J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 16, Number 3 (August 2009), Pages 297–306

GENERALIZED HYERS–ULAM STABILITY OF ADDITIVE FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we obtain the general solution and the generalized Hyers-Ulam stability theorem for an additive functional equation

$$af(x+y)+2f\left(\frac{x}{2}+y\right)+2f\left(x+\frac{y}{2}\right)=(a+3)[f(x)+f(y)]$$

for any fixed integer a.

1. INTRODUCTION

In 1940, S. M. Ulam [11] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms:

We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

One of the most famous functional equations is the additive functional equation which has the following form

$$f(x+y)=f(x)+f(y).$$

It is open called the additive Cauchy functional equation in honor of A. L. Cauchy [1]. The theory of the additive functional equation is frequently applied to the development of theories of other functional equations. Moreover, the properties of the additive equation are powerful tools in almost every field of natural and social

Received by the editors December 24, 2008. Revised July 10, 2009. Accepted July 14, 2009.
2000 *Mathematics Subject Classification.* 39B82, 39B52.
Key words and phrases. Hyers-Ulam stability, additive mapping.
*This study was financially supported by research fund of Chungnam National University in 2008.
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sciences. Every solution of the additive functional equation is called an additive function.

In 1941, D. H. Hyers [4] considered the case of approximately additive mappings $f : E_1 \to E_2$, where E_1 and E_2 are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E_1$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$ and that $L : E_1 \to E_2$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Th. M. Rassias [8] addressed the Hyers' stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference $f(x+y) - f(x) - f(y)$ and proved a considerably generalized result of Hyers. The stability theorem for the case $p > 1$ was proved by Z. Gajda [2]. Let E_1 be a normed space, E_2 a Banach space. Suppose that a mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$ ($x, y \in E_1 \backslash \{0\}$ if $p < 0$), where $\epsilon > 0$ and $p \ne 1$ are constants. Then the limit $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for each $x \in E_1$ and that $T : E_1 \to E_2$ is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \le \frac{2}{|2 - 2^p|}\epsilon\|x\|^p$$

for all $x \in E_1$ ($x \in E_1 \backslash \{0\}$ if $p < 0$). Moreover, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then $T(tx) = tT(x)$ for all $t \in \mathbb{R}$.

It was shown by Z. Gajda [2], as well as by Th. M. Rassias and P. Šemrl [10] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$.

Th. M. Rassias [8] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This concept is known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations [5].

Thereafter, P. Găvruta [3] generalized the stability result of Th. M. Rassias [8]. Let G be an abelian group, E a Banach space and let $\varphi : G \times G \to [0, \infty)$ be a mapping such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. If a mapping $f : G \to E$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique additive mapping $T : G \to E$ satisfying

$$\|f(x) - T(x)\| \leq \frac{1}{2}\tilde{\varphi}(x, x)$$

for all $x \in G$.

During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized *Hyers-Ulam stability* to a number of functional equations and mappings.

Recently, the paper of J. M. Rassias and H.-M. Kim [7] generalizes results obtained for Cauchy-Jensen type mappings and establishes new theorems about the Ulam stability of general Cauchy-Jensen additive mappings.

In this paper, we consider the following generalized functional equation

$$(1) \qquad af(x+y) + 2f\Big(\frac{x}{2} + y\Big) + 2f\Big(x + \frac{y}{2}\Big) = (a+3)[f(x) + f(y)],$$

where a is any fixed integer.

In addition, we will establish the general solution of equation (1) and the generalized Hyers-Ulam stability of the above equation (1).

2. General Solution of Equation (1)

Now, we introduce the following lemma due to A. Najati [6].

Lemma 2.1. *Let X and Y be linear spaces. A mapping $f : X \to Y$ satisfies the equation*

$$\begin{aligned} &f\Big(\frac{x_1 + x_2}{2} + x_3\Big) + f\Big(\frac{x_1 + x_3}{2} + x_2\Big) + f\Big(\frac{x_2 + x_3}{2} + x_1\Big) \\ &= 2[f(x_1) + f(x_2) + f(x_3)] \end{aligned}$$

for all $x_1, x_2, x_3 \in X$ if and only if f is Cauchy additive .

It is noted that the following equation with $x_3 = 0$ in Lemma 2.1

$$\frac{1}{2}f(x_1 + x_2) + f\Big(\frac{x_1}{2} + x_2\Big) + f\Big(x_1 + \frac{x_2}{2}\Big) = 2[f(x_1) + f(x_2)],$$

which is a special case of the equation (1), is equivalent to $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

We will use the following lemma in the proof of the main theorem in Section 3. First of all, we give the general solution of (1).

