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Projected Circular and l-Axial Skew-Normal Distributions

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Abstract

We developed the projected l-axial skew-normal(LASN) family of distributions for l-axial data. The LASN family of distributions contains the semicircular skew-normal(SCSN) and the circular skew-normal(CSN) families of distributions as special cases. The LASN densities are similar to the wrapped skew-normal densities for the small values of the scale parameter. However CSN densities have more heavy tails than those of the wrapped skew-normal densities on the circle. Furthermore the CSN densities have two modes as the scale parameter increases. The LASN distribution has very convenient mathematical features. We extend the LASN family of distributions to a bivariate case.

Keywords: Projection, skewed l-axial data, l-axial distribution, bimodality.

1. Introduction

Circular data are encountered frequently in the fields of astronomy, biology, geology, medicine and meteorology, such as when investigating the origins of comets, solving bird navigational problems, interpreting paleomagnetic currents, assessing variations in the onset of leukemia and analyzing wind directions.

However, modeling circular data on a full circle is sometimes unnecessary. This may occur if the subject of interest is the strike of bedding or the orientation of land forms, such as drumlins. Therefore, rather than a single-headed vector, the observation is actually a double-headed vector. When measuring a strike, for example, N45W and S45E are the same. A similar example is a sea turtle emerging from the ocean in search of a nesting site on dry land. These data are called semicircular data.

As most circular distributions are applicable for observations modulo 2π, another distribution must be assumed. To model semicircular data, there are two symmetric distribution versions. One was developed by Jones (1968) and is essentially a simple transformation of a circular normal distribution. The other is the semicircular normal(SCN) distribution developed by Guardiola (2004). This distribution is obtained by projecting a normal distribution over a semicircular segment.

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Although symmetrically distributed (semi) circular data are rare (Mardia and Jupp, 2000), most work (Jammalamadaka and SenGupta, 2001; Fisher, 1993) has been developed for symmetrical models. Recently, there has been increasing interest in models for skew linear data. Azzalini (1985) provided results for the skew-normal distribution on a line, and Azzalini and Dalla Valle (1996) and Azzalini and Capitanio (1999) considered the multivariate extension of the distribution. Similar to Guardiola (2004), we project a skew-normal distribution over a semicircular segment. The resulting distribution is called the SCSN distribution. Furthermore it is extended to the LASN distribution by a simple transformation. Sometimes, measurements result in any arc of arbitrary length, say $2\pi/l, l = 1, 2, \ldots$. So the LASN family of distributions contains the SCSN and the CSN families of distributions as special cases. Furthermore the SCSN distribution contains the SCN distribution (Guardiola, 2004) as a special case when the shape parameter equals 0.

Recently, some skewed circular models have been developed. Pewsey (2000a, 2006) developed the wrapped skew-normal distribution on a circle. Pewsey (2008) nicely introduced the four-parameter wrapped stable family of unimodal distributions as a highly flexible model for directional data observed on the unit circle. Jammalamadaka and Kozubowski (2004) obtained the wrapped exponential distribution and the wrapped Laplace distribution on a circle. For the comparison purpose, some figures are plotted. Based on these figures, we find the followings. The LASN densities are similar to the wrapped skew-normal densities for the small values of a parameter φ. However CSN densities have more heavy tails than those of the wrapped skew-normal densities on the circle. Furthermore the CSN densities have two modes as a parameter φ increases. One more useful fact is that the LASN distribution can also handle l-axial data.

Section 2 defines the distribution and lists some of its basic properties. In Section 3, we estimate parameters of the LASN distribution using a maximum likelihood method. Two examples are also given. The paper concludes in Section 4 with an extension of the LASN distribution: the bivariate version.

2. The Class of Projected l-Axial Skew-Normal Distributions

2.1. Definition and some basic properties

The SCSN distribution is obtained by projecting a skew-normal distribution over a semicircular segment. Let X have a skew-normal distribution with location parameter 0, scale parameter σ and shape parameter λ, *i.e.*, the density of X is

$$\frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)\Phi\left(\lambda\frac{x}{\sigma}\right), \quad -\infty < x < \infty,\ \sigma \in R^+,\ \lambda \in R, \tag{2.1}$$

where ϕ and Φ are the standard normal density and distribution function, respectively. For brevity, we shall also say that X is $\mathrm{SN}(0, \sigma^2, \lambda)$. For a positive real number r, define the angle θ by $\theta = \tan^{-1}(x/r)$, where $\theta \in (-\pi/2, \pi/2)$. Hence, $x = r\tan(\theta)$. Obviously, the *pdf* of θ is given by

$$\frac{2}{\varphi}\sec^2(\theta)\phi\left(\frac{\tan(\theta)}{\varphi}\right)\Phi\left(\lambda\frac{\tan(\theta)}{\varphi}\right), \quad \varphi = \frac{\sigma}{r},\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{2.2}$$

Occasionally, measurements result in any arc of arbitrary length, say $2\pi/l$, $l = 1, 2, \ldots$, so it is desirable to extend the SCSN distribution. To construct the LASN distribution, we consider the *pdf* (2.2) and use the transformation $\theta^* = 2\theta/l$, $l = 1, 2, \ldots$. The *pdf* of θ^* is given by

$$\frac{l}{\varphi}\sec^2(l\theta^*/2)\,\phi\left(\frac{\tan(l\theta^*/2)}{\varphi}\right)\Phi\left(\lambda\frac{\tan(l\theta^*/2)}{\varphi}\right), \quad -\frac{\pi}{l} < \theta^* < \frac{\pi}{l}. \tag{2.3}$$

