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A Case Analysis on Mathematical Problems Posed by Teachers in Gifted Education[1)]

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Well posed problems for mathematically gifted students provide an effective method to design 'problem solving-centered' classroom activities. In this study, we analyze mathematical problems posed by teachers in distance learning as a part of an advanced training which is an enrichment in-service program for gifted education. The patterns of the teacher-posed problems are classified into three types such as 'familiar,' 'unfamiliar,' and 'fallacious' problems. Based on the analysis on the teacher-posed problems, we then suggest a practical plan for teachers' problem posing practices in distance learning.

I. Introduction

National Council of Teachers of Mathematics (NCTM) (1980) acknowledged that the students most neglected, in terms of realizing full potential, are the gifted students of mathematics. It is well recognized that one of the most important factors to enhance the gifted students' understanding mathematical thinking is the teachers' qualifications. In general, the teachers' qualifications in gifted educational programs are characterized by three aspects such as philosophical understanding and a sense of duty, experts' characteristics, and personalities' characteristics. Moreover, the experts' characteristics consists of expertise at major subjects and ability to teach, ability to enhance the students' creative problem solving skills, ability to perform researches, and expert knowledge and its applications in gifted education (Kim et al., 2000).

The selection and construction of worthwhile mathematical tasks is considered as one of the most important decisions teachers need to make (NCTM, 1991). The tasks teachers pose in their classrooms deserve important consideration because they open or close the students' opportunity for meaningful mathematical learning (Crespo, 2003) and so do the tasks teachers in gifted education pose for mathematically gifted students. Thus when teachers in gifted education are in such a position to pose worthwhile mathematical problems for 'problem-solving centered' activities, it is essential to pose mathematical problems so that they would meet the needs of the mathematically gifted. Furthermore, problem posing can also occur after solving a particular problem, when one might examine

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the conditions of the problem to generate alternative related problems; thus this form of problem posing is associated with 'looking back' stage of problem solving discussed by Polya (Silver et al., 1996).

In this study, we consider the following question with respect to mathematical problems posed by teachers in gifted education: What kinds of resources did they select and how did they adapt them to pose problems?

Hence in the study reported here, we first analyze the patterns of the teacher-posed problems which are classified into three such as 'familiar,' 'unfamiliar,' and 'fallacious' problems. Based on the analysis on the teacher-posed problems, we then suggest a practical plan for teachers' problem posing practices in distance learning within the curriculum of an advanced training for elementary and secondary school (prospective) teachers in gifted education.

II. Background

According to Brown and Walter (1990), problem posing is deeply embedded in the activity of problem solving in two very different ways. First, it is impossible to solve a new problem without first reconstructing the task by posing new problem(s) in the very process of solving. Second, it is frequently the case that after we have supposedly solved a problem, we do not fully understand the significance of what we have done, unless we begin to generate and try to analyze a completely new set of problems. However, while problem solving is easily identified as an important aspect of learning mathematics, problem posing has long been considered a neglected aspect of mathematical inquiry.

In NCTM (1991), a concept of problem posing was described: "students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem." Silver (1994) concluded that students' activities such as generating their own problems or solving preformulated problems have added benefit of providing insight into students' understanding of important mathematical concepts as well as into the nature of their school mathematics activities. In addition, Silver pointed out the term 'problem posing' has been used to refer both to the generation of new problems and to the reformulation of given problems. 'Problem posing' referred in this paper is that involves generating new problems or reformulating given problems in the sense of NCTM (1991) and Silver (1994).

Now, when it comes to the literature on (prospective) teachers' posing problems, Silver et al. (1996) studied on 53 middle school mathematics teachers and 28 prospective secondary school mathematics teachers' posing mathematical problems associated with a reasonably complex task setting and concluded that although some problems were ill-posed or poorly stated, they generated a large number of reasonable problems, thereby suggesting that these teachers and prospective teachers had some personal capacity for mathematical problem posing. Crespo (2003) pointed out that while a lot of attention has been focused on teacher candidates' own ability to solve mathematical problems, little attention has been paid to their ability to construct and pose mathematical problems to their students. In addition, Crespo examined the changes in the problem posing strategies of a group of prospective teachers as they posed problems to students and concluded that their later problem posing practices significantly differed from their earlier ones after taking 11 weeks (twice a week for 1.5 hours each day) course.

On the other hand, Bang (1995) concluded that practices on problem solving had not been fully accomplished in classroom activities even though they were related to mathematical activities because of the insufficient number of mathematical problems. Thus Bang suggested that teachers had to be trained how to pose the original version of mathematical problems for their students.

In order to pose worthwhile mathematical problems, it is also important to understand the background and process of how the original problems are developed. Hence, here we now introduce theoretical frameworks of developing problems for the mathematically gifted through a couple of concrete instructional examples based on the schemes of Sheffield (1999) and Hashimoto and Becker (1999).

