proposition 3.4(i), we have $D(0) = D(0) \wedge 0 = 0(0D(0))$, and so $D(0) \in L_p(X)$.

(ii) Let $a \in L_p(X)$. Then it is known that $ax \in L_p(X)$ and $x(xa) = a$ for all $x \in X$. Hence $D(0)(0a) \in L_p(X)$ since $D(0) \in L_p(X)$ by (i). Then, we obtain

$$\begin{aligned} D(a) &= D(0(0a)) = D(0)(0a) \wedge 0d(0a) \\ &= (0d(0a))((0d(0a))(D(0)(0a))) \\ &= (0d(0a))((0(D(0)(0a)))d(0a)) \\ &= 0(0(D(0)(0a))) \\ &= D(0)(0a) = D(0) + a \in L_p(X). \end{aligned}$$

(iii) Let $a \in L_p(X)$ and $x \in X$. Then we have

$$D(ax) = D(a)x \wedge ad(x) = (ad(x))((ad(x))(D(a)x)) = D(a)x$$

since $ad(x)$, $D(a)x \in L_p(X)$.

(iv) Let $a \in L_p(X)$ and $x \in X$. Using (iii), we have

$$D(a + x) = D(a(0x)) = D(a)(0x) = D(a) + x.$$

(v) follows directly from (iv). □

Proposition 3.6. *Let D be a generalized (r, l)-derivation of a BCI-algebra X. Then*

(i) $(\forall a \in G(X))$ $(D(a) \in G(X))$.
(ii) $(\forall a \in L_p(X))$ $(D(a) \in L_p(X))$.
(iii) $(\forall a \in L_p(X))$ $(D(a) = aD(0) = a + D(0))$.
(iv) $(\forall a, b \in L_p(X))$ $(D(a + b) = D(a) + D(b) - D(0))$.
(v) *D is identity on $L_p(X)$ if and only if $D(0) = 0$.*

Proof. If D is a generalized (r, l)-derivation of X, then there exists an (r, l)-derivation d of X such that $D(xy) = xD(y) \wedge d(x)y$ for all $x, y \in X$. Now let $a \in G(X)$. Then

$$D(a) = D(0a) = 0D(a) \wedge d(0)a = (d(0)a)((d(0)a)(0D(a))) = 0D(a),$$

and so $D(a) \in G(X)$. Hence (i) is valid. For any $a \in L_p(X)$ we get

$$\begin{aligned} D(a) &= D(0(0a)) = 0D(0a) \wedge d(0)(0a) \\ &= (d(0)(0a))((d(0)(0a))(0D(0a))) \\ &= 0D(0a) \in L_p(X), \end{aligned}$$

and

$$\begin{aligned} D(a) &= D(0a) = aD(0) \wedge d(a)0 = d(a)(d(a)(aD(0))) \\ &= aD(0) = a(0D(0)) = a + D(0). \end{aligned}$$

Therefore (ii) and (iii) are valid. Now let $a, b \in L_p(X)$. Then $a + b \in L_p(X)$, and so

$$D(a+b) = a+b+D(0) = a+D(0)+b+D(0)-D(0) = D(a)+D(b)-D(0).$$

Thus (iv) is true. (v) follows directly from (iii). □

Definition 3.7. A BCI-algebra X is said to be *torsion free* if it satisfies:

$$(\forall x \in X)\,(x + x = 0 \;\Rightarrow\; x = 0).$$

By the definition above, we know that if there exists a nonzero element x of a BCI-algebra X such that $x + x = 0$, then X can not be torsion free. Note also that if a BCI-algebra X satisfies:

$$(\exists w(\neq 0) \in X)\,(0w = w),$$

then X can not be torsion free.

Example 3.8. (1) Every BCK-algebra is torsion free.

(2) The BCI-algebra X in Example 3.2 is torsion free.

(3) Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Then X is a torsion free BCI-algebra.

(4) A BCI-algebra $X = \{0, 1, 2, 3\}$ together with the following Cayley table:

	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

is not torsion free because $3(03) = 33 = 0$.

Proposition 3.9. *Let X be a torsion free BCI-algebra and D a generalized derivation. If $D^2 = 0$ on $L_p(X)$, then $D = 0$ on $L_p(X)$.*

Proof. Assume that $D^2 = 0$ on $L_p(X)$. Let $x \in L_p(X)$. Then $x + x \in L_p(X)$, and so

$$\begin{aligned} 0 &= D^2(x+x) = D(D(x+x)) = D(0) + D(x+x) \\ &= D(0) + D(x) + D(x) - D(0) = D(x) + D(x). \end{aligned}$$

Since X is torsion free, it follows that $D(x) = 0$ for all $x \in L_p(X)$ so that $D = 0$ on $L_p(X)$. □