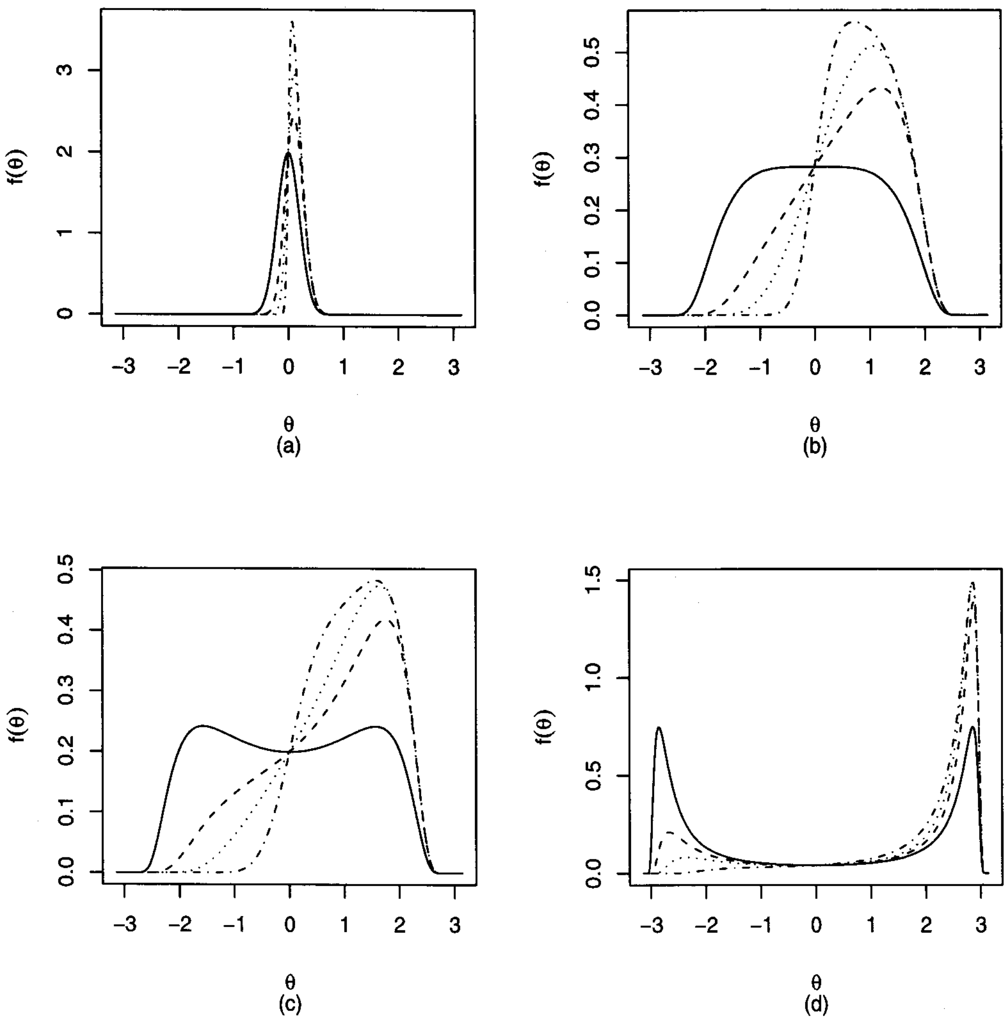

Figure 2.1. CSN densities for different values of λ and φ. For each plot the line types correspond to $\lambda = 0$(—), $\lambda = 1$(− −), $\lambda = 2(\cdots)$ and $\lambda = 5(\cdot - \cdot)$. Figure (a) to (d) are plotted for $\varphi = 0.1$, $\varphi = 1/\sqrt{2}$, $\varphi = 1$ and $\varphi = 5$, respectively.

When $l = 1$, it becomes the *pdf* of the CSN distribution. Note that when $l = 2$, the *pdf* (2.3) is the same as the *pdf* (2.2), the SCSN *pdf.*

More generally, we introduce the parameter μ as the location parameter for the LASN distribution and write the *pdf* as

$$\frac{l}{\varphi}\sec^2(l(\theta^* - \mu)/2)\phi\left(\frac{\tan(l(\theta^* - \mu)/2)}{\varphi}\right)\Phi\left(\lambda\frac{\tan(l(\theta^* - \mu)/2)}{\varphi}\right), \quad -\frac{\pi}{l} < \theta^* < \frac{\pi}{l}, \qquad (2.4)$$

where $-\pi/l < \mu < \pi/l$. Then, we say that θ^* is an LASN random variable with parameters μ, φ^2, and λ; for brevity, we shall also say that θ^* is LASN$(\mu, \varphi^2, \lambda)$. When the shape parameter $\lambda = 0$, the *pdf* becomes that of the *l*-axial normal distribution(LAN) distribution; similarly, we shall say that θ^* is LAN(μ, φ^2).