Jensen (1980) suggested a way to help students creatively increase their mathematical abilities by a heuristic model consists of 'relate, investigate, communicate, evaluate, create.' Using this model, either the teacher or the student suggests a problem to work on, and students begin at any point on the heuristic model and they do not need to follow the steps sequentially. For example, a student may first 'relate' the problem to earlier mathematical ideas and develop a variety of strategies for attacking problem. As the original problem is 'investigated,' a new related problem might be 'created' that the student finds interesting to explore. This new problem is investigated, and the student suggests several hypotheses that are 'evaluated' and new findings are 'communicated' to classmates or the teacher. The original problem might be simple, but it should be interesting and challenging. Sheffield (1999) pointed out that the following list of phrases or questions were good to use when investigating problems: Why?; Why not?; What if ...?; What if not ...?; What patterns do I notice?; What predictions or generalizations can I make?; How is this like ...?; How is this different from ...?; Will that always work?; Will that ever work?; Can I do that another way? How many ways might I?; What is the largest? The smallest?; Convince me. Prove it. Show me; Explore the converse or the inverse of the problem; What other related problems might I explore? In addition, Sheffield suggested the following problem as an instructional example: Find three consecutive integers with a sum of 162.

We now turn to the study of Hashimoto and Becker in 1999. In this practical research, by trial and error and sharing, Japanese researchers and classroom teachers have worked collaboratively to derive some guidelines for developing problems. The following is the list of the guidelines: (1) Choose a physical situation that involves variable quantities in which mathematical relations can be observed; (2) In geometry, instead of asking students to prove a proposition such as 'If P, then Q' change it to 'If P, then what relations can be found?' (3) Show students geometric figures concerning a theorem in geometry, then have them draw other figures like the given one. Then ask them to make conjectures; (4) Show students a table of numbers and ask them to search for some mathematical patterns or rules; (5) Ask students to find a common feature(s) in different figures; (6) Show students several similar exercises or problems. Ask the students to solve them and find properties common to at least two of them; (7) Show students real applications involving variation. Ask the students to create methods for interpreting the variation; (8) Show students a concrete example for which an algebraic structure exists and numerical data are easily collected. Then ask them to find mathematical rules that seem to be true.

The following problem is suggested by Hashimoto and Becker (1999) as an instructional example with respect to the third guideline on developing problems on geometric figures concerning a theorem in geometry: Draw two triangles ABC and DEF. Let J and K be the midpoints of the sides AB and AC respectively and S and T the midpoints of the sides DE and DF respectively. Then what is the relationship between the line segment JK and the side BC in the triangle ABC? What is the relationship between the line segment ST and the side EF in the triangle DEF?

III. Method

3.1. Subjects

The subjects were 20 mathematical problems posed by 6 elementary school teachers (2 males and 4 females) and 4 middle school mathematics teachers (1 male and 3 females) out of 44 (prospective) teachers in gifted education (28 elementary school teachers and 16 secondary school mathematics teachers in gifted education) participated in the advanced training in the academic year 2007 at the institute for gifted education and promotion in 'A' metropolitan city. The specific process of how to select 10 out of 44 teachers is to be discussed later in Section 3.2.

The institute ran educational trainings (basic, advanced, and expert courses) for elementary and secondly school (prospective) teachers in gifted education in order to improve their professional and counseling capabilities of gifted education. In particular, since the advanced training consisted of 25 hours of distance learning, 56 hours of in-class learning, 24 hours of practical in-class teaching based on the prior distance and in-class learning, the teachers had practical opportunities to teach their mathematically gifted students in the classroom.

Almost all 44 participants were teachers either in regional gifted education centers or in their schools' gifted education programs, and their teaching experience ranged from 1 year to more than 20 years. However, little was known to the author about teachers' beliefs, mathematical knowledge, teaching experience in gifted education.

'A creative design for instruction' was the theme of this distance learning program given as a part of the advanced training at the institute in the academic year 2007. In this distance learning program, instead of attending classes, learning resources on the schemes of Sheffield (1999) and

Hashimoto and Becker (1999) including 9 mathematical problems as instructional examples which illustrate how to develop problems for the mathematically gifted were given to the participating teachers nearly for a month. In addition, the teachers were expected to work individually and hence it is of interest to note that the 9 original version of the problems were carefully selected as both learning resources and instructional examples to help the teachers to work alone. The teachers were supposed to pose 9 problems for mathematically gifted students in 5th grade through 7th grade based on the problem suggested by Sheffield with respect to prime numbers and the 8 problems suggested by Hashimoto and Becker as their reports after working over the learning resources on the schemes of Sheffield (1999) and Hashimoto and Becker (1999).

