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A Central Limit Theorem for the Linear Process in a Hilbert Space under Negative Association

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Abstract

We prove a central limit theorem for the negatively associated random variables in a Hilbert space and extend this result to the linear process generated by negatively associated random variables in a Hilbert space. Our result implies an extension of the central limit theorem for the linear process in a real space under negative association to a simplest case of infinite dimensional Hilbert space.

Keywords: Central limit theorem, negatively associated, linear operator, H-valued random variable, linear process.

1. Introduction

Let H be a separable real Hilbert space with the norm $\|\cdot\|_H$ generated by an inner product, $<\cdot,\cdot>$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H. Let $L(H)$ be the class of bounded linear operators from H to H and denote by $\|\cdot\|_{L(H)}$ its usual norm. Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary sequence of H-valued random variables and $\{a_k, k \in \mathbb{Z}\}$ be a sequence of bounded operators in $L(H)$. We define the stationary linear process in a Hilbert space by $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, $k \in \mathbb{Z}$.

Linear processes in a Hilbert space play an important role in global statistics for continuous processes (*cf.* Bosq, 2000). The sequence $\{X_k, k \in \mathbb{Z}\}$ of H-valued linear processes is a natural extension of the multivariate linear processes (Brockwell and Davis, 1987, Chapter 11). We define $S_n = \sum_{k=1}^{n} X_k$. Notice that if $\sum_{j=0}^{\infty} \|a_j\|_{L(H)} < \infty$ and $\{\xi_k, k \in \mathbb{Z}\}$ is a sequence of H-valued *i.i.d.* centered random variables, then the series S_n converges almost surely (see Araujo and Gine, 1980, Chapter 3.2) and the H-valued linear process X_k satisfies the central limit theorem (Bosq, 2000).

A finite family $\{\xi_i, 1 \leq i \leq n\}$ of real-valued random variables is said to be associated if for any coordinatewise increasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$, $\mathrm{Cov}(f(\xi_1, \ldots, \xi_n), g(\xi_1, \ldots, \xi_n)) \geq 0$ whenever this covariance exists. A finite family $\{\xi_i, 1 \leq i \leq n\}$ is said to be negatively associated if for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^A$, g on $\mathbb{R}^B$, $\mathrm{Cov}(f(\xi_k, k \in A), g(\xi_k, k \in B)) \leq 0$ whenever the covariance exists. An infinite family of random variables is associated(negatively associated) if every finite subfamily is associated(negatively associated). These concepts of dependence were introduced by Esary *et al.* (1967) and Joag-Dev and Proschan (1983), respectively.

Newman (1984) studied the central limit theorem for strictly stationary negatively associated sequence and Matula (1992) derived the strong law of large numbers for negatively associated sequence.

As Burton *et al.* (1986) introduced the definition of weak association for random vectors we can give the definition of negative association for random vectors with values in $\mathbb{R}^d$: Let $\{\xi_1, \ldots, \xi_m\}$ be a

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sequence of $\mathbb{R}^d$-valued random vectors. $\{\xi_1, \ldots, \xi_m\}$ is said to be negatively associated if $\mathrm{Cov}(f(\xi_k, k \in A), g(\xi_k, k \in B)) \leq 0$ for any disjoint subsets $A, B \subset \{1, \ldots, n\}$ and for any nondecreasing functions f on $\mathbb{R}^{d|A|}$, g on $\mathbb{R}^{d|B|}$ such that the covariance exists and $|A|$ is the cardinality of A.

Burton *et al.* (1986) also extended the concept of weak association for random vectors with values in $\mathbb{R}^d$ to random vectors with values in a real separable Hilbert space. Similarly, we can also introduce the concept of negative association in a Hilbert space as follows : Let $\{\xi_n, n \geq 1\}$ be a sequence of random variables taking values in a Hilbert space $(H, < \cdot, \cdot >)$. $\{\xi_n, n \geq 1\}$ is said to be negatively associated if, for some orthonormal basis $\{e_k, k \geq 1\}$ in H and for any $d \geq 1$ the d-dimensional sequence $(< \xi_i, e_1 >, \ldots, < \xi_i, e_d >)$, $i \geq 1$ is negatively associated.