Proposition 3.10. *Let X be a torsion free BCI-algebra and let D_1 and D_2 be two generalized derivations. If $D_1 D_2 = 0$ on $L_p(X)$, then $D_2 = 0$ on $L_p(X)$.*

Proof. Let $x \in L_p(X)$. Then $x + x \in L_p(X)$, and thus

$$\begin{aligned} 0 &= (D_1 D_2)(x+x) = D_1(D_2(x+x)) \\ &= D_1(0) + D_2(x+x) \\ &= D_1(0) + D_2(x) + D_2(x) - D_2(0) \\ &= D_1(0) - D_2(0) + D_2(x) + D_2(x) \\ &= D_1(0)D_2(0) + D_2(x) + D_2(x) \\ &= D_1(0)(0D_2(0)) + D_2(x) + D_2(x) \\ &= D_1(0) + D_2(0) + D_2(x) + D_2(x) \\ &= D_1(D_2(0)) + D_2(x) + D_2(x) \\ &= (D_1 D_2)(0) + D_2(x) + D_2(x) \\ &= D_2(x) + D_2(x). \end{aligned}$$

Since X is torsion free, we have $D_2(x) = 0$ for all $x \in L_p(X)$. Hence $D_2 = 0$ on $L_p(X)$. □

Definition 3.11. An (l, r)-derivation (resp., (r, l)-derivation) d of a BCI-algebra X is said to be *regular* if $d(0) = 0$.

Example 3.12. The (l, r)-derivation D in Example 3.2 is regular, but the derivation d in Example 3.2 is not regular.

Definition 3.13. A mapping $D : X \to X$ is called a *regular generalized* (l, r) (resp., (r, l))*-derivation* if there exists a regular (l, r) (resp., (r, l))-derivation d of X such that $D(xy) = D(x)y \wedge xd(y)$ (resp., $D(xy) = xD(y) \wedge d(x)y$) for all $x, y \in X$.

Example 3.14. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCI-algebra described in Example 3.8(3). We give here the $\wedge$-operation on X as follows:

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	0	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	2	4	2	2	4	4
5	3	5	3	3	5	5

Then a map $d : X \to X$ given by

$$d(x) := \begin{cases} 0 & \text{if } x = 0, 1, \\ 2 & \text{if } x = 2, 4, \\ 3 & \text{if } x = 3, 5 \end{cases}$$

is a regular (l, r)-derivation of X. Define a self map D of X by $D(0) = 0$, $D(2) = 2$, $D(3) = 3$, and ordered triple $(D(1), D(4), D(5))$ is $(5, 1, 4)$, $(4, 5, 1)$, $(3, 0, 2)$, $(2, 3, 0)$, $(1, 4, 5)$, or $(0, 2, 3)$. Then D is a regular generalized (l, r)-derivation of X.

If D is a regular generalized (r, l)-derivation of a BCI-algebra X, then there exists a regular (r, l)-derivation d of X such that

$$D(0) = D(00) = 0D(0) \wedge d(0)0 = 0D(0) \wedge 0 = 0(0(0D(0))) = 0D(0).$$

Therefore we have the following proposition.

Proposition 3.15. *If D is a regular generalized (r, l)-derivation of a BCI-algebra X, then $D(0) = 0D(0)$.*

References

[1] M. Aslam, A. B. Thaheem, *A note on p-semisimple BCI-algebras,* Math. Japon. **36(1)**, (1991), 39–45.

[2] M. Bresar, *On the distance of the composition of the two derivations to the generalized derivations,* Glasgow Math. J. **33(1)**, (1991), 89–93.

[3] Y. B. Jun, E. H. Roh, *On the BCI-G part of BCI-algebras,* Math. Japon. **38(4)**, (1993), 697–702.

[4] Y. B. Jun, X. L. Xin, E. H. Roh, *The role of atoms in BCI-algebras,* Soochow J. Math. **30** (2004), no. 4, 491–506.

[5] Y. B. Jun, X. L. Xin, *On derivation of BCI-algebras,* Inform. Sci. **159** (2004), 167–176.

[6] T. D. Lei, C. C. Xi, *p-radical in BCI-algebras,* Math. Japon. **30(4)** (1985), 511–517.

[7] D. J. Meng, *BCI-algebras and abelian groups,* Math. Japon. **32(5)** (1987), 693–696.

[8] J. Meng, Y. B. Jun, E. H. Roh, *BCI-algebras of order 6,* Math. Japon. **47(1)** (1998), 33–43.

[9] E. Posner, *Derivations in prime rings,* Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[10] X. L. Xin, E. H. Roh, J. C. Li, *Some results on the BCI-G part of BCI-algebras,* Far East J. Math. Sci. Special volume (Part III) (1997) 363–370.

[11] Q. Zhang, *Some other characterizations of p-semisimple BCI-algebras,* Math. Japon. **36(5)** (1991) 815–817.

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