3.2. Problem selection

We first selected two kinds of problems on prime numbers and geometric figures out of the 9 kinds of the teacher-posed problems: One relating to prime numbers with respect to the schemes of Sheffield (1999) and the other relating to theorems on geometric figures in geometry according to the third guideline of Hashimoto and Becker (1999). The reason for selecting these two kinds of problems was that we could reveal the patterns of the teachers' choosing and adapting various properties of prime numbers in number theory and theorems on geometric figures in geometry to pose problems for mathematically gifted students. After selecting the prime number and geometric problems, we then examined all 44 teachers' reports and ruled out 34 of them as our research subjects because their problems were neither posed properly in accordance with the schemes of Sheffield (1999) and Hashimoto and Becker (1999) nor related to any properties of prime numbers in number theory or theorems on geometric figures in geometry. As a consequence of our problem selection procedure, we selected 20 teacher-posed problems from 6 elementary school teachers' reports and 4 middle school mathematics teachers' reports as our research subjects. Moreover, the 6 elementary school teachers' posed prime number problems were denoted by EP1, EP2, EP3, EP4, EP5, and EP6 and the 4 middle school mathematics teachers' posed prime number problems were denoted by MP1, MP2, MP3, and MP4. Similarly, the 10 geometric problems were denoted by EG1, EG2, EG3, EG4, EG5, EG6, MG1, MG2, MG3, and MG4. We note that each teacher posed two problems with the same digit number. For example, EP1 and EG1 were posed by the same elementary school teacher. Therefore the 20 teacher-posed problems for mathematically gifted students were the main sources of data for this study.

3.3. Data analysis

Crespo (2003) designed the study to identify the patterns in preservice teachers' selection and adaptation of problems by constructing case studies for each preservice teacher and noticing similarities and differences in their approaches. Crespo attended to several aspects of problem posing discussed in the document of NCTM (1991) when constructing each case. Moreover, she concluded that preservice teachers' beginning approaches to pose problems were as follows, such as 'making problems easy to solve,' 'posing familiar problems,' and 'posing problems blindly,' whereas their changed approaches to pose problems were as follows, such as 'posing unfamiliar problems,' 'posing problems that challenge students' thinking,' and 'posing problems to learn about students' thinking.'

For the analysis of the patterns of the teacher-posed problems in this study, the very nature of the two original version of the problems given by Sheffield (1999) and Hashimoto and Becker (1999) and the initial overview of the teacher-posed problems are considered before we implement both the ways of approaches stated in the study of Crespo (2003) and problem types (routine, non-routine; single answer/computational) and problem features (textbook-like, investigation-like) among several aspects of problem posing discussed in NCTM (1991). Hence we adapt the aspects of the problem types and problem features discussed in NCTM (1991) and Crespo (2003) so that the patterns of the teacher-posed problems are characterized by three categories such as 'familiar,' 'unfamiliar,' and 'fallacious' problems. In addition, each problem type is associated with one or two its own feature(s).

When the problems are classified, it might happen that, for example, there is a problem such that its problem type is both 'unfamiliar' and 'fallacious.' Therefore, the teacher-posed problems are identified into three types according to the following procedure: It is first determined whether the problems are fallacious or not. If not, then they are reclassified into two problem types such as 'familiar' and 'unfamiliar' problems.

For the analysis of the teacher-posed problems relating to prime numbers associated with the problem of Sheffield (1999), three problem types stated earlier are characterized by the following features.

- 'Familiar' problems have two features such as 'similar to a given problem' and 'computational exercises.' Problems similar to a given problem are the ones that modify the conditions of the given problem to somewhat less different conditions. Computational exercises are the ones that can be solved by arithmetic computations.
- 'Unfamiliar' problems have two features such as 'negative answers' and 'less straightforward.' Problems with negative answers are the ones that need to find counterexamples to solve the problems. Less straightforward problems are the ones that require somewhat deeper understanding of mathematics to solve the problems.
- 'Fallacious' problems have two features such as 'poorly stated' and 'ill-posed.' Poorly stated problems are the ones that have oversights in the problem statements. Ill-posed problems are the ones that either have something fundamentally wrong with the problem statements or have no answers. In addition, if a fallacious problem has both 'poorly stated' and 'ill-posed' features, then it is reasonable to classify it as 'ill-posed.'

Furthermore, for the analysis of the teacher-posed problems relating to theorems on geometric figures associated with the problem of Hashimoto and Becker (1999), three problem types have the following feature(s). The 'familiar' problems are characterized by the feature 'textbook-like' because the problem of Hashimoto and Becker is originated in the 'triangle midsegment theorem' in 8th grade mathematics textbooks. Textbook-like problems are the ones that can be typically found in most textbooks as properties, theorems, or exercises. Moreover, 'unfamiliar' problems have the feature 'investigation-like.' Investigation-like problems are the ones that need observations or deeper understanding of mathematics to find mathematically meaningful relations or properties beyond the contents of mathematics textbooks. 'Fallacious' problems have exactly the same features 'poorly stated' and 'ill-posed' as stated before.

IV. Analysis

4.1. Prime number problems

The following is a series of the 10 teacher-posed problems relating to properties of prime numbers in number theory. Recall that the original version of the problem is to find three consecutive integers with a sum of 162.

EP1. If n is a prime greater than 1, then is 2^n-1 always a prime?

EP2. If a natural number n greater than 6 can be expressed as the sum of three primes, then can it be expressed as the sum of two primes?

EP3. Find three consecutive primes with a sum of 177 and the difference of any two consecutive primes is 2.

EP4. Find three consecutive primes whose sum is a prime.

EP5. Find three consecutive primes with a sum of 121.

EP6. Find a pair of natural numbers with the difference is 2 and each natural number has only two divisors.

MP1. When primes are arranged in an increasing order, find three consecutive primes with a sum of 131.

MP2. Find all twin primes p and q such that $pq+4$ is a prime.

