Honam Mathematical J. **31** (2009), No. 4, pp. 633–637

UNIQUENESS OF TOEPLITZ OPERATOR IN THE COMPLEX PLANE

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Abstract. We prove using the Szegő kernel and the Garabedian kernel that a Toeplitz operator on the boundary of C^∞ smoothly bounded domain associated to a smooth symbol vanishes only when the symbol vanishes identically. This gives a generalization of previous results on the unit disk to more general domains in the plane.

1. Introduction

In this paper, we study on uniqueness of Toeplitz operators associated to a C^∞ smooth symbol and a C^∞ smoothly bounded domain in the plane. It is deeply related to commuting Toeplitz operators which are dealt with zero-product problems mostly in the polydisks in $\mathbb{C}^n$. (See [9] and [10]). Using a formula relating classical kernel functions in potential theory, we prove that vanishing property of Toeplitz operator implies vanishing of associated symbols. This is a kind of generalization of previous results to more general domains in the plane.

2. Preliminaries and Notations

In this section, we review briefly on some preliminaries about the classical kernel functions and their relations.

Received November 13, 2009. Accepted December 14, 2009.
2000 Mathematics Subject Classification: [2010]47B35, 30C40.
Key words and phrases: Szegő kernel, Toeplitz operator, Garabedian kernel.
This study was financially supported by Chonnam National University, 2007.

Let Ω be a bounded finitely connected domain in the plane with C^∞ smooth boundary. Let's suppose that Ω is n-connected. Let $\gamma_j, j = 1, \cdots, n$, denote the n non-intersecting C^∞ simple closed curves defining the boundary $b\Omega$ of Ω. We assume that the boundary curve γ_j is parameterized in the standard sense by $z_j(t), 0 \leq t \leq 1$. Let $T(z)$ be the C^∞ function defined on $b\Omega$ by the complex unit tangent vector in the direction of the standard orientation. We shall let $L^2(b\Omega)$ denote the space of complex valued functions on $b\Omega$ that are square integrable with respect to arc length measure ds and let $L^2(\Omega)$ denote the space of complex valued functions on Ω that are square integrable with respect to Lebesgue area measure dA. The Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω shall be written $H^2(b\Omega)$.

The orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$ with respect to the inner product

$$< u, v >_{b\Omega} = \int_{b\Omega} u \; \bar{v} \; ds$$

is called the Szegő projection denoted by P. The Szegő kernel denoted by $S(z, w)$ is the kernel for P. It is well known that $S(z, w)$ extends to the boundary to be in $C^\infty \left((\bar{\Omega} \times \bar{\Omega}) \setminus \{(z, z) : z \in b\Omega\} \right)$.

And it is a holomorphic function of z and an antiholomorphic function of w on $\Omega \times \Omega$. We note that $S(z, z)$ is real and positive for $z \in \Omega$ and $\overline{S(z, w)} = S(w, z)$. The Garabedian kernel $L(z, w)$ is the kernel for the orthogonal projection $P^\perp$ of $L^2(b\Omega)$ onto $H^2(b\Omega)^\perp$ defined by

$$L(z, w) = i\, \overline{S(z, w)\, T(z)}, \qquad \text{for } (z, w) \in b\Omega \times \Omega, \tag{2.1}$$

which is heavily used in the proof of the main theorem in the next section.

The Garabedian kernel satisfies the identity

$$L(z, w) = \frac{1}{2\pi} \frac{1}{z - w} - iSP(\overline{C_w T})(z), \tag{2.2}$$

where

$$C_w(\zeta) = \overline{\frac{1}{2\pi i}\frac{T(\zeta)}{\zeta - w}}, \qquad \zeta \in b\Omega, w \in \Omega$$

is the kernel for the Cauchy transform defining the Cauchy integral. For fixed $w \in \Omega$, $L(z,w)$ is a holomorphic function of z on $\Omega \setminus \{w\}$ with a simple pole at w with residue $1/2\pi$. Furthermore, $L(z,w)$ extends to be in $C^\infty\left((\bar{\Omega} \times \bar{\Omega}) \setminus \{(z,z) : z \in \bar{\Omega}\}\right)$. We also note that $L(w,z) = -L(z,w)$ and $L(z,w)$ is zero-free for all $(z,w) \in \bar{\Omega} \times \Omega$ with $z \neq w$. All of these properties can be found in Bell's book[2]. See also [11].