Burton *et al.* (1986) proved an invariance principle for weakly associated random variable in a Hilbert space. Kim and Ko (2008) proved a central limit theorem for linear process generated by associated random variables in a Hilbert space. Ko *et al.* (2009) showed the almost sure convergence for negatively associated random variables in a Hilbert space and Kim *et al.* (2008) also obtained the almost sure convergence for a linear process generated by negatively associated random variables in a Hilbert space.

In Section 2 we study the maximal inequality and the central limit theorem for negatively associated random variables in a Hilbert space and in Section 3 we will prove the central limit theorem for a strictly stationary linear process generated by H-valued negatively associated random variables by applying this result.

Our result implies the following central limit theorem for the linear process generated by negatively associated random variables.

Theorem 1. (Ko et al., 2006) *Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary sequence of centered and negatively associated random variables having finite second moment and let $\{a_k,\ k \in \mathbb{Z}\}$ be a sequence of numbers such that*

$$\sum_{j=0}^{\infty} |a_j| < \infty.$$

Define $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, k \in \mathbb{Z}$ and $S_n = \sum_{k=1}^{n} X_k$ and assume

$$\sigma^2 = E\xi_1^2 + 2\sum_{j=2}^{\infty} E(\xi_1 \xi_j) < \infty.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \xrightarrow{\mathcal{D}} N\left(0, \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2\right) \quad \text{(See Kim } et\ al.\text{, 2008)},$$

where N is a Gaussian random variable with variance $(\sum_{j=0}^{\infty} a_j)^2 \sigma^2$.

2. Preliminaries

In the proof of Lemma 4 in Matula (1992) we have:

Lemma 1. (Matula, 1992) *Let $\{Y_n, n \geq 1\}$ be a sequence of negatively associated random variables with finite second moments and zero means. Then*

$$E\left(\max_{1 \leq k \leq n} \sum_{i=1}^{k} Y_i\right)^2 \leq \sum_{i=1}^{n} EY_i^2. \tag{2.1}$$

Lemma 2. (Matula, 1992) *Let $\{Y_1, \ldots, Y_m\}$ be a sequence of negatively associated random variables with $E(Y_i^2) < \infty$ and $EY_i = 0$, $i \geq 1$ and let $S_m = Y_1 + \cdots + Y_m$. Then*

$$E\Big(\max(|S_1|, \ldots, |S_m|)\Big)^2 \leq 4 \sum_{i=1}^{m} E\left(Y_i^2\right). \tag{2.2}$$

Lemma 3. (Ko et al., 2009) *Let $\{\xi_n, n \geq 1\}$ be a negatively associated sequence of H-valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty$, $n \geq 1$. Then, we have*

$$E \max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \xi_i \right\|^2 \leq 4 \sum_{i=1}^{n} E\|\xi_i\|^2. \tag{2.3}$$

From the Newman's (1984) central limit theorem for negatively associated sequence we can obtain the following central limit theorem for stationary negatively associated random vectors by means of the simple device due to Cramer Wold technique (*c.f.* Billingsley, 1968).

Theorem 2. *Let $\{\xi_i,\ i \geq 1\}$ be a strictly stationary negatively associated sequence of $\mathbb{R}^d$-valued random vectors with $E\xi_1 = \mathbb{O}$ and $E\|\xi_1\|^2 < \infty$. If*

$$\sigma^2 = E\|\xi_1\|^2 + 2 \sum_{i=2}^{\infty} \sum_{j=1}^{d} E(\xi_{1j}\xi_{ij}) < \infty, \tag{2.4}$$

then

$$n^{-\frac{1}{2}} \sum_{i=1}^{n} \xi_i \xrightarrow{\mathcal{D}} N(\mathbb{O}, \Gamma), \tag{2.5}$$

where N is a Gaussian random vector with covariance matrix $\Gamma = [\sigma_{kj}]$,

$$\sigma_{kj} = E(\xi_{1k}\xi_{1j}) + \sum_{i=2}^{\infty} \Big[E(\xi_{1k}\xi_{ij}) + E(\xi_{1j}\xi_{ik})\Big]. \tag{2.6}$$

