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ON THE ORDERS IN A QUATERNION ALGEBRA OVER A DYADIC LOCAL FIELD

Sungtae Jun and Insuk Kim

Abstract. The orders in a quaternion algebra play a central role of the theory of Hecke operators. In this paper, we study the arithmetic properties of optimal embeddings of orders in a quaternion algebra over a dyadic local field.

1. Introduction

A quaternion algebra over a field k means a semi simple algebra of dimension 4 over k. In a quaternion algebra, there are three kinds of primitive orders in quaternion algebras over a local field. That is, an order Λ of a quaternion algebra A over a local field k is called primitive if it satisfies one of the following conditions. If A is a division algebra, Λ contains the full ring of integers of a quadratic extension field of k. If A is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, then Λ contains a subset which is isomorphic either to $\mathcal{O} \oplus \mathcal{O}$ where $\mathcal{O}$ is the ring of integers in k, or to the full ring of integers in a quadratic extension field of k. The arithmetic properties of first two types of orders were studied by Hijikata, Pizer and Shemanske in [5] and [10] and they solved so called "Basis Problem". More generally, Brezezinski studied bass orders in a quaternion algebra which include the remaining type of orders [1]. In this paper, we compute the number of optimal embeddings of primitive orders containing the full ring of integers in a quadratic extension field of a dyadic local field k with a different method used in [1]. Finally as an application, we constructed theta series associated with these primitive orders, which are modular forms of weight 2 on a certain congruence group. Unlike the theta series

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constructed in [5], characters were not used in this case in defining theta series.

2. Orders

2.1. In this section, we summarize the arithmetic theory of a dyadic local field. Throughout this paper, we assume that k is a dyadic local field. Let $\mathcal{O}$ denote the ring of integers in k, P, the maximal ideal of $\mathcal{O}$. We denote the discriminant of α by $\Delta(\alpha)$. Let L be a quadratic extension field of k. If Γ is an $\mathcal{O}$ algebra of rank 2 contained in L, then $\Gamma = \mathcal{O} + \mathcal{O}x$ and the discriminant of Γ is defined by

$$\Delta(\Gamma) = \Delta(x) \mod U^2,$$

where U is the set of all units in $\mathcal{O}$.

2.2. Let A be a quaternion algebra which is split over a dyadic local field k. That is $A = M_2(k)$. Let L be a quadratic extension field of k. In [8], we have proved that $\left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\}$ is a quaternion algebra over k and $\left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\} = L + \xi L$, where $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\xi\alpha = \overline{\alpha}\xi, \xi^2 = 1$ and $\overline{\xi} = -\xi$. Hence, we can define the norm of an element in A as its determinant.

Definition 1. Let L be a quadratic extension of k.

$$t = t(L) = \mathrm{ord}_k(\Delta(L)) - 1.$$

Remark. If L is an unramified extension field of k, then $t(L) = -1$. On the other hand, if L is a ramified extension field of a field k, then $t(L) \geq 0$. Furthermore, if k is a dyadic local field, then $0 < t \leq 2e$ by 2.3 and 1.3 in [6]. It is easy to see that $\mathrm{ord}_L(x - \bar{x}) \geq \mathrm{ord}_k(\Delta(L)) = t + 1$ for $x \in \mathcal{O}_L$. Let P_L be the prime ideal of $\mathcal{O}_L$, the ring of integers in L.

Proposition 2.1. *Let the notation be as above. Let R be an order of A and L a quadratic extension field in A. Then R contains $\mathcal{O}_L$ if and only if*

$$R = \begin{cases} \mathcal{O}_L + \xi P_L^n & \text{if } L \text{ is an unramified extension field,} \\ \mathcal{O}_L + (1+\xi)P_L^{n-t-1} & \text{if } L \text{ is a ramified extension field ,} \\ \text{or} \quad \mathcal{O}_L + (1-\xi)P_L^{n-t-1} & \end{cases}$$

for some nonnegative integer n and $t = t(L)$.

Proof. See Proposition 3.1 in [8]. □

Remark. If L is unramified, then the index n of $\mathcal{O}_L + \xi P_L^n$ is always even. On the other hand, if L is ramified and $t(L) < 2e$, then $(1+\xi)P_L^{-t-1} = (1-\xi)P_L^{-t-1}$. That is $\mathcal{O}_L + (1+\xi)P_L^{-t-1} = \mathcal{O}_L + (1-\xi)P_L^{-t-1}$. If L is ramified and $t(L) = 2e$, then there are two different maximal orders $\mathcal{O}_L + (1+\xi)P_L^{-t-1}$ and $\mathcal{O}_L + (1-\xi)P_L^{-t-1}$. However, $\mathcal{O}_L + (1+\xi)P_L^{n-t-1} = \mathcal{O}_L + (1-\xi)P_L^{n-t-1}$ for $n \geq 1$.

Definition 2. Let the notation be as above. We define an order of A as follows.

a. If L is unramified,

$$R_{2\nu}(L) = \mathcal{O}_L + \xi P_L^{\nu}.$$

b. If L is ramified,

$$R_\nu(L) = \mathcal{O}_L + (1+\xi)P_L^{\nu-t-1} \text{ or } \overline{R_0(L)} = \mathcal{O}_L + (1-\xi)P_L^{-t-1}$$

for some nonnegative integer ν.

Corollary 2.2. *Let the notations be as above. Then*

1. *if L is unramified,* $\cdots \subset R_{2n}(L) \subset R_{2n-2}(L) \cdots \subset R_0(L)$,
2. *if L is ramified and $t(L) = 2e$* , $\cdots \subset R_n(L) \subset R_{n-1}(L) \cdots \subset R_1(L) \subset \begin{cases} \overline{R_0(L)} \\ R_0(L), \end{cases}$
3. *if L is ramified and $0 < t(L) < 2e$* , $\cdots \subset R_n(L) \subset R_{n-1}(L) \cdots \subset R_1(L) \subset R_0(L)$.

Proof. This is immediate from Definition2 and the remark above. □

Definition 3. Let R be an order in A. The Eichler invariant $e(R)$ of R is defined as follows.

$$e(R) = \begin{cases} 1 & \text{if } R/J(R) \simeq \mathcal{O} \oplus \mathcal{O} \\ 0 & \text{if } R/J(R) \simeq \mathcal{O} \\ -1 & \text{if } R/J(R) \text{ is a quadratic extension of } \mathcal{O} \end{cases}$$

where $j(R)$ is the jacobson radical of R.

Remark. The primitive orders which we are dealing with are classified into either $e(R) = 1$, $e(R) = 0$ or $e(R) = -1$. That is if L is unramified, then $e(R) = -1$. If L is ramified, $R_1(L)$ with $t(L) = 2e$ is of $e(R) = 1$. For the others , $e(R) = 0$.

The followings are crucial in computing the number of optimal embeddings.

Theorem 2.3. *Let L be a quadratic extension field of k and $q = |\mathcal{O}_L/P_L|$. Then*

1. *if L is unramified,*

$$|R_2^\times(L) \setminus R_0^\times(L)| = q^2 - q$$

and

$$|R_{2n+2}^\times(L) \setminus R_{2n}^\times(L)| = q^2 \quad \text{for } n \geq 1,$$

2. *if L is ramified,*

$$|R_{n+1}^\times(L) \setminus R_n^\times(L)| = \begin{cases} q+1 & \text{for } n = 0 \\ q & \text{for } n \geq 1. \end{cases}$$

Proof. See Theorem 3.5 in [8]. □

3. Embeddings

In this section we will discuss the embeddings between orders. By an embedding we mean a k (or $\mathcal{O}_k$, the ring of integers) injective homomorphism. Let L and m be quadratic extensions of k. Then we will now determine all possible embeddings of $R_n(L)$ into $R_m(K)$ for nonnegative integers n and m. Assume that $K \subset A$. Let $\mathcal{O}_K$ be the maximal order of K. We say $\mathcal{O}_K$ is embeddable in $R_n(L)$ if there exists an embedding ϕ of K into A such that $\phi(\mathcal{O}_K) \subset R_n(L)$. According to Theorem 17.3 [13], all maximal orders of A are $A^\times$ conjugate to each other. Hence $\mathcal{O}_K$ is embeddable into $R_0(L)$ and $\mathcal{O}_L$ is embeddable into $R_0(K)$.

Definition 4. Let K and L be quadratic extensions of k contained in B. Then $\mu(K, L)$ is the nonnegative integer or ∞ such that $\mu(K, L) \geq n$ if and only if $\mathcal{O}_K$ is embeddable into $R_n(L)$.