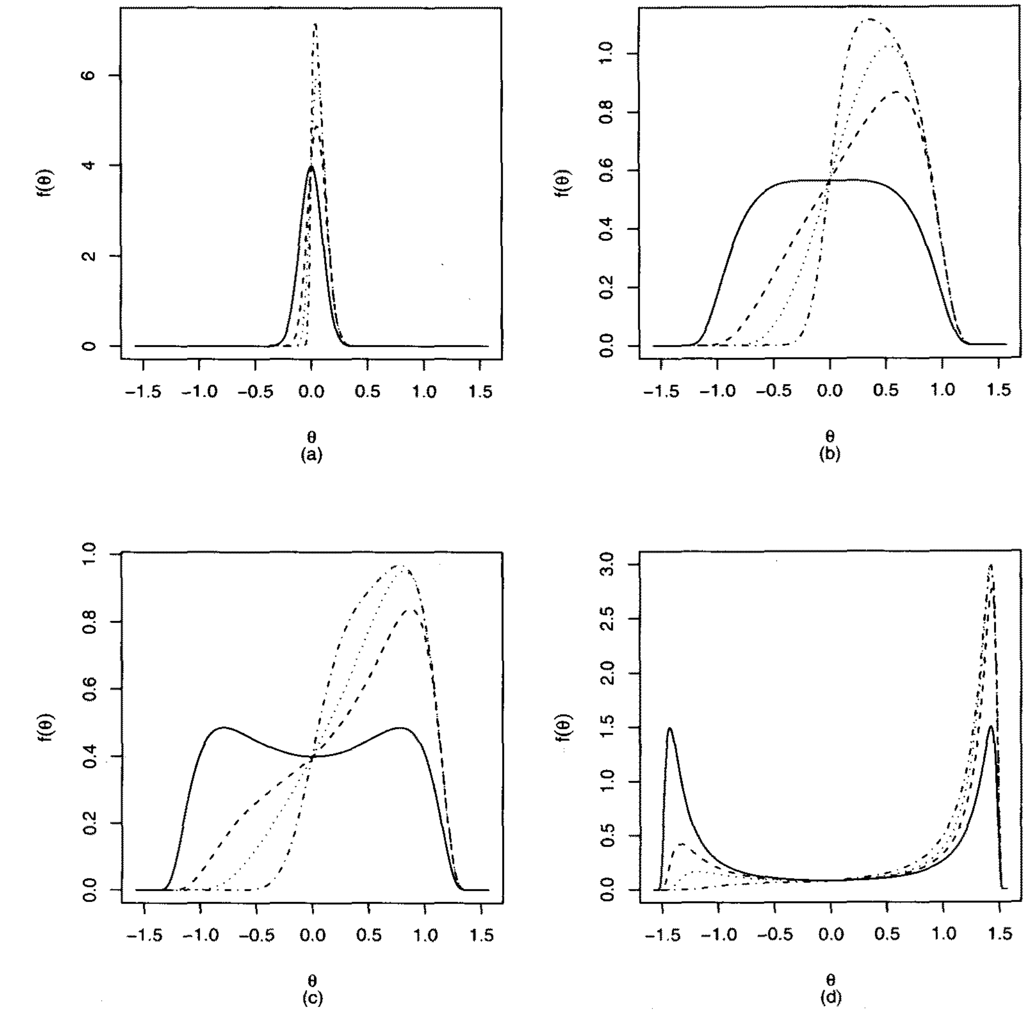

Figure 2.2. SCSN densities for different values of λ and φ. For each plot the line types correspond to $\lambda = 0$(—), $\lambda = 1$(− −), $\lambda = 2(\cdots)$ and $\lambda = 5(\cdot - \cdot)$. Figure (a) to (d) are plotted for $\varphi = 0.1$, $\varphi = 1/\sqrt{2}$, $\varphi = 1$ and $\varphi = 5$, respectively.

Figure 2.1 shows the CSN densities with some combinations of φ and λ. When $\lambda = 0$, the CSN densities denote the CN densities. As φ increases the CSN densities have two modes and large spread. The parameter λ determines the shape of the CSN densities. The parameter μ is the location parameter evidently. Figure 2.2 shows the SCSN densities with the same parameter combinations of Figure 2.1. When $\lambda = 0$, the SCSN densities becomes SCN densities. All parameters have the same role as the CSN densities. For the comparison purpose, Figure 2.3(a) shows the wrapped skew-normal densities (Pewsey, 2000a) for the same parameter combinations of Figure 2.1(b). The density of the wrapped skew-normal distribution is as follows:

$$\frac{2}{\varphi}\sum_{r=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi r}{\varphi}\right)\Phi\left\{\lambda\left(\frac{\theta+2\pi r}{\varphi}\right)\right\}, \quad 0 \le \theta < 2\pi,\ \varphi > 0,\ -\infty < \lambda < \infty.$$

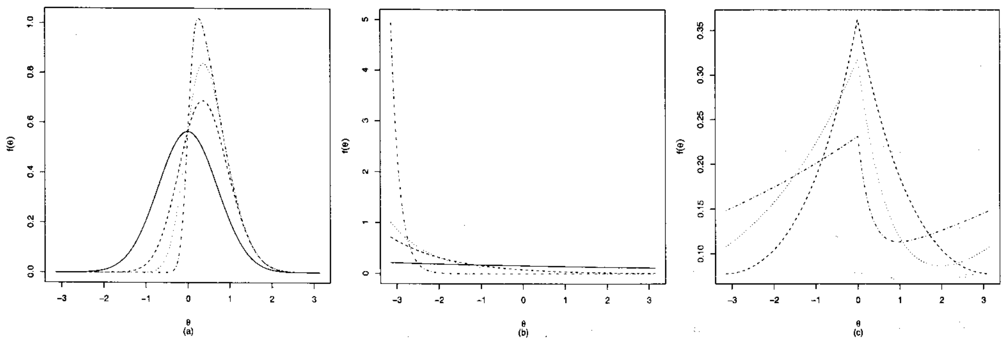

Figure 2.3. (a) Wrapped skew-normal densities for different values of λ and $\varphi = 1/\sqrt{2}$. The line types correspond to $\lambda = 0$(—), $\lambda = 1$(– –), $\lambda = 2$($\cdots$) and $\lambda = 5$($\cdot - \cdot$). (b) Wrapped exponential densities for different values of λ. The line types correspond to $\lambda = 0.1$(—), $\lambda = 1/\sqrt{2}$(– –), $\lambda = 1$($\cdots$) and $\lambda = 5$($\cdot - \cdot$). (c) Wrapped Laplace densities for different values of κ and $\lambda = 1/\sqrt{2}$. The line types correspond to $\kappa = 1$(– –), $\kappa = 2$($\cdots$) and $\kappa = 5$($\cdot - \cdot$).