MP3. If $p\neq 3$ is a prime, then is p^2+2 a prime or a composite?

MP4. Use a prime factorization to verify that 8128 is the sum of its positive divisors excluding 8128. That is, show that 8128 is a perfect number.

<Table IV-1> Classification of Prime Number Problems

Type	Feature	Problem
Familiar	Similar to a given problem	EP5, MP1
	Computational exercises	EP4, EP6, MP4
Unfamiliar	Negative answers	EP1, EP2
	Less straightforward	MP2, MP3
Fallacious	Poorly stated	
	Ill-posed	EP3

4.1.1. Familiar problems

First of all, EP5 is posed similarly to the given problem because no actual adaptations are made to the original version of the problem.

In MP1, the condition that 'when primes are arranged in an increasing order' is not necessary to understand the problem because of the term 'consecutive primes.' Hence MP1 is similar to the given problem because no adaptations are made to the original version of the problem except for the redundant condition mentioned above. MP1 can be rewritten as follows.

• Find three consecutive primes with a sum of 131.

EP4 is a computational exercise because three consecutive primes whose sum is also a prime are easy to be found such as $5+7+11=23$, $7+11+13=31$, $11+13+17=41$, and so on. EP4 could be posed as a less straightforward computational exercise if it is rewritten as follows.

• Find three consecutive primes whose sum is closest to 100.

EP6 is about to find the so-called 'twin primes.' Twin primes are easy to be found such as 3 and 5, 5 and 7, 11 and 13, and so on. Hence it is reasonable to classify EP6 to a computational exercise. EP6 would be less straightforward if it is reposed as follows.

• Find twin primes whose sum is closest to 100.

It is of interest to note that the following property of twin primes could be a useful resource for posing problems. One of the strategies can be applied to verify this property effectively is the 'division algorithm.'

• Every twin prime pair except 3 and 5 is of the form $6n-1$ and $6n+1$ for some positive integer n.

In MP4, the condition 'use a prime factorization to verify that' is likely to make the problem much easier to solve. Furthermore, it might prevent the students from thinking about selecting their own problem solving strategies. Thus MP4 is just a computational exercise. It is worthwhile to recognize that there are some useful relationships between perfect numbers and prime numbers which could be helpful resources for posing problems:

• If n is an integer greater than 1 such that 2^n-1 is a prime, then $2^{n-1}(2^n-1)$ is a perfect number (Koshy, 2002).

• Every even perfect number is of the form $2^{n-1}(2^n-1)$, where 2^n-1 is a prime (Koshy, 2002).

4.1.2. Unfamiliar problems

In EP1, the condition 'greater than 1' is redundant because, by definition, 1 is neither a prime nor a composite. The word 'always' is obviously superfluous. Hence EP1 can be reposed as follows.

• *If n is a prime, then is 2^n-1 a prime?*

It is well known that if n is 2, 3, 5, or 7 then 2^n-1 is a prime, but $2^{11}-1$ is a composite which is a product of 23 and 89. Hence EP1 is directly related to the so-called Mersenne primes. The situation of finding a counterexample as a negative answer to the problem is not common, and hence EP1 could be posed as a problem with an affirmative answer.

• If 2^n-1 is a prime, then is n a prime?

EP2 is associated with the so-called Goldbach's conjecture: All even integers greater than 2 can be expressed as the sum of two primes. It is well known that Goldbach's conjecture is logically equivalent to the statement that every integer greater than 5 is the sum of three primes. Hence the statement that 'if every integer greater than 5 is the sum of three primes, then every even integer greater than 3 is the sum of two primes' is obviously true. Therefore, in order to solve EP2, one has to find a counterexample to the problem, that is, an odd integer which is a sum of three primes, but not a sum of two primes. Hence EP2 can be reposed as problems with affirmative answers as follows.

- If every integer greater than 5 is the sum of three primes, then show that every even integer greater than 2 is the sum of two primes.
- If every even integer greater than 2 is the sum of two primes, then can every integer greater than 5 be expressed as the sum of two primes?

MP2 contains the term 'twin primes' with no explanation of its meaning. In addition, it turns out that MP2 is exactly the same problem as one of the 1987 Korean Mathematical Olympiad problems and therefore it really is difficult to conclude that MP2 is one of the teacher-posed problems. However, since the mathematical problems in this study are supposed to be teacher-posed, we may assume that MP2 is one of the teacher-posed problems. Hence, it might be reasonable to regard MP2 as an unfamiliar problem. MP3 is to show that p^2+2 is a composite if $p \neq 3$ is a prime, and hence it might be appropriate to rewrite MP3 as follows: If $p \neq 3$ is a prime, then is p^2+2 a composite? In fact, since n^2+2 is a composite for every integer n which is not a multiple of 3, MP3 can be generalized such as:

- Show that n^2+2 is a composite for every integer n which is not a multiple of 3.
- If an integer n is not a multiple of 3, then is n^2+2 a composite?
- If n^2+2 is a prime, then is n a multiple of 3?