Lemma 3.1. *Let K and L be the quadratic extensions of k contained in B. Then $\mathcal{O}_K$ is embeddable in $R_n(L)$ if and only if $R_n(K) \simeq R_n(L)$.*

Proof. See Lemma 3.2 in [9]. □

Remark. From Corollary 2.2, it is easy to see the followings. $\mathcal{O}_K$ is embeddable in $R_n(L)$ if and only if $\mathcal{O}_L$ is embeddable in $R_n(K)$. Thus we have Then $\mu(K, L) = \mu(L, K)$.

Let $\mathcal{O}^2 - 4\mathcal{O} = \{s^2 - 4n | s, n \in \mathcal{O}\}$ and let $\Delta_\sigma = (\pi^\sigma U \cap (\mathcal{O}^2 - 4\mathcal{O})/U^2$ for $\sigma = 0, 1, 2 \cdots$.

Definition 5. $\Delta_0^* = \Delta_0 - \{1\}, \Delta_1^* = \Delta_1$ and $\Delta_\sigma^* = \Delta_\sigma - \pi^2\Delta_{\sigma-2}$ for $\sigma \geq 2$.

3.1. Note that $\Delta_\sigma^* \neq \phi$ only if $\sigma = 2\rho, 0 \leq \rho \leq e$ or $\sigma = 2e+1$ where $e = \mathrm{ord}_k(2)$. Let

$$\Delta^* = \cup_{\sigma=0}^{\infty}\Delta_\sigma^* = \left(\cup_{\rho=0}^{e}\Delta_{2\rho}^*\right) \cup \Delta_{2e+1}^*.$$

Then we know that Γ is a maximal order of a quadratic extension field of k if and only if $\Delta(\Gamma) \in \Delta^*$. If $1 \leq \rho \leq e$,

$$\Delta_{2\rho}^* = \pi^{2\rho}(U^2 + \pi^{2e-2\rho+1}U)/U^2.$$

There is a bijective correspondence between elements of Δ^* and quadratic extension field of k given by $\Delta(\Gamma) \to \Gamma \otimes \mathcal{O}_k$ for $\Delta(\Gamma)$ an element of Δ^*.

Thus we can classify all quadratic extension fields of a dyadic local field k as follows: Δ_0^* contains one point which corresponds to the unique unramified quadratic extension of k and

$$\Delta_{2e+1}^* = \pi^{2e+1}U/U^2$$

contains $2q^2$ points representatives where $q = |\mathcal{O}/P|$.

Lemma 3.2. *Let $U = \mathcal{O}^\times$. Then*

$$U^2 = U^2 + P^{2e+1} \subset U^2 + P^{2e} \subset \cdots \subset U^2 + P^2 = U^2 + P = U$$

and

$$(U^2 + P^\sigma)/(U^2 + P^{\sigma+1}) \simeq \begin{cases} 1 & \sigma \text{ even and } < 2e, \\ \mathbb{Z}/2\mathbb{Z} & \sigma = 2e, \\ \bar{k} & \sigma \text{ odd.} \end{cases}$$

Proof. See Proposition 1.4 in [6]. □

3.2. We introduce the following notation;

$$\Delta R_n(L) = \{\Delta(\alpha) \mod U^2 | \alpha \in R_n(L)\},$$

where U is the set of all units in $\mathcal{O}$ and $\Delta(\alpha) = \mathrm{Tr}(\alpha)^2 - 4\mathrm{N}(\alpha)$.

Proposition 3.3. *If $t = t(L) > 0$ and $n \geq 3$, then*

$$\Delta_{t+1}^* \cap \Delta R_{t+n}(L) = \Delta(L)(U^2 + P^{2e+n-t-2}).$$

Proof. See [6]. □

Lemma 3.4. *Let $t(L) = 2e$, $R_n = R_n(L)$, and $0 \leq \sigma \leq e$. Then*

$$\Delta R_0 \cap \Delta^*_{2\sigma} = \Delta^*_{2\sigma}$$

and for $n \geq 1$

$$\Delta R_{2n} \cap \Delta^*_{2\sigma} = \Delta R_{2n+1} \cap \Delta^*_{2\sigma} = \begin{cases} \emptyset \text{ if } & n > \sigma \\ \Delta^*_{2\sigma} \text{ if } & n \leq \sigma. \end{cases}$$

Proof. See [6]. □

Lemma 3.5. *Assume that L and L' are non isomorphic quadratic extensions of k contained in B(i.e. $\Delta(L) \neq \Delta(L')$) and let M be any quadratic extension of k contained in B with $t(M) = 2e$. We have*

1. *if $0 < t(L) < 2e$, then $\mu(L, M) = t(L) + 1$,*
2. *if $0 < t(L) = t(L') < 2e$, then $\mu(L, L') \geq t(L) + 2$,*
3. *if $0 < t(L) = t(L') = 2e$, then $\mu(L, L') \geq 2e + 3$.*

Proof. (1): By Proposition 3.3, $\Delta(L) \in \Delta(R_{2n+1}(M))$ if and only if $2n+1 = t(L)+1$. Hence $\mu(L, M) = t(L)+1$. (2): Letting $t(L) = 2\sigma-1$, we have $\Delta(L) \in \Delta^*_{2\sigma}$. Hence, $\mu(L, L') = t(L) + 2$ which proves (2). By Lemma 3.1, $\nu \leq t(L) + 2$. Then $R_\nu(L) \simeq R_\nu(M) \simeq R_\nu(L')$. So $\mu(L, L') \geq t(L)+2$. (3): we have $\Delta^*_{2e+1} \cap \Delta R_{2e+3}(L) = \Delta(L)U = \Delta^*_{2e+1}$. By Lemma 3.1, $\mu(L, L') \geq t(L) + 3$ which proves (3). □

We are now able to answer the questions about the embeddability as follows.

Theorem 3.6. *Let L' and L be quadratic extensions of k in B. The number $\mu(L', L)$ is determined as follows.*

1. *if $\Delta(L) = \Delta(L')$, then $\mu(L, L') = \infty$,*
2. *if $t(L) \neq t(L')$, then $\mu(L, L') = 2 + min(t(L), t(L'))$,*
3. *if $t(L) = t(L') = t$ and $\Delta(L) \neq \Delta(L')$, there is a unique nonnegative integer i which is either odd and satisfies $2e-t \leq i \leq 2e-1$ or is equal to $2e$ such that $\Delta(L)^{-1}\Delta(L') \in U^2 + P^i, \Delta(L)^{-1}\Delta(L') \notin U^2 + P^{i+1}$. Then $\mu(L, L') = 2t - 2e + 2 + i$.*
4. *Assume $t(L) > 0$. then there is a unique (up to isomorphism) field L' such that $\mu(L, L') = 2t + 2$. Otherwise, $\mu(L, L')$ takes on the values $2\tau + 1$, $0 \leq \tau \leq t$.*

Proof. (1) is trivial, so we consider (2). Let M be any quadratic extension of k with $t(M) = 2e$. Then by Lemma 3.5, $R_{t+2}(L) \simeq R_{t+2}(L') \simeq R_{t+2}(M)$ but $R_{t+3}(L) \not\simeq R_{t+3}(L') \not\simeq R_{t+3}(M)$. Hence

$\mu(L,L') = \min(t(L),t(L')) + 2$. Next, let $t(L) = t(L')$ and $\Delta(L) \neq \Delta(L')$. By Lemma 3.2, $\Delta(L)^{-1}\Delta(L') \in U$ implies that there exists i such that $\Delta(L)^{-1}\Delta(L') \in (U^2+P^i)-(U^2+P^{i+1})$. Finally, since $\Delta(L') \in \Delta R_{t+2}(L)$, let $n \geq 3$. Then by Lemma 3.2, $\Delta(L') \in \Delta R_{t+n}(L) \iff \Delta(L)^{-1}\Delta(L') \in U^2 + P^{2e+n-t-2} \iff i \geq 2e+n-t-2 \iff 2t-2e+2+i \geq t+n$. That is, $\mu(L,L') = 2t-2e+2+i$. □

4. Optimal embedding

Let B be a quaternion algebra over a local field k and let K be a semi simple algebra of dimension 2 over k. Also, let α generate the maximal order $\mathcal{O}+\mathcal{O}\alpha$ of K where $\mathcal{O}$ is the ring of integers of k. By an embedding of K into B, we mean a k (or $\mathcal{O}$) injective homomorphism.

Definition 6. Let α be as above. For a nonnegative integer m,

$$E(\pi^m\alpha, R_n) = \{\phi|\ \phi \text{ is an embedding of } k(\alpha) \text{ into } B \text{ with } \phi(\pi^m\alpha) \in R_n\}$$
$$Eop(\pi^m\alpha, R_n) = \{\phi \in E(\pi^m\alpha, R_n) | \phi(k+k\alpha) \cap R_n = \phi(\mathcal{O}+\mathcal{O}\pi^m\alpha)\}$$

where $n \geq 0$.