Theorem 3. (Ko, 2006) *Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary negatively associated sequence of $\mathbb{R}^d$-valued random vectors with $E\xi_1 = \mathbb{O}$ and $E\|\xi_1\|^2 < \infty$ and let $\{A_j\}$ be a sequence of matrix such that*

$$\sum_{j=0}^{\infty} \|A_j\| < \infty, \qquad \sum_{j=0}^{\infty} A_j \neq \mathbb{O}_{d\times d},$$

where for any $d \times d$ matrix $A = (a_{ij})$, $\|A\| = \sum_{j=1}^{d}\sum_{i=1}^{d} |a_{ij}|$ and $\mathbb{O}_{d\times d}$ denotes the $d \times d$ zero matrix. Define X_k an $\mathbb{R}^d$-valued linear process of the form $X_k = \sum_{j=0}^{\infty} A_j \xi_{k-j}$. If (2.4) holds, then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \xrightarrow{\mathcal{D}} N(\mathbb{O}, T),$$

where N is a Gaussian random vector with covariance matrix $T = (\sum_{j=0}^{\infty} A_j)\ \Gamma(\sum_{j=0}^{\infty} A_j)'$ and Γ is defined in (2.6).

3. Main Results

Theorem 4. *Let $\{\xi_i,\ i \geq 1\}$ be a strictly stationary negatively associated sequence of H-valued centered random variables. If $E\|\xi_1\|^2 < \infty$ and*

$$\sigma^2 = E\|\xi_1\|^2 + 2\sum_{i=2}^{\infty} E(<\xi_1,\ \xi_i>) < \infty, \tag{3.1}$$

then

$$\frac{\sum_{i=1}^{n}\xi_i}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(O, \Gamma), \tag{3.2}$$

where N is a Gaussian random variable with value in a Hilbert space and with covariance operator Γ satisfying

$$\Gamma(e_k, e_l) = E(<\xi_1,\ e_k><\xi_1,\ e_l>) + \sum_{i=2}^{\infty}[E(<\xi_1,\ e_k><\xi_i,\ e_l>)$$
$$+ E(<\xi_1,\ e_l><\xi_i,\ e_k>)], \quad k,l = 1,2,\ldots.$$

Proof: Let (e_k) be the orthonormal basis with respect to which the sequence $\{\xi_j, j \geq 1\}$ is negatively associated in H. For $M \geq 1$, let P_M be the projection on the subspace generated by $e_1, \ldots, e_M$ and $Q_M = I - P_M$.

By stationarity and $E\|\xi_1\|^2 < \infty$ we have

$$n^{-1}\sum_{i=1}^{n} E\|\xi_i\|^2 = n^{-1}\sum_{i=1}^{n} E\sum_{k=1}^{\infty}(<\xi_i,\ e_k>)^2 = E\sum_{k=1}^{\infty}(<\xi_1,\ e_k>)^2$$
$$= E\|\xi_1\|^2 < \infty.$$

Hence, for any $\epsilon > 0$ there exists M_0 such that

$$n^{-1}\sum_{i=1}^{n} E\|Q_M\xi_i\|^2 = n^{-1}\sum_{i=1}^{n} E\sum_{k=M+1}^{\infty}(<\xi_i, e_k>)^2$$
$$= E\sum_{k=M+1}^{\infty}(<\xi_1, e_k>)^2 \leq \epsilon \tag{3.3}$$

for every $M \geq M_0$.