We used $r = 7$ for Figure 2.3(a). Figure 2.3(b) shows the wrapped exponential densities with the following *pdf* (Jammalamadaka and Kozubowski, 2004):

$$\frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}, \quad 0 \le \theta < 2\pi, \quad -\infty < \lambda < \infty.$$

The parameter λ is the scale parameter so we changed it similar to φ of Figure 2.1. Figure 2.3 (c) shows the wrapped Laplace densities with the following *pdf* (Jammalamadaka and Kozubowski, 2004):

$$\frac{\lambda\kappa}{1+\kappa^2}\left(\frac{e^{-\lambda\kappa\theta}}{1 - e^{-2\pi\lambda\kappa}} + \frac{e^{\frac{\lambda}{\kappa}\theta}}{e^{2\pi\frac{\lambda}{\kappa}} - 1}\right), \quad 0 \le \theta < 2\pi, \ \kappa > 0, \quad -\infty < \lambda < \infty.$$

For $\kappa = 1$, we obtain the symmetric wrapped Laplace distribution, so we change the values of κ similar to φ of Figure 2.1. All densities of Figure 2.3 are plotted over the support $[-\pi, \pi)$ for the comparison purpose. The wrapped Laplace densities are shifted for the comparison purpose. From the figures, it is obvious that the wrapped skew-normal densities are similar to the CSN densities for the small values of φ. However CSN densities have more heavy tails than those of the wrapped skew-normal densities on the circle. This difference mainly comes from the absolute value of Jacobian of the CSN distribution. Furthermore CSN densities have two modes as φ increases. Hence the CSN distribution can be used as a model for skewed bimodal data.

It is straightforward to generate samples from an LASN distribution. First, generate samples from a skew-normal distribution (Azzalini and Capitanio, 1999), and then use the inverse transformation, $\theta^* = \mu + 2/l \tan^{-1}(x/r)$, $r = \sigma/\varphi$. Similarly, the cdf of an LASN distribution is $F(\theta^*; \mu, \varphi^2, \lambda) = \Phi\left(1/\varphi \tan(l(\theta^* - \mu)/2)\right) - 2T\left(1/\varphi \tan(l(\theta^* - \mu)/2), \lambda\right)$, where the function $T(h, a)$ is that studied by Owen (1956). The following properties of the density (2.4) with $\mu = 0$ and $\sigma = \varphi = 1$ follow from the definition of an LASN distribution, Azzalini (2005) and the properties of $T(h, a)$ (Owen, 1956).

(a) If $\lambda = 0$, we obtain the LAN(0, 1) density. It's density function is as follows:

$$\frac{l}{2}\sec^2(l\theta^*/2)\,\phi\left(\tan(l\theta^*/2)\right), \quad -\frac{\pi}{l} < \theta^* < \frac{\pi}{l}.$$

(b) If $\theta^* \sim \text{LASN}(0,1,\lambda)$, then $-\theta^* \sim \text{LASN}(0,1,-\lambda)$ by taking simple transformation, $y = -\theta^*$.

(c) As $\lambda \to \infty$, the $\text{LASN}(0,1,\lambda)$ density converges pointwise to the half-LAN(0,1) density, *i.e.*, $l\sec^2(l\theta^*/2)\phi(\tan(l\theta^*/2))$, $0 \le \theta^* < \pi/l$.

(d) If $\theta^* \sim \text{LASN}(0,1,\lambda)$, then $\tan^2(l\theta^*/2) \sim \chi_1^2$.

(e) $F(\theta^*;0,1,-\lambda) = 1 - F(-\theta^*;0,1,\lambda)$ by the properties of $\Phi(\cdot)$ and $T(h,a)$.

(f) $F(\theta^*;0,1,1) = \Phi^2(\tan(l\theta^*/2))$ by the property of $T(h,a)$.

(g) $\sup_{\theta^*} |\Phi(\tan(l\theta^*/2)) - F(\theta^*;0,1,\lambda)| = 1/\pi \tan^{-1}(|\lambda|)$.

(h) If $U \sim N(0,1)$ is independent of $\theta^* \sim \text{LASN}(0,1,\lambda)$, then

$$\frac{aU + b\tan(l\theta^*/2)}{\sqrt{a^2+b^2}} \sim \text{SN}\left(\frac{b\lambda}{\sqrt{a^2(1+\lambda^2)+b^2}}\right)$$

for any $a, b \in R$.

2.2. Trigonometric moments

Similar to those of any circular density, trigonometric moments of the LASN distribution are defined as follows: $\phi_p = Ee^{ip\theta^*} = \alpha_p + i\beta_p = E\cos(p\theta^*) + iE\sin(p\theta^*)$, $p = 0, \pm1, \pm2, \ldots$. We assumed that the location parameter $\mu = 0$ without loss of generality. Even moments of the $\text{SN}(\lambda)$ distribution are independent of the shape parameter λ, *i.e.*, $EX^{2n} = 1 \cdot 3 \cdots (2n-1)$ (Azzalini, 1985). We extend this result to the following Lemma 2.1.

Lemma 2.1. *If $X \sim SN(\lambda)$ and g is an even function, then $Eg(X)$ is independent of λ. Furthermore $Eg(X)$ is equal to $Eg(Z)$, where $Z \sim N(0,1)$.*

Proof.

$$\begin{aligned} Eg(X) &= \int_{-\infty}^{\infty} g(x)2\phi(x)\Phi(\lambda x)dx \\ &= \int_{-\infty}^{\infty} g(x)2\phi(x)(1-\Phi(-\lambda x))dx \\ &= 2\ Eg(Z) - \int_{-\infty}^{\infty} g(x)2\phi(x)\Phi(-\lambda x)dx, \end{aligned}$$

where $Z \sim N(0,1)$. Let $y = -x$ at the last integral and use the assumption that g is an even function, then we have $Eg(X) = 2Eg(Z) - Eg(X)$. So $Eg(X) = Eg(Z)$ which is independent of λ. □

Theorem 2.1. *If $\theta^* \sim LASN(0, \varphi^2, \lambda)$, then $\alpha_p = E\cos(p\theta^*)$ do not depend on the shape parameter λ.*

Proof. After a transformation, $x = 1/\varphi\tan(l\theta^*/2)$, a random variable x follows a skew-normal distribution with a parameter λ. Furthermore $\cos(2p\tan^{-1}(\varphi\ x)/l)$ is an even function of a skew-normal random variable x. Hence the result follows immediately by the Lemma 2.1. □

From now on we concentrate on the trigonometric moments of the SCSN and CSN distributions. Since most of real data are semicircular or circular data. In general, the k^{th} cosine moment, $\alpha_k = E\cos(k\theta^*)$, of the CSN distribution is the same as the $2k^{th}$ cosine moment, $\alpha_{2k} = E\cos(2k\theta)$, of the SCSN distribution. Furthermore the k^{th} sine moment, $\beta_k = E\sin(k\theta^*)$, of the CSN distribution is the same as the $2k^{th}$ sine moment, $\beta_{2k} = E\sin(2k\theta)$, of the SCSN distribution. Based on this fact, we concentrate on the trigonometric moments of the SCSN distribution afterwards. The following Lemma 2.2 is needed to derive the trigonometric moments of the SCSN distribution.