4.1.3. Fallacious problems

In EP3, it is of interest to note that the only 'three consecutive primes such that the difference of any two consecutive primes is 2' are 3, 5, and 7 among all prime numbers less than 100. Hence the answer to the problem does not exist. The intention for posing EP3 is most likely to find three primes with a sum of 177 within the set of twin primes. If it really is the case, then one can find three primes 43, 61, 73 whose sum is 177 within the set of twin primes 41 and 43, 59 and 61, 71 and 73. However, since twin primes are defined by pairs of primes, the condition 'three consecutive primes such that the difference of any two consecutive primes is 2' causes EP3 to be ill-posed. Hence EP3 could be reposed as follows.

- Find three primes with a sum of 177 within the set of twin primes.

4.2. Geometric problems

The following is a series of the 10 teacher-posed problems relating to theorems on geometric figures in geometry. Recall that the given problem is: In two triangles ABC and DEF let J and K be the midpoints of the sides AB and AC respectively and S and T the midpoints of the sides DE and DF respectively. Then what is the relationship between the line segment JK and the side BC in the triangle ABC? What is the relationship between the line segment ST and the side EF in the triangle DEF? This problem is just a restatement of the 'triangle midsegment theorem' in 8th grade mathematics textbooks. It is of interest to note that this problem is suggested so that students could make conjectures on properties of geometric figures through abduction.

EG1. In a triangle ABC let I be the point of intersection of two angle bisectors for the angles A and B and let D, E, and F be the feet of perpendiculars from I to the sides of the triangle ABC. How do you know that the triangles made here are congruent?

EG2. Draw two triangles with side lengths 3cm, 4cm, 5cm and 6cm, 8cm, 10cm respectively. What kind of triangles do you get? Find the relationship between the longest and the shortest sides of each triangle.

EG3. Draw two parallelograms $ABCD$ and $EFGH$. Let I, J, K, and L be the midpoints of the line segments AB, CD, EF, and HG respectively. How are the line segments IJ and KL related to each other?

EG4. Draw two squares $ABCD$ and $EFGH$. Let J, K, L, and M be the midpoints of the line segments AB, CD, EF, and HG respectively. What is the relationship between the line segments JK and CD? What is the relationship between the line segments LM and GH?

EG5. Let M and N be any points in the bisectors of the line segments AB and CD respectively. What is the relationship between the line segments MA and MB? What is the relationship between the line segments NC and ND?

EG6. Draw a circle with the center O and a diameter 4. Draw a line through O and let A and B be the point of intersection of the circle and the line. Let C be a point on the circle which is different from A and B. Draw a triangle ABC. Let D be a point on the circle which is different from A, B, and C. Draw a triangle ABD. What properties do the magnitudes of the angles ACB and ADB have?

MG1. After drawing a trapezoid $ABCD$, let E, G, and I be the midpoint, trisection point, and quadrisection point of the side AB respectively and let F, H, and J be the midpoint, trisection point, and quadrisection point of the side CD respectively. How are the line segments EF, GH, and IJ related?

MG2. Draw two triangles ABC and DEF. Draw a line segment PQ which is parallel to the line segment BC and passes through the line segments AB and AC. Draw a line segment ST which is parallel to the line segment EF and passes through the line segments DE and EF. What is the relationship between the line segments PQ and BC? What is the relationship between the line segments ST and EF?

MG3. In a triangle ABC let D and E be two points on the line segments AB and AC respectively such that D and E are parallel to the line segment BC. What is the relationship between two triangles ABC and ADF? How are the line segments related to each other?

MG4. Draw a circle with the center O. Let A and B be two arbitrary points on the circle and draw $\angle AOB$. Let P be a point on the circle different from A and B and draw $\angle APB$. What is the relationship between $\angle AOB$ and $\angle APB$?

<Table IV-2> Classification of Geometric Problems

Type	Feature	Problem
Familiar	Textbook-like	EG2, EG5, EG6, MG4
Unfamiliar	Investigation-like	
Fallacious	Poorly stated	EG1, EG4, MG1
	Ill-posed	EG3, MG2, MG3

4.2.1. Familiar problems

EG2 is related to the 'Pythagorean theorem' in 9th grade mathematics textbooks. In EG2, both the unit 'cm' of each side length and the question 'What kind of triangles do you get?' are redundant. Instead of the two similar triangles, suggesting two non-similar triangles would make EG2 somewhat more challenging.

EG5 is just a restatement of the 'perpendicular bisector theorem' in 7th grade mathematics textbooks.

EG6 can be regarded as a corollary of the 'inscribed angle theorem' in 9th grade mathematics textbooks. We recall that any line segment containing the center and its endpoints on the circle is called 'a diameter' of the circle; its length is called 'the diameter' and hence the expression 'a diameter of 4' is not appropriate. In addition, the condition 'draw a line through O and let A and B be the points of intersection of the circle and the line.' could be rewritten as 'let A and B be the endpoints of a diameter.' Hence EG6 could be simplified as follows.

- Let A, B, C, and D be distinct points on a circle with a diameter AB. How are the magnitudes of the angles ACB and ADB related?

MG4 is just a restatement of the 'inscribed angle theorem' in 9th grade mathematics textbooks and it could be simplified as follows.

- Let A, B, and C be distinct points on a circle with the center O. What is the relationship between $\angle AOB$ and $\angle AOC$?