From Theorem 2 we have that $P_M(n^{-1/2}\sum_{i=1}^{n}\xi_i)$ converges in distribution to $P_M N$. Hence it remains to show that $Q_M(n^{-1/2}\sum_{i=1}^{n}\xi_i)$(and $Q_M N$) becomes small as $M \to \infty$. By Lemma 2 we also have

$$E\max_{1\leq k\leq n}\left(\sum_{i=1}^{k} <\xi_i, e_j>\right)^2 \leq 4\sum_{i=1}^{n} E(<\xi_i, e_j>)^2. \tag{3.4}$$

Hence, it follows from (3.3) and (3.4) that

$$
\begin{aligned}
En^{-1}\max_{1\le k\le n}\left\|\sum_{i=1}^{k} Q_M\xi_i\right\|^2 &\le n^{-1}\sum_{j=M+1}^{\infty} E\max_{1\le k\le n}\left(\sum_{i=1}^{k} <\xi_i, e_j>\right)^2 \\
&\le 4n^{-1}\sum_{j=M+1}^{\infty}\sum_{i=1}^{n} E\left(<\xi_i, e_j>\right)^2 \\
&= 4n^{-1}\sum_{i=1}^{n} E\|Q_M\xi_i\|^2 \le \epsilon.
\end{aligned}
$$

A similar estimation holds for $Q_M N$. Let

$$
n^{-\frac{1}{2}}\sum_{i=1}^{n}\xi_i = P_M\left(n^{-\frac{1}{2}}\sum_{i=1}^{n}\xi_i\right) + Q_M\left(n^{-\frac{1}{2}}\sum_{i=1}^{n}\xi_i\right) \quad \text{and} \quad N = P_M N + Q_M N,
$$

where N is a Gaussian random variable on H. From the above considerations we obtain $P_M N \xrightarrow{\mathcal{D}} N$ *as* $M \to \infty$, $n^{-1/2}\sum_{i=1}^{n}\xi_i \xrightarrow{\mathcal{D}} P_M(n^{-1/2}\sum_{i=1}^{n}\xi_i)$ and $P_M(n^{-1/2}\sum_{i=1}^{n}\xi_i) \xrightarrow{\mathcal{D}} P_M N$. Hence, (3.2) holds, *i.e.*, the proof is completed. □

Theorem 5. *Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary negatively associated sequence of H-valued centered random variables with $E\|\xi_1\|^2 < \infty$. Let $\{a_k, k \in \mathbb{Z}\}$ be a sequence of bounded linear operators on H satisfying*

$$
\sum_{j=0}^{\infty}\|a_j\|_{L(H)} < \infty. \tag{3.5}
$$

We define the linear process in a Hilbert space by $X_k = \sum_{j=0}^{\infty} a_j\xi_{k-j}$.

If (3.1) holds, then

$$
\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(O, A\Gamma A^*),
$$

where N is a Gaussian random variable on H, Γ is defined in Theorem 4, $A = \sum_{j=0}^{\infty} a_j$ and A^* denotes the adjoint operator of A.

Proof: Let

$$
\widetilde{X_k} = \left(\sum_{j=0}^{\infty} a_j\right)\xi_k \quad \text{and} \quad \widetilde{S_n} = \sum_{k=1}^{n}\widetilde{X_k}.
$$

It is clear that

$$
\begin{aligned}
\widetilde{S_n} &= \sum_{k=1}^{n}\widetilde{X_k} \\
&= \sum_{k=1}^{n}\left(\sum_{j=0}^{n-k} a_j\right)\xi_k + \sum_{k=1}^{n}\left(\sum_{j=n-k+1}^{\infty} a_j\right)\xi_k \\
&= \sum_{k=1}^{n}\left(\sum_{j=0}^{k-1} a_j\xi_{k-j}\right) + \sum_{k=1}^{n}\left(\sum_{j=n-k+1}^{\infty} a_j\right)\xi_k.
\end{aligned}
$$