Lemma 2.2. *Using $x = \tan(\theta)$, the multiple-angle formulas are given in terms of x by*

$$\begin{aligned}\cos(p\theta) &= \sum_{k=0}^{p}\binom{p}{k}c^1_{p-k}x^{p-k}(1+x^2)^{-\frac{p}{2}},\\ \sin(p\theta) &= \sum_{k=0}^{p}\binom{p}{k}c^2_{p-k}x^{p-k}(1+x^2)^{-\frac{p}{2}},\end{aligned}$$

where $\sin(\theta) = x/\sqrt{1+x^2}$, $\cos(\theta) = 1/\sqrt{1+x^2}$,

$$\cos\left[(p-k)\frac{\pi}{2}\right] = c^1_{p-k} = \begin{cases} 1, & \text{if } p-k = 4m,\\ 0, & \text{if } p-k = 2m+1,\\ -1, & \text{if } p-k = 4m+2,\end{cases}$$

and

$$\sin\left[(p-k)\frac{\pi}{2}\right] = c^2_{p-k} = \begin{cases} 1, & \text{if } p-k = 4m+1,\\ 0, & \text{if } p-k = 2m,\\ -1, & \text{if } p-k = 4m+3,\end{cases}$$

where $m = 0, 1, 2, \ldots$.

Proof. The proof is straightforward using the multiple-angle formulas and a transformation $x = \tan(\theta)$. □

Theorem 2.2. *Let $\theta \sim SCSN(0, \varphi^2, \lambda)$, then the cosine moments are as follows:*

$$\alpha_p = \frac{1}{\sqrt{2\pi}\varphi}\sum_{k\in R_c}\binom{p}{k}c^1_{p-k}\Gamma\left(\frac{p-k+1}{2}\right)\Psi\left(\frac{p-k+1}{2}, \frac{3-k}{2}; \frac{1}{2\varphi^2}\right),$$

$$\alpha_{-p} = \alpha_p, \quad p \in \aleph. \text{ Furthermore } \alpha_0 = 1, \tag{2.5}$$

where $\Psi(\alpha, \gamma; z)$ has an integral representation as $1/\Gamma(\alpha)\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}dt$ (the integral formula 9.211.4 of Gradshteyn and Ryzhik, 2007). $\Psi(\alpha, \gamma; z)$ is related to a confluent hypergeometric function $\Psi(\alpha, \gamma; z)$ (the formula 9.210.2 of Gradshteyn and Ryzhik, 2007) as follows:

$$\Psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}z^{1-\gamma}\Phi(\alpha-\gamma+1, 2-\gamma; z).$$

A confluent hypergeometric function has a second notation ${}_1F_1(\alpha; \gamma; z)$. R_c denote the set of values such that $\{k | k = 0, 1, \ldots, p$ satisfying $\{p-k = 2m,\ m = 0, 1, 2, \ldots\}\}$.

Proof. By the Theorem 2.1, the cosine moments do not depend on the shape parameter and are the same as those of the SCN cosine moments. We first use the transformation $x = \tan(\theta)$. So $\cos(p\theta)$ can be expressed as a function of x using Lemma 2.2. The integrand is an even function of x when $p-k$ is even so we use this property. Note that when $p-k$ is odd, the integral is 0 since the integrand is an odd function of x. We change the order of the summation and integration. And apply a transformation, $y = x^2$, then an intermediate expression is

$$\alpha_p = \frac{1}{\sqrt{2\pi}\varphi} \sum_{k\in R_c} \binom{p}{k} c^1_{p-k} \int_0^\infty y^{\frac{p-k-1}{2}} (1+y)^{-\frac{p}{2}} \exp\left(-\frac{y}{2\varphi^2}\right) dy,$$

where R_c denote the set of values such that $\{k|k = 0, 1, \ldots, p$ satisfying $\{p-k = 2m, m = 0, 1, 2, \ldots\}\}$. The result follows immediately by the integral formula 9.211.4 (Gradshteyn and Ryzhik, 2007). By the property of cosine function, the remaining results are obvious. □

For example, the first and second cosine moment can be derived in more simpler forms using slightly different approach. The first and second $\alpha_p, p = 1, 2$ are given by Guardiola (2004), who only obtained these two moments:

$$\begin{aligned}\alpha_1 &= \frac{1}{\sqrt{2\pi}\varphi} \exp\left(\frac{1}{4\varphi^2}\right) K_0\left(\frac{1}{4\varphi^2}\right),\\ \alpha_2 &= \frac{2\sqrt{2\pi}}{\varphi} \exp\left(\frac{1}{2\varphi^2}\right)\left(1-\Phi\left(\frac{1}{\varphi}\right)\right)-1,\end{aligned} \tag{2.6}$$

where K_0 is the modified Bessel function of the second kind (Section 9.6 of Abramowitz and Stegun, 1972). Guardiola (2004) used Mathematica to derive two trigonometric moments. The analytical proof is the process of using some transformations. First, use the transformation $x = \tan(\theta)$ and then use $y = 2\sinh^{-1}(x)$. The result α_1 follows by the integral formula 9.6.24 of Abramowitz and Stegun (1972). To obtain α_2, we use the transformation $x = \tan(\theta)$ followed by $y = x^2$ and the result is immediate by the integral formula 7.4.9 of Abramowitz and Stegun (1972). Equality of (2.5) and (2.6) can be proved by direct application of Lemma 2.2.