Lemma 4.1. *Let the notations be as in Definition 6 above. Then for $m \geq 1$,*

$$Eop(\pi^m\alpha, R_\nu) = E(\pi^m\alpha, R_\nu) - E(\pi^{m-1}\alpha, R_\nu).$$

Proof. If $\varphi \in Eop(\pi^m\alpha, R_\nu)$, then $\varphi(k+k\alpha)\cap R_\nu = \mathcal{O}+\mathcal{O}\pi^m\varphi(\alpha)$. Suppose $\varphi(\pi^{m-1}\alpha) \in \mathcal{O}+\mathcal{O}\pi^m\varphi(\alpha)$. Let $\varphi(\pi^{m-1}\alpha) = x + y\pi^m\varphi(\alpha)$ for $x, y \in \mathcal{O}$. Then $\pi^{m-1}\varphi(\alpha) = x(y\pi-1)^{-1} \in \mathcal{O}$, which implies $\varphi(\alpha) \in k$. This contradicts that $k+k\alpha$ is a semisimple algebra of rank 2 over k. Hence $\varphi(\pi^{m-1}\alpha) \notin R_\nu$. That is $\varphi \notin E(\pi^{m-1}\alpha, R_\nu)$.

Conversely, if $\varphi \in E(\pi^m\alpha, R_\nu) - E(\pi^{m-1}\alpha, R_\nu)$, then $\varphi(k+k\alpha)\cap R_\nu = (k+k\varphi(\alpha))\cap R_\nu \subset \mathcal{O}+\mathcal{O}\varphi(\alpha)$. Let $x+y\varphi(\alpha) \in (k+k\varphi(\alpha))\cap R_\nu$ with $x, y \in \mathcal{O}$. Then $y\varphi(\alpha) \in R_\nu$ and $\pi^{m-1}\varphi(\alpha) \notin R_\nu$ imply $y \in P^m$. Thus $(k+k\varphi(\alpha))\cap R_\nu \subset \mathcal{O}+\mathcal{O}\pi^m\varphi(\alpha)$. It is clear that $\mathcal{O}+\mathcal{O}\pi^m\varphi(\alpha) \subset k+k\varphi(\alpha))\cap R_\nu$. That is $\varphi \in Eop(\pi^m\alpha, R_\nu)$. □

Corollary 4.2. *If α is embeddable in R_ν, then $Eop(\alpha, R_\nu) = E(\alpha, R_\nu)$.*

Proof. It is clear from Definition 6. □

Lemma 4.3. *Assume that $m \geq 1$.*

$$E(\pi^m\alpha, R_\nu) = E(\pi^{m-1}\alpha, R_{\nu-2}),$$

where $\nu \geq 2$.

Proof. First, we prove the unramified case. If L is unramified, then ν is even. Let $\varphi(\pi^m\alpha) = a + \xi b \in \mathcal{O}_L + \xi P_L^{\frac{\nu}{2}}$, then $N(\varphi(\pi^m\alpha)) = N(a) - N(b)$ implies that $a \in \pi_L\mathcal{O}_L$.

$$\varphi(\pi^{m-1}\alpha) = \pi^{-1}\varphi(\pi^m\alpha) = \pi^{-1}a + \xi\pi^{-1}b \in \mathcal{O}_L + \xi\pi^{-1}P_L^{\nu-2} = R_{\nu-2}.$$

Second, if L is ramified, then for $\varphi \in E(\pi^m\alpha, R_\nu)$ we have $\varphi(\pi^m\alpha) = a+(1+\xi)b \in R_\nu$ with $a \in \mathcal{O}_L$ and $b \in P_L^{\nu-t-1}$. $N(a+(1+\xi)b) = N(a)+Tr(a\bar{b}) \in P^{2m}$. Since $\nu \geq 2$, $b \in P_L^{\nu-t-1}$, $Tr(a\bar{b}) \in P$ by Proposition 4 in pp.142 [14]. Thus $N(a) \in P$ and $a \in P_L$. Let $a = \pi_L a'$ with $a' \in \mathcal{O}$. Since $a' = (\pi^{m-1}\varphi(\alpha)+(1+\xi)b\pi_L^{-2})\pi_L$ and $\pi^{m-1}\varphi(\alpha)+(1+\xi)b\pi_L^{-2} \in R_0$, $a' \in P_L$.

Hence,

$$\begin{aligned}\varphi(\pi^{m-1}\alpha) &= \pi^{-1}\varphi(\pi^m\alpha) = \pi^{-1}a + (1+\xi)\pi^{-1}b \\ &\in \mathcal{O}_L + (1+\xi)\pi^{-1}P_L^{\nu-t-1} \subset R_{\nu-2}.\end{aligned}$$

□

Corollary 4.4. *Assume that $m \geq 1$.*

1. *If $t(L) = -1$ and $\nu \geq 1$,*

$$E(\pi^m\alpha, R_{2\nu}) = \begin{cases} E(\pi^{m-\nu}\alpha, R_0) & \text{if } m \geq \nu, \\ E(\alpha, R_{2\nu-2m}) & \text{if } m < \nu. \end{cases}$$

2. *If $t(L) > 0$,*

$$E(\pi^m\alpha, R_\nu) = \begin{cases} E(\pi^{m-\frac{\nu}{2}}\alpha, R_0) & \text{if } \nu \text{ is even and } \nu \leq 2m, \\ E(\pi^{m-\frac{\nu-1}{2}}\alpha, R_1) & \text{if } \nu \text{ is odd and } \nu \leq 2m, \\ E(\alpha, R_{\nu-2m}) & \text{if } \nu > 2m. \end{cases}$$

Proof. This is immediate from Lemma 3.5. □

Lemma 4.5. *Let $K = k+k\alpha$ be a quadratic extension of k contained in B. Then*

$$E(\alpha, R_n) \simeq \{h \in B^\times | h\alpha h^{-1} \in R_n\}/K^\times$$

where the bijection is induced by the map,

$$\phi \in E(\alpha, R_n) \to g \in B^\times \text{ where } \phi(\alpha) = g\alpha g^{-1}. \tag{4.1}$$

Proof. For each $\phi \in E(\alpha, R_n)$, there exists $g \in B^\times$ such that $\phi(\alpha) = g\alpha g^{-1}$ by Neother Scholem theorem. Since the centralizer of α in B is K, we can define a map f from $E(\alpha, R_n)$ into $B^\times$ induced by $\phi \to g$ with $\phi(\alpha) = g\alpha g^{-1}$, namely, $f(\phi) = gK^\times$. Then it is easy to see that $E(\alpha, R_n) \approx \{h \in B^\times | h\alpha h^{-1} \in R_n\}/K^\times$. □

Corollary 4.6. *Let $K = k + k\alpha$ be a quadratic extension of k contained in B. If α is embeddable in R_n, then $E(\alpha, R_n)$ is bijective to the set of cosets, $\{h \in B^\times | hR_nh^{-1} = R_n\}/K^\times$ by the map induced by (4.1).*

Proof. By Lemma 4.5, it suffices to show that $\{h \in B^\times | h\alpha h^{-1} \in R_n\}/K^\times = \{h \in B^\times | hR_nh^{-1} \subset R_n\}/K^\times$. If $xR_nx^{-1} \subset R_n$, Then $x\alpha x^{-1} \in R_n$. Conversely, if $x\alpha x^{-1} \in R_n$, $\mathcal{O}_\alpha = \mathcal{O} + \mathcal{O}\alpha \subset x^{-1}R_nx$ and $\mathcal{O}_\alpha \subset R_n$. By Corollary 2.2, the n-th(or $\frac{n}{2}$ -th) largest nonmaximal order containing $\mathcal{O}_\alpha$ is unique. We have $R_n = x^{-1}R_nx$. Thus $\{h \in B^\times | h\alpha h^{-1} \in R_n\}/K^\times = \{h \in B^\times | hR_nh^{-1} \subset R_n\}/K^\times$. □

4.1. For the computational convenience, we introduce a new notation, $\mathcal{N}(R_\nu) = \{x \in R_0(L)^\times | x^{-1}R_\nu x = R_\nu\}$.

