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SOME RESULTS ON IFP NEAR-RINGS

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Abstract. In this paper, we begin with to introduce the concepts of IFP and strong IFP in near-rings and then give some characterizations of IFP in near-rings.

Next we derive reversible IFP, and then equivalences of the concepts of strong IFP and strong reversibility.

Finally, we obtain some conditions to become strong IFP in right permutable near-rings and strongly reversible near-rings.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $(a+b)c = ac + bc$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*.

A (two sided) *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $a(I+b)-ab \subset I$ for all $a,\ b \in R$, (iii) $(I+a)b-ab \subset I$ for all $a,\ b \in R$, equivalently, $IR \subset I$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

On the other hand, a *(two-sided) R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R.

Also, a subset H of R together with (i) $RH \subset H$ and (ii) $HR \subset H$ is called an *R-subset* of R. If this H satisfies (i) then it is called a *left R-subset* of R, and H satisfies (ii) then it is called a *right R-subset* of R.

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We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid a0 = a\}$ which is called the *constant part* of R.

Obviously, we see that R_0 and R_c are subnear-rings of R. Thus a near-ring R is called *zero symmetric*, in case $R = R_0$ also, in case $R = R_c$, R is called *constant*. From the Pierce decomposition theorem, we obtain that $R = R_0 \oplus R_c$ as additive groups. So every element $a \in R$ has a unique representation of the form $a = b + c$, where $b \in R_0$ and $c \in R_c$.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $\theta(a + b) = \theta a + \theta b$, (ii) $\theta(ab) = \theta a \theta b$. Also, we can define the usual meanings of monomorphism, epimorphism, isomorphism, endomorphism and automorphism as for rings ([1]).

For the remainder concepts and results on near-rings, we refer to [5] and [6].

2. Results

We say that a near-ring R has the *insertion of factors property* (briefly, *IFP*) provided that for all a, b, x in R with $ab = 0$ implies $axb = 0$. Moreover, a near-ring R is said to fulfill the *strong IFP* if every homomorphic image of R has the IFP. These notions are introduced in the book [6].

A near-ring R is called *right permutable* if for all a, b, c in R, we have $abc = acb$. Every constant near-ring is right permutable. Similarly, we can define left permutable.

Also, we say that R is *reduced* if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. In ring theory, McCoy proved that R is reduced if and only if for each a in R, $a^2 = 0$ implies $a = 0$.

On the other hand, a near-ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Furthermore, R is said to be *strongly reversible* if for any $a, b \in R$ and for each ideal I of R, $ab \in I$ implies $ba \in I$. All reversible near-rings and strongly reversible near-rings are zero symmetric.

At our convenience, we can define the following compound concept.

We say that R has the *reversible IFP* in case R has the IFP and is reversible. Sometimes, this near-ring R is called *reversible IFP.*

Now, we consider the following quotient substructures of R and some relations between them.

Let X and Y be non-empty subsets of R. We can define the following quotient substructures of R.

$$(X : Y) := \{a \in R \mid aX \subset Y\}.$$

We abbreviate that for $x \in R$, $(\{x\} : Y) =: (x : Y)$. Similarly for $(Y : x)$. Sometimes, we say that $(X : o)$ is the *annihilator* of X, and denoted it by $Ann(X)$.

In the above notation, note that if Y is a subgroup (resp. normal subgroup, left R-subgroup, left ideal) of R, then so is $(X : Y)$ in R. Moreover, we have the following basic statements.

LEMMA 2.1. *Let K_1 and K_2 be non-empty subsets of a near-ring R. Then we have the following conditions:*

(1) *If K_2 is a normal left R-subgroup of R, then $(K_1 : K_2)$ is a normal left R-subgroup of R.*
(2) *If K_1 is a left R-subset of R and K_2 is a left R-subgroup of R, then $(K_1 : K_2)$ is a two-sided R-subgroup of R.*
(3) *If K_1 is a left R-subset of R and K_2 is a left ideal of R, then $(K_1 : K_2)$ is a two-sided ideal of R.*

Proof. (1) and (2) are easily proved by simple calculation.

Now, we prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal left R-subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$(ar)K_1 = a(rK_1) \subset aK_1 \subset K_2,$$

because K_1 is a left R-subset of R, so that $ar \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a right ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$\{r_1(a + r_2) - r_1r_2\}k = r_1(ak + r_2k)r_2 - r_1r_2k \in K_2$$

for all $k \in K_1$, since $aK_1 \subset K_2$ and K_2 is a left ideal of R. Thus $(K_1 : K_2)$ is a left ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R. □

Corollary 2.2. *Let R be a near-ring.*

(1) *([5]) For any $a \in R$, $Ann(a)$ is a left ideal of R.*
(2) *([5]) For any left R-subgroup K of R, $Ann(K)$ is a two-sided ideal of R.*
(3) *For any left R-subset K of R, $Ann(K)$ is a two-sided ideal of R.*
(4) *For any non-empty subset K of R, $Ann(K) = \bigcap_{a \in K} Ann(a)$.*

From Lemma 2.1 and Corollary 2.2, we have the following important conditions for IFP.