Unlike any other symmetric circular density, $\beta_p = E\sin(p\theta)$ are not zero and depend on the shape parameter λ. Obviously, if the shape parameter $\lambda = 0$, then all $\beta_p = E\sin(p\theta)$ are 0 as the density is symmetric about 0. To evaluate integrals, we first denote the cdf of the standard normal distribution as an infinite sum (Section 26.2 of Abramowitz and Stegun, 1972) as follows:

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! 2^n (2n+1)}, \quad x \geq 0. \tag{2.7}$$

Theorem 2.3. *Let $\theta \sim SCSN(0, \varphi^2, \lambda)$, then the sine moments are as follows:*

$$\beta_p = \sum_{k,n\in R_s} \binom{p}{k} c^2_{p-k}\, c_n\, \Gamma\left(n+1+\frac{p-k}{2}\right) \Psi\left(n+1+\frac{p-k}{2}, n+2-\frac{k}{2}; \frac{1}{2\varphi^2}\right)$$

$$\beta_{-p} = -\beta_p, \quad p \in \aleph. \quad \textit{Furthermore } \beta_0 = 0,$$

where $c_n = \{(-1)^n \lambda^{2n+1}\}/\{\pi n! 2^n (2n+1)\varphi^{2n+2}\}$ is a constant depending on n and $\Gamma(\cdot)$ is a standard gamma function. R_s denote the set of values such that $\{k, n|k = 0, 1, \ldots, p,\ n = 0, 1, \ldots, \infty$ satisfying $\{2n + p - k + 1 = 2m,\ m = 0, 1, 2, \ldots\}\}$.

Proof. We use the transformation $x = \tan(\theta)$. So $\sin(p\theta)$ and the cdf of the standard normal distribution can be expressed as functions of x using Lemma 2.2 and (2.7), respectively. Furthermore the first integrand is an even function of x when $p-k$ is even so we use this property and in this case c_{p-k}^2 is 0. Note that $p-k$ is odd the integral is 0. Hence the first integral multiplied by c_{p-k}^2 is 0 whether $p-k$ is odd or even. The second integrand is odd function of x when $2n+p-k+1$ is odd so the integral is 0. And the second integrand is even function of x when $2n+p-k+1$ is even so we use this property. We change the order of the summation and integration. And apply a transformation, $y = x^2$, then the result follows immediately by the integral formula 9.211.4 (Gradshteyn and Ryzhik, 2007). By the property of sine function, the remaining results are obvious.

For example we may also derive the first two sine moments in more simpler forms using direct integration. To derive β_1, first take $x = \tan(\theta)$ and then use (2.7) followed by $y = x^2$. Using the integral formula 9.211.4 and the formula 9.210.2 (Gradshteyn and Ryzhik, 2007), the result follows.

$$\beta_1 = \sum_{n=0}^{\infty} c_n \left\{ 2^{n+1}\varphi^{2n+2}\Gamma(1+n)\,{}_1F_1\left(\frac{1}{2};-n;\frac{1}{2\varphi^2}\right) + \frac{1}{\sqrt{\pi}}\Gamma(-1-n)\Gamma\left(\frac{3}{2}+n\right)\;{}_1F_1\left(\frac{3}{2}+n;2+n;\frac{1}{2\varphi^2}\right)\right\},$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function (Section 13.2 of Abramowitz and Stegun, 1972).

$$\beta_2 = 2\exp\left(\frac{1}{2\varphi^2}\right)\sum_{n=0}^{\infty} c_n\Gamma\left(\frac{3}{2}+n\right)\Gamma\left(-\frac{1}{2}-n,\frac{1}{2\varphi^2}\right),$$

where $\Gamma(\cdot,\cdot)$ is the standard incomplete gamma function. To show β_2, first take $x = \tan(\theta)$ and then use (2.7) followed by $y = x^2$. Using the integral formula 3.383.10 (Gradshteyn and Ryzhik, 2007), the result follows. Similar to the equality of (2.5) and (2.6), these sine moments are equal by direct application of Lemma 2.2.

Note that $\beta_p, i = 1,2$ depend on the shape parameter λ and the scale parameter φ. Hence, if the shape parameter λ equals 0, then $\beta_p, i = 1,2$ equals 0 as in any other circular sine moment.

2.3. Asymptotics

We consider the behavior of the LASN distribution when $\varphi \to 0$. Suppose the distribution of θ^* follows LASN$(\mu, \varphi^2, \lambda)$. Let $\alpha = l(\theta^* - \mu)/(2\varphi)$, and then use the change of variable technique. For sufficiently small φ, we have $\tan(\alpha\varphi) \simeq \alpha\varphi$, and $\sec(\alpha\varphi) \simeq 1$ by the first order approximation of the Taylor series expansion. Hence, the distribution of θ^* becomes SN$(\mu, 4\varphi^2/l^2, \lambda)$.

Another useful asymptotic distribution is the sample dispersion about the population location parameter μ, *i.e.*, $n - V = n - \sum_{i=1}^{n}\cos(\theta_i^* - \mu)$. If $\theta_1^*, \ldots, \theta_n^*$ are a random sample from LASN$(\mu, 4\varphi^2/l^2, \lambda)$, and then $\theta_i^* - \mu \sim$ SN$(0, 4\varphi^2/l^2, \lambda)$ for sufficiently small φ. Since $\cos(\theta_i^* - \mu) \simeq 1 - (\theta_i^* - \mu)^2/2$ for small $\theta_i^* - \mu$ by the small angle theory, $\{l(\theta_i^* - \mu)/(2\varphi)\}^2 \simeq l^2(1 - \cos(\theta_i^* - \mu))/(2\varphi^2) \sim \chi_1^2$ from the property H of the skew-normal distribution (Azzalini, 1985). Hence, the distribution of $n - V$ follows $2\varphi^2/l^2\chi_n^2$ by the additive property of χ^2 distribution.