Lemma 4.7. *Let $K = k + k\alpha$ be a quadratic extension field of k contained in B. Then $\mathcal{O}_\alpha = \mathcal{O} + \mathcal{O}\alpha$ is an order in K. If α is embeddable in R_n, $\{h \in B^\times | hR_nh^{-1} = R_n\}/K^\times \approx \mathcal{N}(R_n)/\mathcal{O}_\alpha^\times$ for $\nu \geq 1$.*

Proof. It is easy to see $\{h \in B^\times | hR_nh^{-1} \subset R_n\} = \{h \in B^\times | hR_nh^{-1} = R_n\}$. Without loss of generality, we assume $\alpha \in R_n$. Let $M(R_n) = \{h \in B^\times | hR_nh^{-1} = R_n\}$. Since $M(R_0) = k^\times R_0^\times$, $M(R_{2\nu}) \subset M(R_0) = K^\times R_0^\times$ for $\nu \geq 1$. $M(R_{2\nu}) = \{g \in R_0^\times K^\times | gR_{2\nu}g^{-1} = R_{2\nu}\} = M(R_{2\nu})K^\times$. Thus each coset in $M(R_{2\nu})/K^\times$ corresponds to a coset in $M(R_{2\nu})K^\times/K^\times$. That is $M(R_{2\nu})/K^\times \simeq \mathcal{N}(R_{2\nu})K^\times/K^\times$. Next, by the map $f:\ x\mathcal{O}_\alpha^\times \to xK^\times$ $\mathcal{N}(R_{2\nu})K^\times/K^\times \approx \mathcal{N}(R_{2\nu})/\mathcal{O}_\alpha^\times$. □

Proposition 4.8. *Let L be a ramified quadratic extension field of k, i.e. $0 < t(L)$. Then we have*

$$\mathcal{N}(R_\nu) = \begin{cases} R_0^\times & \text{if } \nu = 0, \\ R_{[\frac{1}{2}\nu]}^\times & \text{if } 0 < \nu \leq 2t+2, \\ R_{\nu-t-1}^\times \cup \xi R_{\nu-t-1}^\times & \text{if } 2t+2 < \nu, \end{cases}$$

where $[x]$ is the largest integer not greater than x.

Proof. See Theorem 4.3 in [7]. □

Remark. Two different embeddings $\phi_1, \phi_2 \in E(\pi^m\alpha, R_n)$ are said to $R_n^\times$ equivalent if there exists $\gamma \in R_n^\times$ such that $\phi_2 = \gamma\phi_1\gamma^{-1}$ for all $x \in \mathcal{O} + \mathcal{O}\pi^m\alpha$.

4.2. In a quaternion algebra, it is known that all maximal orders are $B^\times$ conjugate each other [13]. It is known that the number of $R_0^\times$ equivalent classes of optimal embeddings of α into R_0 is 1 (See [5] or [6]).

Thus we are able to write it as $R_0^\times g\mathcal{O}_L^\times$ for some $g \in B^\times$. On the other hand, in cases of $t(L) = 2e$, $R_1^\times$ equivalent classes of optimal embeddings of α into R_1 is 2 (See 2.2 pp.65 in [5]). That is, $R_1^\times$ equivalent classes of optimal embeddings can be written as $R_1^\times g_1\mathcal{O}_L^\times \cup R_1^\times g_2\mathcal{O}_L^\times$ for some $g_1, g_2 \in B^\times$.

We are now able to compute the number of optimal embeddings. For the theoretical reason, we divided the cases into three parts according to L. Namely, $t(L) = -1$, $0 < t(L) < 2e$ and $t(L) = 2e$ cases. First we consider the $t(L) = -1$ case.

Theorem 4.9. *Assume that $t(L) = -1$. The number of $R_{2\nu}^\times$ equivalence classes of $Eop(\pi^m\alpha, R_{2\nu})$ is given as follows.*

	$\Delta(\alpha) \in \Delta_0^*$	$\Delta(\alpha) \notin \Delta_0^*$
$m > \nu$	$q^{\nu+1} - q^\nu$	$q^{\nu+1} - q^\nu$
$m = \nu > 0$	$q^m - 2q^{m-1}$	q^m
$m = \nu = 0$	1	1
$0 < m < \nu$	$2q^{m+1} - 2q^m$	0
$0 = m < \nu$	2	0

where $q = |\mathcal{O}/P|$.

Proof. See [8]. □

Lemma 4.10. *Assume $0 < t(L) < 2e$. Let α be an integral element of degree 2 over $\mathcal{O}$ which generates the maximal order of an algebra $k(\alpha)$ and let m be a nonnegative integer. Then*

$$E(\pi^m\alpha, R_1) = E(\pi^m\alpha, R_2).$$

Proof. Assume $m \geq 1$. Let $\pi^m\varphi(\alpha) = a + (1+\xi)b \in R_1$ for $\varphi \in E(\pi^m\alpha, R_1)$.

$$\mathrm{ord}_L\Delta(\pi^m\varphi(\alpha)) = \mathrm{ord}_L(a + b - \bar{a} - \bar{b} + 2\xi b)^2.$$

Since $\mathrm{ord}_L(a - \bar{a}) \geq t + 1$ and $\mathrm{ord}_L(2\xi b) \geq 2e - t \geq 1$, $\mathrm{ord}_L(b - \bar{b}) \geq 1$ which implies that $b \in P_L^{2-t-1}$. i.e. $\pi^m\varphi(\alpha) \in R_2$. Thus

$$E(\pi^m\alpha, R_1) = E(\pi^m\alpha, R_2).$$

If $m = 0$, then $\varphi(\alpha) \in R_1$ means that $\mu(k(\alpha), L) \geq 1$. By Theorem 3.6, we have $\mu(k(\alpha), L) \geq 2$ which implies $\varphi(\alpha) \in R_2$. □

Corollary 4.11. *Assume $0 < t(L) < 2e$ and $2 \leq m$. Let α be as in Lemma 4.10. Then*

$$Eop(\pi^m\alpha, R_1) = Eop(\pi^{m-1}\alpha, R_0).$$

Proof. If $m \geq 2$, this is immediate from Lemma 4.3 and Lemma 4.10. □

Corollary 4.12. *Assume $0 < t(L) < 2e$. Let α be as in Lemma 4.10. Then*

$$Eop(\pi\alpha, R_1) = \begin{cases} E(\alpha, R_0), & \mu(k(\alpha), L) = 0, \\ \emptyset, & \mu(k(\alpha), L) \geq 3. \end{cases}$$

Proof. By Lemma 4.1 and Corollary 4.4,

$$Eop(\pi\alpha, R_1) = E(\pi\alpha, R_1) - E(\alpha, R_1) = E(\alpha, R_0) - E(\alpha, R_1) = \emptyset.$$

□

Theorem 4.13. *Assume that $0 < t(L) < 2e$. Let α be an element of semisimle algebra K of dimension 2 over k such that $\mathcal{O} + \mathcal{O}\alpha$ is the maximal order of K. Then $R_\rho^\times$ equivalence classes of $Eop(\pi^m\alpha, R_\rho)$ is as follows, where $\rho = 2\nu + 2$ or $2\nu + 1$.*

1. *If $m > \nu + 1$,*

$$R_\rho^\times \setminus R_0^\times / (\mathcal{O} + \mathcal{O}\pi^{m-\nu-1} g\alpha g^{-1})^\times$$

where $\pi^{m-\nu-1} g\alpha g^{-1} \in R_0 - R_2$ for some $g \in B^\times$.

2. *If $m = \nu + 1$,*

$$R_{2\nu+1} \text{ case : } \begin{cases} R_{2\nu+1}^\times \setminus R_0^\times / \mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) = 0, \\ \emptyset & \text{if } \mu(k(\alpha), L) \geq 3, \end{cases}$$

$$R_{2\nu+2} \text{ case : } \begin{cases} R_{2\nu+2}^\times \setminus R_0^\times / \mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) = 0, \\ R_{2\nu+2}^\times \setminus (\mathcal{N}(R_0) - \mathcal{N}(R_2)) / \mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) \geq 3. \end{cases}$$

3. *If $m \leq \nu$,*
   a. *$\mu(k(\alpha), L) = 0$,*

$$\emptyset,$$

   b. *$0 < \mu(k(\alpha), L) < \infty$,*

$$\begin{cases} \emptyset & \text{if } \mu(k(\alpha), L) < \rho - 2m, \\ R_\rho^\times \setminus \mathcal{N}(R_{\rho-2m}) / \mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) = \rho - 2m, \\ R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2})) / \mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) > \rho - 2m. \end{cases}$$

   c. *$\mu = \infty$,*

$$R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2})) / \mathcal{O}_\alpha^\times.$$

Proof. If $m > \nu+1$, Lemma 4.3 and Corollary 4.4, $Eop(\pi^m\alpha, R_\rho) = Eop(\pi^{m-\nu}\alpha, R_{\rho-2\nu})$. By 4.2, $R_0^\times$ equivalence classes of $Eop(\pi^{m-\nu}\alpha, R_{\rho-2\nu})$ is $R_0^\times \setminus R_0^\times/(\mathcal{O}+\mathcal{O}\pi^{m-\nu}g\alpha g^{-1})^\times$.