Then

$$\widetilde{S_k} - S_k = -\sum_{i=1}^{k}\left(\sum_{j=i}^{\infty} a_j \xi_{i-j}\right) + \sum_{i=1}^{k}\left(\sum_{j=k-i+1}^{\infty} a_j\right)\xi_i =: A + B.$$

First we will prove

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \left\|\widetilde{S_k} - S_k\right\| \xrightarrow{P} 0. \tag{3.6}$$

In order to prove (3.6), we need only to show

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|A\| \xrightarrow{P} 0 \tag{3.7}$$

and

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|B\| \xrightarrow{P} 0. \tag{3.8}$$

Using the Minkowski inequality, finite second moment, Lemma 3 and the dominated convergence theorem

$$\begin{aligned} n^{-1}E \max_{1\le k\le n}\left\|\sum_{i=1}^{k}\left(\sum_{j=i}^{\infty} a_j\xi_{i-j}\right)\right\|^2 &= n^{-1}E\max_{1\le k\le n}\left\|\sum_{j=1}^{\infty}\left(\sum_{i=1}^{j\wedge k} a_j\xi_{i-j}\right)\right\|^2 \\ &\le n^{-1}E\left(\sum_{j=1}^{\infty}\|a_j\|_{L(H)} \max_{1\le k\le n}\left\|\sum_{i=1}^{j\wedge k}\xi_{i-j}\right\|\right)^2 \\ &\le n^{-1}\left\{\sum_{j=1}^{\infty}\|a_j\|_{L(H)}\left(E\max_{1\le k\le n}\left\|\sum_{i=1}^{j\wedge k}\xi_{i-j}\right\|^2\right)^{\frac{1}{2}}\right\}^2 \\ &\le 4n^{-1}\left\{\sum_{j=1}^{\infty}\|a_j\|_{L(H)}\left(\sum_{i=1}^{j\wedge n}E\|\xi_{i-j}\|^2\right)^{\frac{1}{2}}\right\}^2 \\ &\le Cn^{-1}\left(\sum_{j=1}^{\infty}\|a_j\|_{L(H)}(j\wedge n)^{\frac{1}{2}}\right)^2 = o(1) \end{aligned} \tag{3.9}$$

which yields (3.7). Next,

$$\begin{aligned} B &= \sum_{i=1}^{k}\left(\sum_{j=k-i+1}^{\infty} a_j\right)\xi_i \\ &= \sum_{j=1}^{k} a_j \sum_{i=k-j+1}^{k}\xi_i + \sum_{j=k+1}^{\infty} a_j\sum_{i=1}^{k}\xi_i \\ &=: B_1 + B_2. \end{aligned} \tag{3.10}$$

Let p_n be a positive integer such that $p_n \to \infty$ and $p_n/n \to 0$.

$$\begin{aligned} & n^{-\frac{1}{2}} \max_{1\le k\le n} \|B_2\|_H \\ & \le \sum_{j=0}^{\infty} \|a_j\|_{L(H)} n^{-\frac{1}{2}} \max_{1\le k\le p_n} \left\| \sum_{i=1}^{k} \xi_i \right\| + \left(\sum_{j=p_n+1}^{\infty} \|a_j\|_{L(H)} \right) n^{-\frac{1}{2}} \max_{1\le k\le n} \left\| \sum_{i=1}^{k} \xi_i \right\| \\ & =: \|B_{21}\|_H + \|B_{22}\|_H. \end{aligned} \tag{3.11}$$

From Lemma 3 and finite second moment we have

$$E\|B_{21}\|_H^2 \le C \left(\sum_{j=0}^{\infty} \|a_j\|_{L(H)} \right)^2 \left(\frac{p_n}{n} \right) = o(1) \tag{3.12}$$

and

$$E\|B_{22}\|_H^2 \le C \left(\sum_{j=p_n+1}^{\infty} \|a_j\|_{L(H)} \right)^2 = o(1). \tag{3.13}$$

By (3.11), (3.12) and (3.13), we have

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|B_2\|_H \xrightarrow{P} 0. \tag{3.14}$$