3. Parameter Estimation and Examples

In this section, we will assume that $r = 1$ without loss of generality, so $\varphi = \sigma$.

3.1. Maximum likelihood estimation

The LASN distribution is a simple transformed one, so we first consider the parameter estimation of the SN distribution. Parameter estimation of a skew-normal distribution is cumbersome, as the direct parameterization of density (2.1) is parameter redundant for the important case of the normal distribution. This case corresponds to $\lambda = 0$. The implications of parameter redundancy are discussed in Azzalini (1985), Azzalini and Capitanio (1999) and Pewsey (2000a, 2000b). To resolve this problem, Azzalini (1985) introduced the centered parameterization of the density (2.1). Under this different parameterization,

$$Y_C = \nu + \tau\left(\frac{Z - E(Z)}{\sqrt{\mathrm{var}(Z)}}\right), \quad -\infty < \nu < \infty,\ \tau > 0$$

is a skew-normal random variable with mean ν, standard deviation τ, and coefficient of skewness γ_1, where $-0.99527 < \gamma_1 < 0.99527$ and $Z \sim \mathrm{SN}(0, 1, \lambda)$. We will denote this relation by $Y_C \sim \mathrm{SN}_C(\nu, \tau^2, \gamma_1)$; here and subsequently, the subscript "C" indicates a reference to the centered parameterization. Note that $Y_D = X = \sigma Z \sim \mathrm{SN}_D(0, \sigma^2, \lambda)$. The subscript "D" indicates a reference to the direct parameterization. The direct parameters are related to the centered parameters according to

$$\nu = c\gamma_1^{\frac{1}{3}}\tau, \qquad \sigma = \tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}, \qquad \lambda = \frac{c\gamma_1^{\frac{1}{3}}}{\sqrt{b^2 + c^2(b^2 - 1)\gamma_1^{\frac{2}{3}}}}, \tag{3.1}$$

where $b = \sqrt{2/\pi}$ and $c = (2/(4 - \pi))^{1/3}$ since the location parameter of the direct parameterization is 0. Hence, the density of Y_C is given by

$$\frac{2}{\tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}}\phi\left(\frac{y}{\tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}}\right)\Phi\left(\frac{c\gamma_1^{\frac{1}{3}}}{\sqrt{b^2 + c^2(b^2 - 1)\gamma_1^{\frac{2}{3}}}}\frac{y}{\tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}}\right).$$

As mentioned above, similar problems occur for the density (2.4). To avoid such problems, it is desirable to project Y_C over a semicircular segment. And then we use the transformation $\theta^* = 2\theta/l$ to cover l-axial data. So the projected density after introducing a location parameter μ is given by

$$\frac{l\sec^2(l(\theta^* - \mu)/2)}{\tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}}\phi\left(\frac{\tan(l(\theta^* - \mu)/2)}{\tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}}\right)\Phi\left(\frac{c\gamma_1^{\frac{1}{3}}}{\sqrt{b^2 + c^2(b^2 - 1)\gamma_1^{\frac{2}{3}}}}\frac{\tan(l(\theta^* - \mu)/2)}{\tau\sqrt{1 + c^2\gamma_1^{\frac{2}{3}}}}\right), \tag{3.2}$$

for $-\pi/l < \theta^* < \pi/l$ and $-\pi/l < \mu < \pi/l$. We denote this distribution as $\theta_C^* \sim \mathrm{LASN}_C(\mu, \tau^2, \gamma_1)$. The negative log-likelihood for a random sample of size n, $\theta^* = (\theta_1^*, \ldots, \theta_n^*)$, from the LASN distribution with centered parameters τ^2 and γ_1 and the location parameter μ is given by

$$-l\left(\mu, \tau^2, \gamma_1; \theta^*\right) = -n\log\left(\frac{l}{\sqrt{2\pi}\tau}\right) + \frac{n}{2}\log\left(1 + c^2\gamma_1^{\frac{2}{3}}\right) + \frac{\sum_{i=1}^{n}\tan^2(l(\theta_i^* - \mu)/2)}{2\tau^2\left(1 + c^2\gamma_1^{\frac{2}{3}}\right)}$$

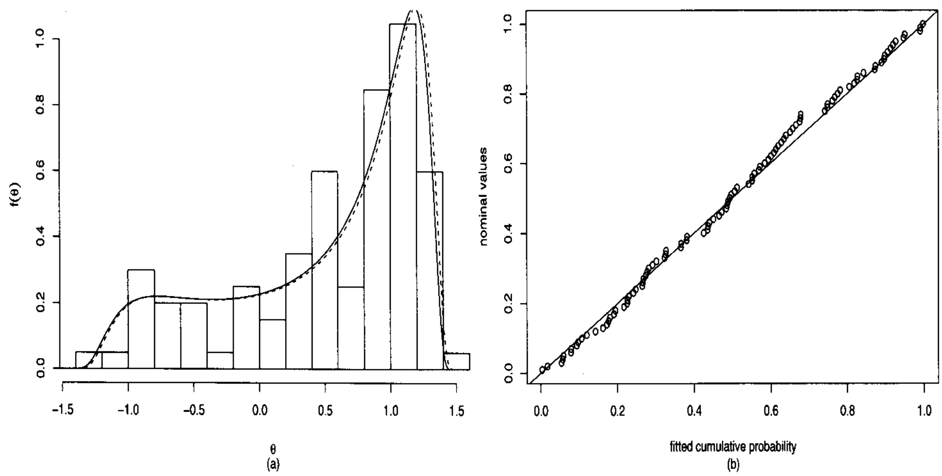

Figure 3.1. Histogram with *pdfs* (the solid line represents the original *pdf* and the dashed line corresponds to the fitted *pdf*) and Healy's plot

$$+\sum_{i=1}^{n}\log\left(\cos^2(l(\theta_i^*-\mu)/2)\right)-\sum_{i=1}^{n}\log\Phi\left(\frac{c\gamma_1^{\frac{1}{3}}}{\sqrt{b^2+c^2(b^2-1)\gamma_1^{\frac{2}{3}}}}\frac{\tan(l(\theta_i^*-\mu)/2)}{\tau\sqrt{1+c^2\gamma_1^{\frac{2}{3}}}}\right).$$

For the following examples, the corresponding estimates have been computed by the Nelder-Mead simplex method (Nelder and Mead, 1965) combined with a grid of starting values. Using a grid of initial values rather than a single starting value is advisable since multiple minima can occur on the negative log-likelihood surface. After the negative log-likelihood function has been minimized, it seems preferable to convert back the estimates from (3.2) to the simpler form (2.4) using the relationship (3.1).