If $m = \nu+1$ and $\rho = 2\nu+1$, by Corollary 4.4 and Corollary 4.11,

$$op(\pi^m\alpha, R_\rho) = E(\pi\alpha, R_1) \approx \begin{cases} R_0^\times/\mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) = 0, \\ \emptyset & \text{if } \mu(k(\alpha), L) \geq 3. \end{cases}$$

If $m = \nu+1$ and $\rho = 2\nu+2$,

$$\begin{aligned} Eop(\pi^m\alpha, R_\rho) =& E(\pi^m\alpha, R_{2\nu+2}) - E(\pi^{m-1}\alpha, R_{2\nu+2}) \\ =& E(\alpha, R_0) - E(\alpha, R_2) \\ \approx& \begin{cases} R_0^\times/\mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) = 0, \\ R_0^\times/\mathcal{O}_\alpha^\times - \mathcal{N}(R_2)/\mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) \geq 3. \end{cases} \end{aligned}$$

If $m \leq \nu$, $Eop(\pi^m\alpha, R_\rho) = E(\pi^m\alpha, R_\rho) - E(\pi^{m-1}\alpha, R_\rho) = E(\alpha, R_{\rho-2m}) - E(\alpha, R_{\rho-2m+2})$.

Thus if $0 < \mu(k(\alpha), L) < \infty$,

$$\begin{aligned} & E(\alpha, R_{\rho-2m}) - E(\alpha, R_{\rho-2m+2}) \\ \approx & \begin{cases} \emptyset & \text{if } 2\tau+1 < \rho-2m, \\ \mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times & \text{if } 2\tau+1 = \rho-2m, \\ (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2}))/\mathcal{O}_\alpha^\times & \text{if } 2\tau+1 > \rho-2m. \end{cases} \end{aligned}$$

If $\mu = \infty$, $R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2})/\mathcal{O}_\alpha^\times$. □

Lemma 4.14. *Let $\mathcal{N}_\sigma^* = \mathcal{N}(R_\sigma) - \mathcal{N}(R_{\sigma+1})$. If $t(L) > 0$ and $\nu \geq \sigma \geq 2$, then*

$$R_\nu^\times \setminus \mathcal{N}_\sigma^*/\mathcal{O}_\alpha^\times \approx (R_{\nu'}^\times \setminus R_{\nu'-1}^\times) \times (R_{\nu'-2}^\times \setminus R_{\nu'-3}^\times) \times \cdots \times (R_\sigma^\times \setminus \mathcal{N}_\sigma^*)$$

where $\nu' = \begin{cases} \nu & \text{if } \nu \equiv \sigma \mod 2, \\ \nu-1 & \text{if } \nu \not\equiv \sigma \mod 2. \end{cases}$

That is, the product of representatives of right hand side gives the representation of $R_\nu^\times \setminus \mathcal{N}_\sigma^*/\mathcal{O}_\alpha^\times$.

Proof. We prove this lemma by induction on ν. Without loss of generality, assume that $\alpha \in \mathcal{N}_\sigma^*$. If $\nu = \sigma$, for $x \in \mathcal{N}^*$, $R_\sigma^\times x\mathcal{O}_\alpha^\times = xR_\sigma^\times\mathcal{O}_\alpha^\times = R_\sigma^\times x$. Therefore, $R_\sigma^\times \setminus \mathcal{N}_\sigma^*/\mathcal{O}_\alpha^\times = R_\sigma^\times \setminus \mathcal{N}_\sigma^*$. Next, assume the result holds $\nu = \tau \geq \sigma$.

Let $h_1, h_2, \cdots, h_n$ be the representatives of $R_\tau^\times \setminus \mathcal{N}_\sigma^*/\mathcal{O}_\alpha$ in $\mathcal{N}_\sigma^*$ where $n = |R_\tau^\times \setminus \mathcal{N}_\sigma^*/\mathcal{O}_\alpha^\times|$. Fixing $h = h_i$ for some i, we need to determine

$R_{\tau+1}^{\times} \setminus R_{\tau}^{\times} h\mathcal{O}_{\alpha}^{\times}/\mathcal{O}_{\alpha}^{\times}$. By Corollary 4.6,

$$R_{\tau+1}^{\times} \setminus R_{\tau}^{\times} h\mathcal{O}_{\alpha}^{\times}/\mathcal{O}_{\alpha}^{\times} \approx R_{\tau+1}^{\times} \setminus R_{\tau}^{\times}/\{h\mathcal{O}_{\alpha}^{\times}h^{-1} \cap R_{\tau}^{\times}\}$$
$$\approx \begin{cases} R_{\tau+1}^{\times} \setminus R_{\tau}^{\times} & \text{if } h\mathcal{O}_{\alpha}^{\times}h^{-1} \cap R_{\tau}^{\times} \subset R_{\tau+1}^{\times}, \\ 1 & \text{if } h\mathcal{O}_{\alpha}^{\times}h^{-1} \cap R_{\tau}^{\times} \not\subset R_{\tau+1}^{\times}. \end{cases}$$

Now, $h\mathcal{O}_{\alpha}^{\times}h^{-1} \cap R_{\tau}^{\times} \not\subset R_{\tau+1}^{\times}$ if and only if there exists a nonnegative integer s such that $h\pi^s\alpha h^{-1} \in R_{\tau} - R_{\tau+1}$. Since $\alpha \in R_{\sigma}$, there exists a nonnegative integer s such that $\sigma = \tau - 2s$. Thus

$$R_{\tau+1}^{\times} \setminus R_{\tau}^{\times}/R_{\tau}^{\times} \cap h\mathcal{O}_{\alpha}^{\times}h^{-1} = \begin{cases} R_{\tau+1}^{\times} \setminus R_{\tau}^{\times} & \text{if } \tau = \sigma \mod 2 \\ 1 & \text{if } \tau \neq \sigma \mod 2. \end{cases}$$

Thus if $\tau \equiv \sigma \mod 2$, then $\pi^s h\alpha h^{-1} \in R_{\tau+1}$, otherwise, $\pi^s h\alpha h^{-1} \notin R_{\tau+1}$. i.e. $\tau \equiv \sigma \mod 2$. □

Lemma 4.15. *Assume that $y \in R_{\sigma} - R_{\sigma+2}$. If $t(L) > 0$ and $\nu \geq 2$, then*

$$R_{\nu}^{\times} \setminus R_{\sigma}^{\times}/(\mathcal{O}+\mathcal{O}y)^{\times} \approx (R_{\nu'}^{\times} \setminus R_{\nu'-1}^{\times}) \times (R_{\nu'-2}^{\times} \setminus R_{\nu'-3}^{\times}) \times \cdots \times (R_{\sigma+1}^{\times} \setminus R_{\sigma}^{\times})$$

where $\nu' = \begin{cases} \nu & \text{if } \nu \not\equiv \sigma \mod 2, \\ \nu - 1 & \text{if } \nu \equiv \sigma \mod 2. \end{cases}$

That is, the product of representatives of right hand side gives the representation of $R_{\nu}^{\times} \setminus R_{\sigma}^{\times}/(\mathcal{O} + \mathcal{O}y)^{\times}$.

Proof. We prove this lemma by induction on ν. Let $\mathcal{O}_y = \mathcal{O} + \mathcal{O}y$ for the notational convenience. If $\nu = \sigma$, we are done. So assume $\nu > \sigma$ and let $\sigma \leq k \leq \nu - 1$. Assume $g_1, g_2, \cdots, g_n$ are representatives of $R_k^{\times} \setminus R_{\sigma}/\mathcal{O}_y^{\times}$. Fixing $g = g_i$ for some i, we need to determine representatives of $R_{k+1}^{\times} \setminus R_k g\mathcal{O}_y^{\times}/\mathcal{O}_y^{\times} \approx R_{k+1}^{\times} \setminus R_k^{\times}/R_k^{\times} \cap g\mathcal{O}_y^{\times}g^{-1}$. By Corollary 4.6,

$$R_{k+1}^{\times} \setminus R_k^{\times}/R_k^{\times} \cap g\mathcal{O}_y^{\times}g^{-1} \approx \begin{cases} R_{k+1}^{\times} \setminus R_k^{\times} & \text{if } g\mathcal{O}_y^{\times}g^{-1} \cap R_k^{\times} \subset R_{k+1}^{\times}, \\ 1 & \text{if } g\mathcal{O}_y^{\times}g^{-1} \cap R_k^{\times} \not\subset R_{k+1}^{\times}. \end{cases}$$

Now, $g\mathcal{O}_y^{\times}g^{-1} \cap R_k^{\times} \not\subset R_{k+1}^{\times}$ if and only if there exists a nonnegative integer s such that $g\pi^s yg^{-1} \in R_k - R_{k+1}$. i.e. There exists a nonnegative integer s such that $\sigma = k - 2s$. Thus

$$R_{k+1}^{\times} \setminus R_k^{\times}/R_k^{\times} \cap g\mathcal{O}_y^{\times}g^{-1} = \begin{cases} R_{k+1}^{\times} \setminus R_k^{\times} & \text{if } k = \sigma \mod 2, \\ 1 & \text{if } k \neq \sigma \mod 2. \end{cases}$$