Theorem 2.3. *Let R be a near-ring. Then the following conditions are equivalent:*

(1) *R has the IFP.*
(2) *For any $a \in R$, $Ann(a)$ is an ideal of R.*
(3) *For any non-empty subset S of R, $Ann(S)$ is an ideal of R.*

Proof. (1) $\Rightarrow$ (2). Let $x \in Ann(a)$. Then $xa = 0$, by definition. Since R has the IFP, $xra = 0$ for each $r \in R$, that is, $xr \in Ann(a)$. Hence $Ann(a)$ is a right ideal of R. On the other hand, by Corollary 2.2 (1), $Ann(a)$ is a left ideal of R. Consequently, $Ann(a)$ is an ideal of R.

(2) $\Rightarrow$ (3). Assume the condition (2). Then because of $Ann(a)$ is an ideal of R, for any $a \in R$, and by Corollary 2.2 (4), $Ann(S) = \bigcap_{a \in S} Ann(a)$, obviously, $Ann(S)$ is an ideal of R.

(3) $\Rightarrow$ (1). Suppose that the condition (3) is true, and $ab = 0$ for $a,\ b$ in R. Then we see that $a \in Ann(b)$. From the condition (3), $Ann(b)$ is an ideal of R, so that $ax \in Ann(b)$, for any $x \in R$. That is $axb = 0$, for any $x \in R$. Therefore R has the IFP. □

Lemma 2.4 [6].

(1) *Let I be a two-sided ideal of a near-ring R. Then the canonical map $\pi : R \to R/I$ defined by $a \rightsquigarrow a + I$ is a near-ring epimorphism. So R/I is a homomorphic image of R, and $ker\pi = I$.*
(2) *Let the map $\phi : R \to S$ be a near-ring epimorphism. Then $ker\phi$ is a two-sided ideal of R and $R/ker\phi \cong S$.*

Clearly, every homomorphic image of zero symmetric near-ring is zero symmetric.

From the Lemma 2.4, obviously, we have the following conditions.

Proposition 2.5. *Let R be a near-ring. Then we have the following conditions:*

(1) *R has the strong IFP if and only if for any ideal I of R, for all a, b, x in R with $ab \in I$ implies $axb \in I$.*
(2) *R is strongly reversible if and only if every homomorphic image of R has the IFP.*
(3) *Every homomorphic image of right permutable is right permutable.*

Theorem 2.6. *Every zero-symmetric reduced near-ring R has the reversible IFP.*

Proof. Suppose that a, b in R such that $ab = 0$. Then, since R is zero-symmetric, we have $(ba)^2 = baba = b0a = b0 = 0$. Reducibility implies that $ba = 0$.

Next, assume that for all a, b, x in R with $ab = 0$. Then $(axb)^2 = axbaxb = ax0xb = ax0 = 0$ This implies $axb = 0$, by reducibility. Hence R has the IFP. Consequently, R has the reversible IFP. □

Theorem 2.7.

(1) *Every right permutable near-ring R has the strong IFP.*
(2) *Every strongly reversible near-ring R has the strong IFP.*

Proof. (1) From the Proposition 2.5 (3), every right permutable is inherited to the homomorphic images. So it suffices to show that R has the IFP in this case: Let a, b in R such that $ab = 0$. Then for all x in R, $axb = abx = 0x = 0$. Hence R has the IFP.
(2) Assume that R is strongly reversible and I is an ideal of R. Consider a, b in R such that $ab \in I$. Then since $ba \in I$, using the right ideality of I, we have that $b(ax) = (ba)x \in I$. Strong reversibility implies that $(ax)b \in I$. Consequently, R is a strong IFP near-ring. □

References

1. F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, Springer-Verlag, New York, Heidelberg, Berlin, 1974.
2. Y. U. Cho, *R-homomorphisms and R-homogeneous maps*, J. Korean Math. Soc. **42** (2005), 1153-1167.

3. C.G. Lyons and J.D.P. Meldrum, *Characterizing series for faithful d.g. near-rings*, Proc. Amer. Math. Soc. **72** (1978), 221-227.
4. J.D.P. Meldrum, *Upper faithful d.g. near-rings*, Proc. Edinburgh Math. Soc. **26** (1983), 361-370.
5. J.D.P, Meldrum, *Near-rings and their links with groups*, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1985.
6. G. Pilz, *Near-rings*, North Holland Publishing Company, Amsterdam, New York, Oxford, 1983.

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