4.2.2. Fallacious problems

First of all, EG1 is related to the 'incenter theorem' in 8th grade mathematics textbooks. Note that the condition 'D, E, and F are the feet of perpendiculars from I to the sides of the triangle ABC' is not clear because the sides of the triangle ABC containing D, E, and F respectively are not stated explicitly. Moreover, the phrase 'the triangles made here' needs to be stated more specifically so that one could understand what triangles are referred to.

In EG4, the fact that the line segments JK and LM are perpendiculars of the sides CD and GH respectively is trivial and hence there is nothing to prove. Therefore it is reasonable to conclude that EG4 is poorly stated in the sense that it does not meet the needs of the mathematically gifted.

In MG1, it is not given that which two sides are parallel. Moreover, since the side AB has two trisection and three quadricsection points respectively, it is not clear which trisection and quadrisection points are referred to G and I respectively. Similarly, the location of the points F and J on the side CD is not clear either.

In EG3, there is no mathematically meaningful relationship between the line segments IJ and KL. Thus we conclude that E3 is ill-posed.

Now, both MG2 and MG3 are related to the 'side splitting theorem' in 8th grade mathematics textbooks. MG2 is both poorly stated and ill-posed by the following three reasons. First, in the condition 'a line segment PQ which is parallel to the line segment BC and passes through the line segments AB and AC' is not clearly stated such that the two points P and Q are on the line segments AB and AC respectively. Second, the condition 'a line segment ST which is parallel to the line segment EF and passes through the line segments DE and EF' does not make any mathematical sense at all because such line segment ST does not exist. Third, it is to ask the relationship between the line segments PQ and BC in spite of the fact that they are being parallel is already given as a condition of the problem.

Finally, in MG3, the condition 'D and E are two points on the line segments AB and AC respectively such that D and E are parallel to the line segment BC' does not make any mathematical sense because the term 'parallel' can be defined between two lines, but not between points and a line. In addition, the question 'How are the line segments are related to each other?' has to be stated more specifically so that one could understand what line segments are referred to.

V. Conclusion

In this study, we selected 20 teacher-posed problems relating to prime numbers in number theory or theorems on geometric figures in geometry associated with instructional examples of Sheffield (1999) and Hashimoto and Becker (1999) in order to investigate the patterns of the teacher-posed problems in gifted education and then suggest a practical plan for teachers' problem posing in distance learning within the curriculum of an advanced training for elementary and secondary school (prospective) teachers in gifted education. The patterns of the teacher-posed problems are classified into three types such as 'familiar,' 'unfamiliar,' and 'fallacious' problems. In addition, since properties of prime numbers in number theory are not within the scope of school mathematics curriculum whereas the theorems on geometric figures in geometry are commonly discussed themes in school mathematics curriculum, the patterns of the teacher-posed problems are also characterized by the selection and adaptations of resources for the problems.

5.1. Prime number problems

As indicated in Table IV-1, the patterns of the teacher-posed problems are identified into three problem types, which may reveal the teachers' tendency to pose problems. It is also immediate to verify from Table IV-1 that all problems except EP3 are either familiar or unfamiliar and the problems of each group of the elementary school teachers and the middle school mathematics teachers are somewhat fairly distributed into such two problem types. The analysis on the patterns of the teacher-posed prime number problems can be summarized as follows.

First of all, EP3, EP4, EP5, and MP1 are about to find three consecutive primes satisfying certain condition(s). They are posed similarly to the given problem and basically no adaptations are made to the original version of the problem except for the use of prime numbers. Thus they can be classified into 'familiar' problems with an exception of EP3. In fact, EP3 is ill-posed. It turns out that the answer to EP3 does not exist because the condition 'three consecutive primes with a sum of 177 and the difference of any two consecutive primes is 2' does not make any mathematical sense at all. This critical error is most likely due to a lack of understanding of the very definition of twin primes.

Now, it is of interest to note that although EP4 looks like a problem similar to the given problem, it is regarded as a computational exercise because the answers to EP4 are easy to be found by arithmetic computations. Moreover, both EP6 and MP4 are considered as straightforward computational exercises by exactly the same reason as that of EP4. Meanwhile, each EP1 and EP2 is related to the situation of finding a counterexample as a negative answer to the problem. In general, coping with this kind of problems is not an easy task even for mathematicians. Hence it might be better to pose them as problems with affirmative answers as suggested in Section 4.1.1. On the other hand, although MP2 is exactly the same problem as one of the 1987 Korean Mathematical Olympiad problems and hence it is difficult to conclude that MP2 is one of the teacher-posed problems, it would be a good source for posing problems if it is somewhat adapted. Finally, MP3 is kind of a problem that leaves a room for its generalization as stated in Section 4.1.4.

Now as far as the resources on prime number problems are concerned, EP1, EP2, EP3, EP6, MP2, and MP4 are related to one of the following contents: Mersenne numbers, Goldbach's conjecture, twin primes, and perfect numbers. In addition, EP1, EP6, and MP4 are just restatements of well known mathematical facts with no adaptations. It is worthwhile to note that although EP2 is posed with some adaptations to the so-called Goldbach's conjecture, it is not recommendable to use conjectures as resources on problem posing.