It remains to prove

$$L_n = n^{-\frac{1}{2}} \max_{1\le k\le n} \|B_1\|_H \xrightarrow{P} 0. \tag{3.15}$$

For each $m \ge 1$ let

$$B_{1,m} = \sum_{j=1}^{k} b_j \sum_{i=k-j+1}^{k} \xi_i,$$

where $b_j = a_j I(j \le m)$ and

$$L_{n,m} = n^{-\frac{1}{2}} \max_{1\le k\le n} \|B_{1,m}\|.$$

Then, for each $m \ge 1$

$$L_{n,m} \le (\|a_1\|_{L(H)} + \cdots + \|a_m\|_{L(H)}) n^{-\frac{1}{2}} (\|\xi_1\|_H + \cdots + \|\xi_m\|_H) \xrightarrow{P} 0 \tag{3.16}$$

as $n \to \infty$.

$\forall \epsilon > 0$ by Lemma 3 and finite second moment we have

$$\begin{aligned} P(\|L_n - L_{n,m}\|_H > \epsilon) & \le \epsilon^{-2} E\|L_n - L_{n,m}\|_H^2 \\ & \le \epsilon^{-2} n^{-1} E \max_{m\le k\le n} \left\| \sum_{j=1}^{k} (a_j - b_j)(\xi_k + \cdots + \xi_{k-j+1}) \right\|_H^2 \\ & \le \epsilon^{-2} n^{-1} E \max_{m\le k\le n} \left\| \sum_{j=m+1}^{k} \|a_j\|_{L(H)} \left(\sum_{j=1}^{k} \xi_j - \sum_{j=1}^{k-j} \xi_j \right) \right\|_H^2 \end{aligned}$$

$$\leq 4\epsilon^{-2}\left(\sum_{j=m+1}^{\infty}\|a_j\|_{L(H)}\right)^2 n^{-1}E\max_{1\leq k\leq n}\left\|\sum_{j=1}^{k}\xi_j\right\|_H^2$$

$$\leq 16\epsilon^{-2}\left(\sum_{j=m+1}^{\infty}\|a_j\|_{L(H)}\right)^2 n^{-1}\sum_{j=1}^{n}E\|\xi_j\|_H^2$$

$$\leq C\left(\sum_{j=m+1}^{\infty}\|a_j\|_{L(H)}\right)^2 \to 0 \quad as\ m\to\infty. \tag{3.17}$$

Hence by (3.17)

$$\|L_n - L_{n,m}\| \xrightarrow{P} 0. \tag{3.18}$$

Using (3.16) and (3.18) we have (3.15). By (3.14), (3.15) and (3.10), we have (3.8). Therefore we prove (3.6) and the proof is completed(see proof of Theorem 5 in Kim *et al.* (2008)).

Finally, we consider the sufficient conditions that the sequence $\{X_k\}$ of H-valued linear process satisfies the central limit theorem if the corresponding result for $\{\xi_k\}$ is true. □

Theorem 6. *Let $\{\xi_k,\ k\geq 1\}$ be a strictly stationary negatively associated sequence of H-valued centered random variables with $E\|\xi_1\|^2<\infty$ and $\{a_k\}$ be a sequence of bounded linear operators on H satisfying (3.5). Then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i \xrightarrow{\mathcal{D}} N(O,\Gamma)\ \text{implies}\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i \xrightarrow{\mathcal{D}} N(O,A\Gamma A^*),$$

where N is a Gaussian random variable on H, $A=\sum_{j=0}^{\infty}a_j$ and A^* denotes the adjoint operator of A.