□

Lemma 4.16. *Assume $0 < t = t(L) < 2e$ and set $R_\sigma = R_\sigma(L)$. Then*

a. $n_\sigma = |R_\sigma^\times \setminus \mathcal{N}(R_\sigma)| = \begin{cases} 1 & \text{if } \sigma = 0, \\ q+1 & \text{if } \sigma = 1, \\ q^{[\frac{\sigma+1}{2}]} & \text{if } 2 \le \sigma \le 2t+2, \\ 2q^{t+1} & \text{if } \sigma \ge 2t+3. \end{cases}$

b. $\tilde{n}_\sigma = |R_\sigma^\times \setminus \mathcal{N}(R_{\sigma+1})| = \begin{cases} q^{[\frac{\sigma}{2}]} & \text{if } 0 \le \sigma \le 2t+1, \\ 2q^t & \text{if } 2t+2 \le \sigma. \end{cases}$

c. $n_\sigma^* = n_\sigma - \tilde{n}_\sigma = \begin{cases} 0 & \text{if } 0 \le \sigma < 2t+2 \text{ and } \sigma \text{ is even}, \\ q & \text{if } 1 \le \sigma < 2t+2 \text{ and } \sigma \text{ is odd}, \\ q^{t+1} - 2q^t & \text{if } \sigma = 2t+2, \\ 2q^{t+1} - 2q^t & \text{if } \sigma \ge 2t+3. \end{cases}$

Proof. This is immediate from Proposition 4.8 and Theorem 4.13. □

Theorem 4.17. *Assume $0 < t(L) < 2e$ and set $R_\sigma = R_\sigma(L)$. then the number of $R_\nu^\times$ equivalence classes of $Eop(\alpha, R_\nu)$ is as follows:*

1. $\mu = 0$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m \ge \nu+1$	$q^\nu + q^{\nu-1}$	$q^\nu + q^{\nu-1}$
$m = \nu$	$q^\nu + q^{\nu-1}$	0
$m < \nu$	0	0

2. $\mu = 2\tau + 1$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m > \nu+1$	$q^\nu + q^{\nu-1}$	$q^{\nu+1} + q^\nu$
$m = \nu+1$	q^ν	$q^{\nu+1}$
$\tau > \nu - m, t = \nu - m > 0$	q^{m+1}	$q^{m+1} + q^{m+t+1} - 2q^{m+t}$
$\tau > \nu - m, t > \nu - m > 0$	q^{m+1}	q^{m+1}
$\tau > \nu - m = t+1$	$3q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\tau > \nu - m > t+1$	$4q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\tau = \nu - m > t+1$	$2q^{m+t+1}$	$2q^{m+t+1}$
$\tau = \nu - m = t+1$	$q^{m+[\frac{\tau}{2}]}$	$2q^{m+t+1}$
$\tau = \nu - m < t+1$	$q^{m+[\frac{\tau}{2}]}$	$q^{m+[\frac{\tau}{2}]}$
$\tau = \nu - m = 1$	$q^{m+1} + q^m$	$q^{\nu+1}$
$\tau < \nu - m$	0	0
$\nu = m$	q^{m+1}	q^m

3. $\mu = 2t + 2$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m > \nu+1$	$q^\nu + q^{\nu-1}$	$q^{\nu+1} + q^\nu$
$m = \nu+1$	q^ν	$q^{\nu+1}$
$m = \nu$	q^ν	q^{m+1}
$t \geq \nu - m > 0$	q^{m+1}	q^{m+1}
$\nu - m = t+1$	$q^{m+[\frac{t}{2}]+1}$	0
$\nu - m > t+1$	0	0

4. $\mu = \infty$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m > \nu+1$	$q^\nu + q^{\nu-1}$	$q^{\nu+1} + q^\nu$
$m = \nu+1$	$q^{\nu+1}$	$q^{\nu+2}$
$\nu - m > t+1$	$4q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\nu - m = t+1$	$3q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\nu - m = t$	q^{m+1}	$q^{m+1} + q^{m+t+1} - 2q^{m+t}$
$\nu - m < t$	q^{m+1}	q^{m+1}

Proof. It suffices to compute the number of double cosets given in Theorem 4.13.

1. $m > \nu + 1$.
   The number of $R_{2\nu}^\times$ equivalence classes of $Eop(\pi^m\alpha, R_{2\nu})$ is $|R_{2\nu}^\times \setminus R_0^\times/(\mathcal{O} + \mathcal{O}\pi^{m-\nu-1}g\alpha g)^\times|$. By Lemma 4.17 and Theorem 2.4, it is $q^{\nu-1}(q+1)$. Similarly, the number of $R_{2\nu+1}^\times$ equivalence classes of $Eop(\pi^m\alpha, R_{2\nu+1})$ is $q^{\nu+1} + q^\nu$.
2. $m = \nu + 1$.
   If $\alpha \in R_0 - R_2$, by Lemma 4.10, $\alpha \in R_0 - R_1$. By Corollary 2.8 , Theorem 4.15 and Lemma 4.17, $|R_{2\nu+1} \setminus R_0/\mathcal{O}_\alpha^\times| = q^{\nu-1}(q+1)$ for $\mu = 0$. On the other hand, $|R_{2\nu} \setminus \mathcal{N}(R_0)/\mathcal{O}_\alpha^\times - R_{2\nu} \setminus \mathcal{N}(R_2)/\mathcal{O}_\alpha^\times| = q^\nu \cdot n_0^* + q^{\nu-1} \cdot n_1^* = q^\nu$ by Lemma 4.18 for $\mu \geq 3$.
3. $m \leq \nu$.
   We divide it into four cases.
   1. $\mu = 0$.
      If $m = \nu$, by Corollary 2.8 , Theorem 4.15 and Lemma 4.17 $|R_{2\nu} \setminus R_0/\mathcal{O}_\alpha^\times| = q^{\nu-1}(q+1)$. If $m < \nu$, it is 0.
   2. $\mu(k(\alpha), L) = 2\tau + 1$.
      We need to compute

$$|R_\rho \setminus \mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times - R_\rho \setminus \mathcal{N}(R_{\rho-2m+2})/\mathcal{O}_\alpha^\times| \tag{4.2}$$

$$=|R_\rho \setminus (\mathcal{N}^*(R_{\rho-2m}) \cup \mathcal{N}^*(R_{\rho-2m+1}))/\mathcal{O}_\alpha^\times| \tag{4.3}$$

a. $\tau > \nu - m$.

By (4.3) and Lemma 4.18,

$$q^{m-1}\cdot n^*_{\rho-2m}+q^m\cdot n^*_{\rho-2m+1}$$

$$=q^m\cdot\begin{cases}0+q & \text{if } \rho-2m=0,\\ q+0 & \text{if } \rho-2m=1,\\ 0+q & \text{if } 2\le\rho-2m\le 2t,\\ q+(q^{t+1}-2q^t) & \text{if } \rho-2m=2t+1,\\ (q^{t+1}-2q^t)+(2q^{t+1}-2q^t) & \text{if } \rho-2m=2t+2,\\ (2q^{t+1}-2q^t)+(2q^{t+1}-2q^t) & \text{if } \rho-2m>2t+2.\end{cases}$$

b. $\tau=\nu-m$.

By (4.3),

$$|R^\times_\rho\setminus\mathcal{N}(R_{\rho-2m})/\mathcal{O}^\times_\alpha|$$

$$=\begin{cases}|R^\times_\rho\setminus R^\times_{[\frac{1}{2}\rho]-m}/\mathcal{O}^\times_\alpha| & \text{if } \rho-2m\le 2t+2\\ R^\times_\rho\setminus R^\times_{\rho-m-t-1}\cup\xi R^\times_{\rho-m-t-1}/\mathcal{O}^\times_\alpha| & \text{if } \rho-2m>2t+2\end{cases}$$

$$=q^m\cdot\begin{cases}q+1 & \text{if } \rho-2m=0,\\ q+1 & \text{if } \rho-2m=1,\\ q^{[\frac{\rho+1}{2}]-m} & \text{if } 2\le\rho-2m\le 2t+2,\\ 2q^{t+1} & \text{if } \rho-2m\ge 2t+3.\end{cases}$$

c. $\tau<\nu-m$.

(4.3) is 0.