5.2. Geometric problems

As Table IV-2 shows, EG2, EG5, EG6, and MG4 are familiar problems, while EG1, EG3, EG4, MG1, MG2, and MG3 are fallacious ones. It is of interest to note that there are no unfamiliar geometric problems. The problems posed by the elementary school teachers are fairly distributed into the other two problem types, but 3 problems posed by the middle school mathematics teachers are fallacious. We now summarize the teacher-posed geometric problems as follows.

When it comes to the conditions of the problems, textbook-like problems EG2, EG6, and MG4 have some redundant condition(s), whereas all poorly stated problems EG1, EG4, and MG1 have some missing condition(s). Hence the given condition(s) of the problems would be better to be checked carefully whether such conditions are all necessary or sufficient, for example, by drawing the corresponding geometric figures satisfying such conditions. In addition, in EG3, EG4, MG2, and MG3, the term 'line segment' should be better replaced by the term 'sides' because a polygon is said to be a geometric figure formed by enclosing region with 'line segments' and these line segments are called the 'sides' of the polygon.

We now consider ill-posed problems EG3, MG2, and MG3. Obviously EG3 is about to find any relationship between two geometric figures which does not exist. This critical oversight is most likely due to the fact that EG3 is posed without being solved beforehand. MG2 contains a couple of conditions which do not make any mathematical sense as indicated in Section 4.2.2. On the other hand, in MG3, the concept of two lines being parallel is seriously misused. This kind of critical oversight is most likely due to a lack of understanding of the concept of two lines being parallel.

Now as far as the theorems on geometric figures are concerned, each EG1, EG2, EG5, EG6, MG2, MG3, and MG4 is related to one of the following theorems in 7th grade through 9th grade mathematics textbooks: The incenter theorem, the Pythagorean theorem, the perpendicular bisector theorem, the inscribed angle theorem, the side splitting theorem. In fact, EG5 is the only problem relating to the 'perpendicular bisector theorem' in 7th grade mathematics textbooks and others are related to the theorems in either 8th grade or 9th grade mathematics textbooks. Since the problems are posed for 5th grade through 7th grade mathematically gifted students, those problems originated in 8th grade or 9th grade mathematics textbooks might be less straightforward for them.

On the other hand, the 'triangle midsegment theorem' is applied to parallelograms in EG3, squares in EG4, and a trapezoid in MG1 respectively. However, EG3, EG4, and MG1 can be all generalized by applying the 'triangle midsegment theorem' to any quadrilaterals. Finally, since EG4 is too trivial to solve, it is not appropriate for the mathematically gifted.

5.3. Overview of prime number problems and geometric problems

As indicated in Table V-1, the teacher-posed prime number and geometric problems are compared to each other with respect to their problem types. Since each group of 'familiar (unfamiliar)' prime number problems and 'familiar (unfamiliar, respectively)' geometric problems do not have the same features, it is not easy to compare them based on the same criterion. For example, there is no 'unfamiliar' geometric problem mainly because the feature of unfamiliar geometric problems and that of unfamiliar prime number problems are different. However, there are some relationships between the distribution of the prime number and geometric problems.

<Table V-1> Overview of Prime Number and Geometric Problems

Type	Prime Number Problem	Geometric Problem
Familiar	EP4, EP5, EP6, MP1, MP4	EG2, EG5, EG6, MG4
Unfamiliar	EP1, EP2, MP2, MP3	
Fallacious	EP3	EG1, EG3 EG4, MG1, MG2, MG3

First of all, regardless of the prime number and geometric problems, EP5, EG5, EP6, EG6, MP4, and MG4 are all familiar problems and both EP3 and EG3 are fallacious problems. In fact, EP3 and EG3 are ill-posed. Thus it is evident that 3 teachers (2 elementary school teachers and 1 middle school mathematics teacher) posed both familiar problems and 1 elementary school teacher posed both fallacious problems. On the other hand, it really is an unexpected result that the 6 geometric problems are turned out to be fallacious: EG1, EG4, and MG1 are poorly stated and EG3, MG2, and MG3 are ill-posed. We have already discussed that the critical oversights are most likely due to either the fact that they are posed without being solved beforehand or lacks of understanding of mathematical concepts.

We recall that most properties of prime numbers in number theory are not within the scope of school mathematics curriculum, whereas the theorems on geometric figures in geometry are commonly discussed themes in school mathematics curriculum. Hence the patterns of the teacher-posed problems are also characterized by the selection and adaptations of the resources for problems. As mentioned in Sections 5.1 and 5.2, each EP1, EP2, EP3, EP6, MP2, and MP4 is related to one of the following contents: Mersenne numbers, Goldbach's conjecture, twin primes, and perfect numbers. In addition, EP1, EP6, and MP4 are just restatements of well known mathematical facts. Furthermore, almost all the geometric problems are related to theorems in either 8th grade or 9th grade mathematics textbooks. Hence it is evident that the most resources for the geometric problems come directly from school mathematics textbooks.