Proof: The proof is similar to that of Theorem 5. □

Corollary 1. (Ko, 2006) *Let $\{\xi_k,\ k\in\mathbb{Z}\}$ be a strictly stationary negatively associated sequence of $\mathbb{R}^d$-valued random vectors with $E\xi_1=\mathbb{O}$ and $E\|\xi_1\|^2<\infty$ and let $\{A_j\}$ be a sequence of matrix such that*

$$\sum_{j=0}^{\infty}\|A_j\|<\infty,\quad \sum_{j=0}^{\infty}A_j\neq\mathbb{O}_{d\times d},$$

where for any $d\times d$ matrix $A=(a_{ij})$, $\|A\|=\sum_{j=1}^{d}\sum_{i=1}^{d}|a_{ij}|$ and $\mathbb{O}_{d\times d}$ denotes the $d\times d$ zero matrix. Define X_k an $\mathbb{R}^d$-valued linear process of the form $X_k=\sum_{j=0}^{\infty}A_j\xi_{k-j}$. Then

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\xi_k \xrightarrow{\mathcal{D}} N(\mathbb{O},\Gamma)\ \text{implies}\ \frac{1}{\sqrt{n}}\sum_{k=1}^{n}X_k \xrightarrow{\mathcal{D}} N(\mathbb{O},T),$$

where N is a Gaussian random vector with covariance matrix $T=(\sum_{j=0}^{\infty}A_j)\,\Gamma(\sum_{j=0}^{\infty}A_j)'$ and Γ is defined in (2.6).

Example 1. Let $X_k-\mu=\sum_{j=0}^{\infty}a_j\xi_{k-j}$ be a strictly stationary linear process in a Hilbert space, where μ is the unknown mean and $\{\xi_k\}$ is a strictly stationary negatively associated sequence of H-valued random variables with $E\xi_1=0$, $E\|\xi_1\|^2<\infty$ and satisfying (3.1). Here we suppose that the linear operator a_j are geometrically bounded, in the sense that there exist real constant $b > 0$ and $0 < \rho < 1$ such that $\|a_j\| \leq b\rho^j$ for all $j \geq 0$. Then this process satisfies Theorem 5.

Note that this class of a linear process in a Hilbert space is a basic object in time series analysis and contains a stationary Hilbertian auto regressive process.

Confidence Region: Suppose that one wishes to construct a confidence set for unknown mean μ in a Hilbert space, based on $\{X_k\}$ in Example 1, for which the probability of coverage is at least $1-\alpha(0 < \alpha < 1)$. To obtain confidence regions for μ, Theorem 5 is useful, that is, the following result is useful :

$$n^{\frac{1}{2}}\left(\bar{X}_n - \mu\right) \xrightarrow{\mathcal{D}} N(0, A\Gamma A^*), \tag{3.19}$$

where $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$, N denotes a Gaussian random variable on H, $\xrightarrow{\mathcal{D}}$ indicates convergence in distribution, Γ is defined in Theorem 4, $A = \sum_{j=0}^{\infty} a_j$ and A^* denotes the adjoint operator of A. Based on (3.19), if $A\Gamma A^*$ is known and $A\Gamma A^*$ is nonsingular, then an asymptotic $(1-\alpha)$ confidence set for μ is $\{\theta : n(\bar{X}_n - \theta)'(A\Gamma A^*)^{-1}(\bar{X}_n - \theta) \leq \chi^2_{1-\alpha}\}$, where $\chi^2_{1-\alpha}$ is the upper $1-\alpha$ point of a chi-square distribution.

Concluding Remark: In the future, we will attempt to obtain a new method of the proof of Theorem 5 by proving the following proposition: Let $\{\xi_n, n \geq 1\}$ be a strictly stationary sequence of H-valued NA random variables. Assume that there exists a positive constant such that for every sequence of linear bounded operators $\{d_k, k \in \mathbb{Z}\}$ on H, and for every $0 \leq p < q < \infty$,

$$E\left\|\sum_{j=p}^{q} d_j\xi_j\right\| \leq K\sum_{j=p}^{q} \|d_j\|^2 < \infty. \tag{3.20}$$

Then

$$\frac{1}{n}E\left\|\sum_{k=1}^{n} X_k - A\sum_{k=1}^{n} \xi_k\right\|^2 \to 0 \quad as\ n \to \infty.$$

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