3. Main Results

Let Ω be a bounded finitely connected domain in the plane with C^∞ smooth boundary and let $\alpha \in C^\infty(b\Omega)$. We will without of generality think of the function α as a function $\alpha \in C^\infty(\Omega)$ by extending to $\overline{\Omega}$.

M. Schiffer[8](See also [1]) proved the following orthogonal decomposition for α.

Lemma 3.1. *Suppose that Ω is a bounded domain in the plane with C^∞ smooth boundary. Let T be a unit tangent vector function on the boundary of Ω and let P be the Szegő projection associated to Ω.*

If u is a function in $L^2(b\Omega)$ then it has an orthogonal decomposition

$$u = P(u) + \overline{T}\,\overline{P(\overline{u}\,\overline{T})}.$$

We also need a boundary regularity of the Szegő projection as follows. (See [3].)

Lemma 3.2. *Suppose that Ω is a bounded domain in the plane with C^∞ smooth boundary. and suppose P is the Szegő projection associated to Ω. Then P maps $C^\infty(b\Omega)$ into $C^\infty(b\Omega)$.*

Now let u be equal to α in Lemma 3.1. Then we have

$$\alpha = P(\alpha) + \overline{T}\,\overline{P(\overline{\alpha}\,\overline{T})}, \tag{3.1}$$

where $P(\alpha) \in H^2(b\Omega)$ and $\overline{P(\overline{\alpha}\,\overline{T})} \in H^2(b\Omega)^{\perp}$. In particular, by boundary regularity of the Szegő projection(Lemma 3.2), the functions $P(\alpha)$ and $\overline{P(\overline{\alpha}\,\overline{T})}$ are in $C^\infty(\Omega)$. (See [2].)

The Toeplitz operator T_α associated to the symbol α is defined by

$$T_\alpha(f) = P(\alpha f)$$

as a bounded linear operator on the Hardy space $H^2(b\Omega)$. As mentioned in the Introduction, we want to find a condition for vanishing of the Toeplitz operator. Suppose that α is fixed and suppose that T_α vanishes identically on $H^2(b\Omega)$.

Let $a \in \Omega$ be fixed. Then

$$P(\alpha S(\cdot, a)) = 0.$$

It follows from (3.1) that

$$\alpha(z)S(z,a) = \overline{T(z)}\ \overline{P\left(\overline{\alpha}\,\overline{S(\cdot,a)}\,\overline{T}\right)}(z), \quad z \in b\Omega.$$

Note that the T is a unit vector and hence we have from (2.1) that

$$\mathrm{i}\,\alpha(z)\overline{L(z,a)} = \overline{P\left(\overline{\alpha}\,\overline{S(\cdot,a)}\,\overline{T}\right)}(z). \tag{3.2}$$

Notice it is well known that the Szegő projection P maps $C^\infty(b\Omega)$ into itself and the Garabedian kernel $L(z,a)$ is a meromorphic function of z on Ω with a single simple pole at a.

Thus it follows easily that the identity (3.2) holds for $z \in \Omega$ and by letting $z \to a$, we have

$$\alpha(a) = 0$$

which proves α vanishes identically on Ω because a was arbitrary.

Theorem 3.3. *Ω is a bounded domain in the plane with C^∞ simple closed curves. Let $\alpha \in C^\infty(b\Omega)$. Suppose that the Toeplitz operator T_α associated to α vanishes on $H^2(b\Omega)$.*

Then the symbol α vanishes identically.

Acknowledgment

I would like to express gratitude to Professor Young-Joo Lee for motivation of this problem.

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