3. $\mu(L,k(\alpha))=2t+2$.

$$\begin{cases}\emptyset & \text{if } 2t+2<\rho-2m,\\ R^\times_\rho\setminus\mathcal{N}(R_{\rho-2m})/\mathcal{O}^\times_\alpha, & \text{if } 2t+2=\rho-2m,\\ R^\times_\rho\setminus(\mathcal{N}(R_{\rho-2m})-\mathcal{N}(R_{\rho-2m+2}))/\mathcal{O}^\times_\alpha & \text{if } 2t+2>\rho-2m,\end{cases}$$

$$=\begin{cases}\emptyset, & \text{if } 2t+2<\rho-2m,\\ R^\times_\rho\setminus R^\times_{\frac{\rho-2m}{2}}/\mathcal{O}^\times_\alpha, & \text{if } 2t+2=\rho-2m,\\ q^m\cdot(n^*_{\rho-2m}+n^*_{\rho-2m+1}), & \text{if } 2t+2>\rho-2m.\end{cases}$$

$$=\begin{cases}\emptyset, & \text{if } 2t+2<\rho-2m,\\ q^{m+[\frac{\rho+1}{2}]}, & \text{if } 2t+2=\rho-2m,\\ q^{m+1}, & \text{if } 2t+2>\rho-2m.\end{cases}$$

4. $\mu(k(\alpha),L)=\infty$.

If $m\ge\nu$, the proof is exactly same as the above case.

$$\text{If } m < \nu, \begin{cases} q^m \cdot (4q^{t+1} - 4q^t), & \text{if } \rho - 2m \geq 2t+3, \\ q^m \cdot (3q^{t+1} - 4q^t), & \text{if } \rho - 2m = 2t+2, \\ q^m \cdot (q + q^{t+1} - 2q^t), & \text{if } \rho - 2m = 2t+1, \\ q^m \cdot q & \text{if } \rho - 2m \leq 2t. \end{cases}$$

□

Theorem 4.18. *Assume that $t(L) = 2e$. Let α be an element of semisimle algebra K of dimension 2 over k such that $\mathcal{O} + \mathcal{O}\alpha$ is the maximal order of K. Assume that $\alpha \in R_0(L)$.*

1. *If $m > \nu$, let $G_i = g_i \alpha g_i^{-1}$,*

$$Eop(\pi^m\alpha, R_{2\nu}) \approx R_0^\times/(\mathcal{O} + \mathcal{O}\pi^{m-\nu} g\alpha g^{-1})^\times$$

where $\pi^{m-\nu} g\alpha g^{-1} \in R_0 - R_2$ for some $g \in B^\times$,

$$Eop(\pi^m\alpha, R_{2\nu+1}) \approx R_1^\times/(\mathcal{O} + \mathcal{O}\pi^{m-\nu}G_1)^\times \cup R_1^\times/(\mathcal{O} + \mathcal{O}\pi^{m-\nu}G_2)^\times,$$

where $g_i \pi^m \alpha g_i^{-1} \in R_1 - R_3$ for some $g_i \in B^\times$ for each $i = 1, 2$.

2. *If $m = \nu$,*

$$Eop(\pi^m\alpha, R_{2\nu}) \approx \begin{cases} R_0^\times/\mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) = 0, \\ \mathcal{N}(R_0)/\mathcal{O}_\alpha^\times - \mathcal{N}(R_2)/\mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) \geq 3. \end{cases}$$

and

$$Eop(\pi^m\alpha, R_{2\nu+1}) \approx \begin{cases} \emptyset & \text{if } \mu(k(\alpha), L) = 0, \\ \mathcal{N}(R_1)/\mathcal{O}_\alpha^\times - \mathcal{N}(R_3)/\mathcal{O}_\alpha^\times & \text{if } \mu(k(\alpha), L) \geq 3. \end{cases}$$

3. *If $m < \nu$, let $\rho = 2\nu$ or $2\nu + 1$.*
   a. *$0 < \mu(k(\alpha), L) < \infty$. Then*

$$Eop(\pi^m\alpha, R_\rho) \approx \begin{cases} \emptyset, & \text{if } 2\tau + 1 < \rho - 2m, \\ \mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times, & \text{if } 2\tau + 1 = \rho - 2m, \\ \mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times - \mathcal{N}(R_{\rho-2m+2})/\mathcal{O}_\alpha^\times, & \text{if } 2\tau + 1 > \rho - 2m. \end{cases}$$

   b. *$\mu(k(\alpha), L) = \infty$.*

$$\mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times - \mathcal{N}(R_{\rho-2m+2})/\mathcal{O}_\alpha^\times, \quad \text{if } 2t + 2 > \rho - 2m.$$

Proof. 1. $m > \nu$. By Corollary 4.4,

$$Eop(\pi^m\alpha, R_{2\nu}) = Eop(\pi^{m-\nu}\alpha, R_0) \approx R_0^\times/(\mathcal{O} + \mathcal{O}\pi^{m-\nu} g\alpha g^{-1})^\times$$

where $\pi^{m-\nu} g\alpha g^{-1} \in R_0 - R_2$ for some $g \in B^\times$. By Lemma 4.3,

$$Eop(\pi^m\alpha, R_{2\nu+1}) = Eop(\pi^{m-\nu}\alpha, R_1).$$

Since $R_1 = R_0 \cap \bar{R}_0$, $R_1 \simeq \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ P & \mathcal{O} \end{pmatrix}$, the number of $R_1^\times$ equivalence classes of $Eop(\pi^m\alpha, R_1)$ is 2 by 2.2 in [5]. Thus there exist g_1 and g_2 in $B^\times$ such that $Eop(\pi^m\alpha, R_1) \approx R_1^\times g_1 K^\times/K^\times \cup R_1^\times g_2 K^\times/K^\times$, where $g_i \pi^m \alpha g_i^{-1} \in R_1 - R_3$ for each $i = 1, 2$. Since

$$\begin{aligned} R_1^\times gK^\times/K^\times &\approx R_1^\times gK^\times g^{-1}/gK^\times g^{-1} \approx R_1^\times/R_1^\times \cap gK^\times g^{-1} \\ &\approx R_1^\times/(\mathcal{O} + \mathcal{O}\pi^m g\alpha g^{-1})^\times, \end{aligned}$$

we have

$$Eop(\pi^m\alpha, R_{2\nu+1}) \approx R_1^\times/(\mathcal{O}+\mathcal{O}\pi^{m-\nu}g_1\alpha g_1^{-1})^\times \cup R_1^\times/(\mathcal{O}+\mathcal{O}\pi^{m-\nu}g_2\alpha g_2^{-1})^\times.$$

2. $m = \nu$. $Eop(\pi^m\alpha, R_{2\nu}) = E(\alpha, R_0) - E(\alpha, R_2) \approx R_0^\times/\mathcal{O}_\alpha^\times - \mathcal{N}(R_2)^\times/\mathcal{O}_\alpha^\times$. On the other hand,

$$\begin{aligned} Eop(\pi^m\alpha, R_{2\nu+1}) &= E(\alpha, R_1) - E(\alpha, R_3) \\ &\approx \begin{cases} \emptyset, & \text{if } \mu = 0, \\ E(\alpha, R_1) - E(\alpha, R_1) & \\ \mathcal{N}(R_1)/\mathcal{O}_\alpha^\times - \mathcal{N}(R_3)/\mathcal{O}_\alpha^\times, & \text{if } \mu \geq 2, \end{cases} \end{aligned}$$

where $g_i\alpha g_i^{-1} \in R_1 - R_3$ for each $i = 1, 2$.

3. Assume that $m < \nu$.

$$\begin{aligned} &Eop(\pi^m\alpha, R_\rho) = E(\pi^m\alpha, R_\rho) - E(\pi^{m-1}\alpha, R_\rho) \\ &= E(\alpha, R_{\rho-2m}) - E(\alpha, R_{\rho-2m+2}) \\ &\approx \begin{cases} \mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times, & \text{if } \mu(k(\alpha), L) \leq \rho - 2m + 1, \\ (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2}))/\mathcal{O}_\alpha^\times, & \text{if } \mu(k(\alpha), L) > \rho - 2m + 1. \end{cases} \end{aligned}$$

We now consider $m \leq \nu$.
If $\mu(k(\alpha), L) = 0$, it is $\emptyset$.
If $0 < \mu(k(\alpha), L) < \infty$, then

$$\begin{aligned} &E(\alpha, R_{\rho-2m}) - E(\alpha, R_{\rho-2m+2}) \\ &\approx \begin{cases} \emptyset, & \text{if } 2\tau + 1 < \rho - 2m, \\ \mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times, & \text{if } 2\tau + 1 = \rho - 2m, \\ (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2}))/\mathcal{O}_\alpha^\times, & \text{if } 2\tau + 1 > \rho - 2m. \end{cases} \end{aligned}$$

If $\mu = \infty$, $R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2})/\mathcal{O}_\alpha^\times$. □