3.2. Examples

EXAMPLE 3.1. We simulated a data set of size 100 from an SCSN distribution with $\mu = 0$, $\varphi = 1.75$, and $\lambda = 1$. For this data set, the corresponding maximum likelihood estimates are given by $\hat{\mu} = -0.01$, $\hat{\varphi} = 1.75$ and $\hat{\lambda} = 0.99$. Histogram with *pdfs* and Healy's plot (Healy, 1968) are shown in Figure 3.1. A visual inspection of Figure 3.1 indicates a satisfactory fit of the density to the data. Healy's plot is based on

$$d_i = \frac{\tan^2(l(\theta_i^*-\mu)/2)}{\varphi^2}, \quad (i = 1, \dots, n) \tag{3.3}$$

and is sampled from a χ_1^2−distribution if the fitted model is appropriate, using property (d). In practice, estimates must replace the exact parameter values in Equation (3.3). d_i above must be sorted and plotted against the χ_1^2 percentage points. Equivalently, the cumulative χ_1^2− probabilities can be plotted against their nominal values $1/n, 2/n, \dots, 1$; the points should lie on the bisection line of the quadrant.

EXAMPLE 3.2. We simulated a data set of size 100 from an CSN distribution with $\mu = 0$, $\varphi = 5$ and $\lambda = 1$. For this data set, the corresponding maximum likelihood estimates are given by $\hat{\mu} = -0.009$,

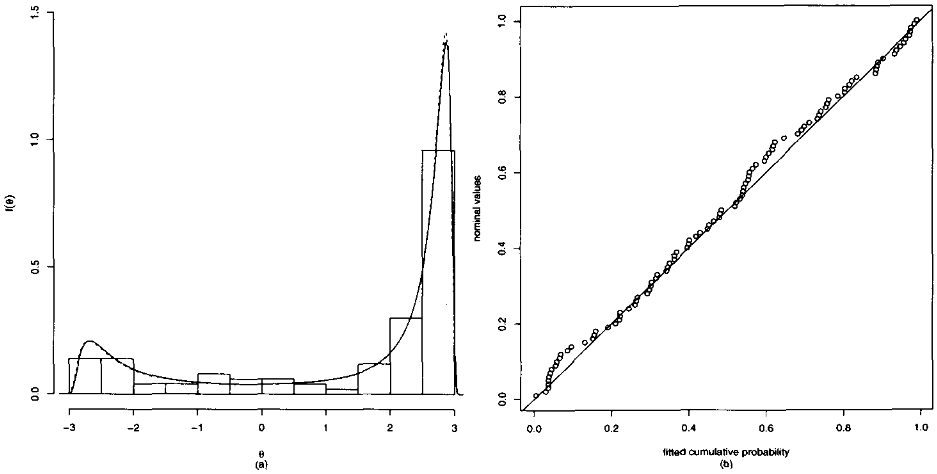

Figure 3.2. Histogram with *pdfs* (the solid line represents the original *pdf* and the dashed line corresponds to the fitted *pdf*) and Healy's plot

$\hat{\varphi} = 5.12$ and $\hat{\lambda} = 1.02$. Histogram with *pdfs* and Healy's plot (Healy, 1968) are shown in Figure 3.2. A visual inspection of Figure 3.2 indicates a satisfactory fit of the density to the data.

4. An Extension and Discussion

We have an extension of the suggested model (2.4). The bivariate version of the LASN distribution is developed which is applicable to any arc of arbitrary length say $2\pi/l$ for $l = 1, 2, \ldots$ in a bivariate context.

We can construct a bivariate LASN distribution in a manner similar to the construction of a univariate LASN distribution. We shall use the same semicircular transformation applied in a bivariate context. The density function of the LASN distribution is defined as

$$\frac{l^2}{2}\sec^2(l\theta_1^*/2)\sec^2(l\theta_2^*/2)\phi_2\left(\begin{pmatrix}\tan(l\theta_1^*/2)\\ \tan(l\theta_2^*/2)\end{pmatrix};\Omega_r\right)\Phi\left(\lambda_1\frac{\tan(l\theta_1^*/2)}{\sqrt{\varphi_{11}}}+\lambda_2\frac{\tan(l\theta_2^*/2)}{\sqrt{\varphi_{22}}}\right),$$

where $-\pi/l < \theta_i^* < \pi/l$, $i = 1, 2$, $\Omega_r = (\varphi_{ij}) = (\sigma_{ij}/r^2)$ and $\Omega = (\sigma_{ij})$ is a 2×2 positive definite matrix. To construct this density, we begin with the bivariate skew-normal density

$$2\phi_2(x;\Omega)\Phi(\lambda^t w^{-1}x), \quad x \in R^2,$$

where $\lambda \in R^2$, and w is the diagonal matrix formed by the standard deviations of Ω. Consider the transformation $x_i = r\tan(\theta_i)$, $i = 1, 2$ and then apply $\theta_i^* = 2\theta_i/l$, $i = 1, 2$ to cover l-axial data. Consequently, the *pdf* of a bivariate LASN distribution is obtained using simple algebra. Similar to univariate case, we may introduce the location parameters. Plugging in $\theta_i^* - \mu_i$, $i = 1, 2$ instead of θ_i^*, we have a bivariate LASN distribution with location parameters, where $-\pi/l < \mu_i < \pi/l$, $i = 1, 2$.

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