5.4. Implications

This study investigates the patterns of the teacher-posed problems in gifted education. Although the investigation is based on 20 teacher-posed problems with somewhat situative resources on number theory and geometry, the analysis of the patterns of the teacher-posed problems suggests that there are several ways to improve teachers' problem posing practices in distance learning within the curriculum of an advanced training for elementary and secondary school (prospective) teachers in gifted education.

As Table V-1 displays, 1 prime number problem and 6 geometric problems out of the 20 problems are fallacious. Considering the procedure of our data selection, the frequency of the fallacious problems is much higher than expected. Unfortunately, the reason is not revealed by this study. However, it might be reasonable to conclude that it is most likely due to the teachers' lack of substantial educational experience with problem posing.

According to Silver et al. (1996) middle school teachers and prospective secondly school teachers' lack of substantial educational experience with problem posing was not a barrier to their being able to use problem posing with their students in spite of many problems were ill-posed or poorly stated. If follows that the unusual number of the fallacious problems does not allow us to conclude that those teachers would have difficulty in posing problems to mathematically gifted students even though it really is due to the teachers' lack of substantial experience with problem posing. Therefore a further study is needed to investigate whether lack of educational experience with problem posing is not a barrier to teachers' being able to use problem posing with their students in spite of many of the teacher-posed problems are fallacious.

Although little is known about teachers' substantial educational experience with problem posing in gifted education, an effective teachers' problem posing in distance learning can be suggested based on the results of our investigation on the teacher-posed problems. Since our suggestions to improve teachers' problem posing in distance learning are obviously related to the curriculum of the advanced training which includes distance learning as a part of it, it is helpful to recall the outline of the advanced training in terms of its time and process as indicated in Section 3.1. The advanced training consists of 56 hours of in-class learning, 25 hours of distance learning, 24 hours of practical in-class teaching based on the prior in-class and distance learning. In particular, it is of interest to recall that the teachers have practical opportunities to teach their mathematically gifted students in the classroom because of those 24 hours of practical in-class teaching.

Overall, from the advanced learning point of view, it would seem beneficial to include the contents of 'learning to pose problems' in the curriculum of the advanced training according to the following sequential steps: (1) In-class learning to pose problems; (2) Practice to pose problems in distance learning and get some feedback from experts; (3) Get some practical experience through in-class teaching and get some feedback from students' responses. Hence the whole process of learning to pose problems is performed by systematic planning and evaluating process as an organic whole.

As indicated in Ball (1990), the mathematical understanding prospective teachers bring their school and college mathematics courses was inadequate for teaching elementary and second school mathematics. In addition, Crespo (2003) concluded that teacher education courses must be designed to not only extend prospective teachers' mathematical knowledge, but to also provide opportunities to change and revise unexamined knowledge and beliefs about subject matter and about teaching and learning. This suggests that it would be effective for prospective teachers in gifted education to acquire a deeper understanding of mathematical contents and learn problem posing in their teacher education courses so that they have a substantial experience with problem posing.

Apart from the overall process of problem posing in distance learning within the curriculum of the advanced training, problem posing in distance learning in this study can be improved as follows:

First, distance learning program must include the provision of two-way communication between the teachers and the teacher educator.

Second, well selected the original version of mathematical problems or resources would play key factors in problem posing. Thus it is important to find good sources for the original version of problems and resources with respect to their problems and problem types and features indicated as aspects of problem posing discussed in NCTM (1991). Hence the distance learning provides teachers with an optional self-tutorial on how to find good mathematical problems and resources for the mathematically gifted.

Third, teacher-posed problems should be evaluated to see how well they meet the needs of the mathematically gifted before posing them to the students. Hence effective assessment instruments for problem posing in distance learning should be developed.

Fourth, it would be beneficial to run both problem posing in distance learning and teachers' in-class teaching with the mathematically gifted at the same time so that the teachers could get an immediate feedback from students' responses. Hence the teachers could monitor their problem posing practices in distance learning.

Last, but not least, how 'problem posing in distance learning' positively affect teachers' problem posing abilities should be evaluated by examining the changes in the patterns of their earlier posed problems and later ones.

But most of all, it is evident that the extent to which the teachers in gifted education are willing to engage in problem posing in distance learning plays an important role. Hence a further study is needed to devise a plan for the teachers' voluntarily involvement in problem posing in distance learning.

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수학영재 지도교사의 문제만들기 사례분석

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수학영재의 지적 욕구를 충족시키고 창의성을 신장시키는 문제해결 중심의 수업활동을 하기 위해서는 수학영재의 수준에 맞게 만들어진 문제가 필수적이다. 본 논문의 목적은 수학영재 지도교사의 교수 능력을 신장시키기 위한 심화 연수 과정의 일부인 원격 연수에 참여한 수학영재 지도교사가 만든 문제의 형태를 문제 접근 방법에 따라 '익숙한 문제', '익숙하지 않은 문제', '오류가 있는 문제'로 나누어 분석하여 수학영재 지도교사를 위한 원격 연수에서 교사의 문제만들기에 대한 실천적 방안을 제시하는데 있다.

* key words : mathematically gifted students(수학영재), teachers in gifted education(영재 지도교사), teacher-posed problems(교사가 만든 문제)

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