Theorem 4.19. *Assume $t(L) = 2e$. Let $\mu = \mu(L, k(\alpha))$. If $m \geq 1$ and $\nu \geq 2m+1$, then the number of $R_\nu^\times$ equivalence classes of $Eop(\alpha, R_\nu)$ is as follows:*

1. $\mu = 0$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m \geq \nu+1$	$q^\nu + q^{\nu-1}$	$2q^{\nu+1}$
$m = \nu$	$q^\nu + q^{\nu-1}$	0
$m < \nu$	0	0

2. $\mu = 2\tau + 1$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m \geq \nu+1$	$q^{\nu+1} + q^\nu$	$2q^{\nu+1}$
$m = \nu$	q^ν	q^ν
$\tau > \nu - m, t = \nu - m > 0$	q^{m+1}	$q^{m+1} + q^{m+t+1} - 2q^{m+t}$
$\tau > \nu - m, t > \nu - m > 0$	q^{m+1}	q^{m+1}
$\tau > \nu - m = t+1$	$3q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\tau > \nu - m > t+1$	$4q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\tau = \nu - m > t+1$	$2q^{m+t+1}$	$4q^{m+t+1} - 4q^{m+t}$
$\tau = \nu - m = t+1$	q^{m+t}	$2q^{m+t+1}$
$\tau = \nu - m < t+1$	q^ν	$q^{\nu+1}$
$\tau < \nu - m$	0	0

3. $\mu = 2t + 2$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m \geq \nu+1$	$q^{\nu+1} + q^\nu$	$2q^{\nu+1}$
$m = \nu$	q^ν	q^ν
$t+1 = \nu - m > 0$	$q^{m+\nu}$	$q^{m+\nu+1}$
$t = \nu - m > 0$	q^{m+1}	q^{m+1}
$\nu - m > t+1$	0	0

4. $\mu = \infty$.

	$Eop(\pi^m\alpha, R_{2\nu})$	$Eop(\pi^m\alpha, R_{2\nu+1})$
$m > \nu$	$q^{\nu+1} + q^\nu$	$2q^{\nu+1}$
$m = \nu$	q^ν	q^ν
$t \geq \nu - m > 0$	q^{m+1}	q^{m+1}
$\nu - m = t+1$	$3q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$
$\nu - m > t+1$	$4q^{m+t+1} - 4q^{m+t}$	$4q^{m+t+1} - 4q^{m+t}$

Proof. By Theorem 4.20 and Lemma 4.18, we are able to compute the following procedures.

1. $m > \nu$.

$|R_{2\nu}^\times \setminus R_0^\times/(\mathcal{O} + \mathcal{O}\pi^{m-\nu}g\alpha g^{-1})^\times| = q^{\nu-1}(q+1)$ and $|R_{2\nu+1}^\times \setminus R_1^\times/(\mathcal{O}+\mathcal{O}\pi^{m-\nu}g_1\alpha g_1^{-1})^\times| + |R_{2\nu+1}^\times \setminus R_1^\times/(\mathcal{O}+\mathcal{O}\pi^{m-\nu}g_2\alpha g_2^{-1})^\times| = 2q^{\nu-1}\cdot q$.

2. $m=\nu$.
$|R_{2\nu}^{\times}\setminus R_0^{\times}/\mathcal{O}_\alpha^{\times}| = q^{\nu-1}(q+1)$ for $\mu=0$ and $|R_{2\nu}^{\times}\setminus\mathcal{N}(R_0)/\mathcal{O}_\alpha^{\times} - R_{2\nu}^{\times}\setminus\mathcal{N}(R_2)/\mathcal{O}_\alpha^{\times}| = q^{\nu-1}(n_0^*+n_1^*) = q^{\nu}$ for $\mu(k(\alpha),L)\geq 3$. Moreover, $|R_{2\nu+1}^{\times}\setminus\mathcal{N}(R_1)/\mathcal{O}_\alpha^{\times} - R_{2\nu+1}^{\times}\setminus\mathcal{N}(R_3)/\mathcal{O}_\alpha^{\times}| = q^{\nu-1}(n_1^*+n_2^*) = q^{\nu}$ for $\mu(k(\alpha),L)\geq 3$.
3. $m<\nu$.
We divide it into four different cases.
(a) $\mu=0$. It is 0 by Definition 4.
(b) $\mu(k(\alpha),L)=2\tau+1$.
We need to compute

$$\begin{aligned}&|R_\rho\setminus\mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^{\times} - R_\rho\setminus\mathcal{N}(R_{\rho-2m+2})/\mathcal{O}_\alpha^{\times}|\\ =&|R_\rho\setminus(\mathcal{N}^*(R_{\rho-2m})\cup\mathcal{N}^*(R_{\rho-2m+1}))/\mathcal{O}_\alpha^{\times}|\end{aligned}$$

a. $\tau>\nu-m$, By (4.3) and Lemma 4.18,

$$\begin{aligned}&q^m\cdot(n_{\rho-2m}^*+n_{\rho-2m+1}^*)\\ =&q^m\cdot\begin{cases}0+q & \text{if } \rho-2m=0,\\ q+0 & \text{if } \rho-2m=1,\\ 0+q & \text{if } 2\leq\rho-2m\leq 2t,\\ q+q^{t+1}-2q^t & \text{if } \rho-2m=2t+1,\\ 3q^{t+1}-4q^t & \text{if } \rho-2m=2t+2,\\ 4q^{t+1}-4q^t & \text{if } \rho-2m>2t+2.\end{cases}\end{aligned}$$

b. $\tau=\nu-m$ By (4.3),

$$\begin{aligned}&|R_\rho^{\times}\setminus\mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^{\times}|\\ =&\begin{cases}|R_\rho^{\times}\setminus R_{[\frac{1}{2}\rho]-m}^{\times}/\mathcal{O}_\alpha^{\times}| & \text{if } \rho-2m\leq 2t+2\\ R_\rho^{\times}\setminus R_{\rho-m-t-1}^{\times}\cup\xi R_{\rho-m-t-1}^{\times}/\mathcal{O}_\alpha^{\times}| & \text{if } \rho-2m>2t+2\end{cases}\\ &q^m\cdot\begin{cases}q+1 & \text{if } \rho-2m=0,\\ q+1 & \text{if } \rho-2m=1,\\ q^{[\frac{\rho+1}{2}]-m} & \text{if } 2\leq\rho-2m\leq 2t+2,\\ 2q^{t+1} & \text{if } \rho-2m\geq 2t+3.\end{cases}\end{aligned}$$

c. $\tau<\nu-m$, $(4-3)$ is 0.

(c) $\mu(L, k(\alpha)) = 2t + 2$. By Lemma 4.18,

$$\begin{cases} \emptyset & \text{if } 2t+2 < \rho - 2m, \\ R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m})/\mathcal{O}_\alpha^\times, & \text{if } 2t+2 = \rho - 2m, \\ R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2}))/\mathcal{O}_\alpha^\times & \text{if } 2t+2 > \rho - 2m, \end{cases}$$

$$= \begin{cases} \emptyset, & \text{if } 2t+2 < \rho - 2m, \\ R_\rho^\times \setminus R^\times_{\frac{\rho-2m}{2}}/\mathcal{O}_\alpha^\times, & \text{if } 2t+2 = \rho - 2m, \\ q^m \cdot (n^*_{\rho-2m} + n^*_{\rho-2m+1}), & \text{if } 2t+2 > \rho - 2m. \end{cases}$$

$$= \begin{cases} \emptyset, & \text{if } 2t+2 < \rho - 2m, \\ q^{m+[\frac{\rho+1}{2}]}, & \text{if } 2t+2 = \rho - 2m, \\ q^{m+1}, & \text{if } 2t+2 > \rho - 2m. \end{cases}$$

(d) $\mu(k(\alpha), L) = \infty$.

$$R_\rho^\times \setminus (\mathcal{N}(R_{\rho-2m}) - \mathcal{N}(R_{\rho-2m+2}))/\mathcal{O}_\alpha^\times = q^m \cdot (n^*_{\rho-2m} + n^*_{\rho-2m+1})$$

$$= \begin{cases} q^m \cdot (4q^{t+1} - 4q^t), & \text{if } \rho - 2m \geq 2t+3 \\ q^m \cdot (3q^{t+1} - 4q^t), & \text{if } \rho - 2m = 2t+2 \\ q^m \cdot (q + q^{t+1} - 2q^t), & \text{if } \rho - 2m = 2t+1 \\ q^m \cdot q, & \text{if } \rho - 2m \leq 2t. \end{cases}$$

□

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Division of Mathematics and Computer sicience
Konkuk University,
Choongju, Choongbuk, 380-151, Korea
E-mail: sjun@kku.ac.kr

Department of Mathematics education
Wonkwang University,
Iksan, Jeonbuk, 540-749, Korea
E-mail: iki@wonkwang.ac.kr