Lemma 2.2. *Let X and Y be vector spaces. For any fixed integer $a \neq -2$, a mapping $f : X \to Y$ satisfies the equation (1) for all $x, y \in X$ if and only if $f : X \to Y$ is additive.*

Proof. Assume that f satisfies (1) for any integer $a \neq -2$. Putting $x = y = 0$ in (1) yields

$$(a+2)f(0) = 0$$

and we get $f(0) = 0$.

If we put $y := 0$ in (1), we have

$$2f\left(\frac{x}{2}\right) = f(x) \tag{2}$$

for all $x \in X$. Replacing x by $2x$ in (2) yields

$$2f(x) = f(2x) \tag{3}$$

for all $x \in X$. Substituting $-x$ for y in (1) and using (3) yields

$$(a+2)[f(x) + f(-x)] = 0$$

and so we have

$$f(-x) = -f(x)$$

for all $x \in X$.

Applying (3) to the equation (1), we obtain

$$af(x+y) + f(x+2y) + f(2x+y) = (a+3)[f(x) + f(y)] \tag{4}$$

for all $x, y \in X$. Replacing x,y by $x+y$,$-y$ in (4) and using the oddness of f, one gets that

$$af(x) + f(x-y) + f(2x+y) = (a+3)[f(x+y) - f(y)] \tag{5}$$

for all $x, y \in X$. Letting x, y by y, x in (5), respectively, we have

$$af(y) - f(x-y) + f(x+2y) = (a+3)[f(x+y) - f(x)] \tag{6}$$

for all $x, y \in X$. Adding (5) to (6), we arrive at

$$f(2x+y) + f(x+2y) = 2(a+3)f(x+y) - (2a+3)[f(x) + f(y)] \tag{7}$$

for all $x, y \in X$. From (4) and (7) it follows that

$$(3a+6)f(x+y) = (3a+6)[f(x) + f(y)]$$

for all $x, y \in X$. Since for any integer $a \neq -2$, one has

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. So the mapping f is additive.

Conversely, if f is additive, then it is obvious that f satisfies the equation (1). □

Now, we are going to establish the solution of (1) in the case $a = -2$.

Lemma 2.3. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$-2f(x+y) + 2f\Big(\frac{x}{2}+y\Big) + 2f\Big(x+\frac{y}{2}\Big) = f(x) + f(y) \tag{8}$$

for all $x, y \in X$ if and only if there exists an additive mapping $A : X \to Y$ such that $f(x) = A(x) + f(0)$ for all $x \in X$.

Proof. Assume that f satisfies (8) and let $A(x) := f(x) - f(0)$. Then we get $A(0) = 0$ and

$$-2A(x+y) + 2A\Big(\frac{x}{2}+y\Big) + 2A\Big(x+\frac{y}{2}\Big) = A(x) + A(y) \tag{9}$$

for all $x, y \in X$.

By the similar way to the proof of Lemma 2.2, we see that the mapping A is additive. Hence $f(x) = A(x) + f(0)$ for all $x \in X$.

Conversely, it is obvious that if there exists an additive mapping $A : X \to Y$ such that $f(x) = A(x) + f(0)$ for all $x \in X$, then f satisfies the equation (8). □

3. Generalized Hyers-Ulam Stability of Equation (1)

Throughout this section X and Y will be a vector space and a Banach space, respectively, unless we give any specific reference. Given $f : X \to Y$, we set

$$Df_a(x,y) := af(x+y) + 2f\Big(\frac{x}{2}+y\Big) + 2f\Big(x+\frac{y}{2}\Big) - (a+3)[f(x)+f(y)]$$

for all $x, y \in X$ and for any fixed integer a.

Let $\varphi : X \times X \to [0, \infty)$ be a mapping satisfying one of the conditions

$$\Phi(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k}\varphi(2^k x, 2^k y) < \infty, \tag{a}$$

$$\Psi(x,y) = \sum_{k=0}^{\infty} 2^k \varphi\Big(\frac{x}{2^k}, \frac{y}{2^k}\Big) < \infty, \tag{b}$$

for all $x, y \in X$.

Theorem 3.1. *Let $f : X \to Y$ be a function such that*

$$\|Df_a(x,y)\| \le \varphi(x,y) \tag{10}$$

for all $x, y \in X$. If φ satisfies the condition (a), then there exists a unique additive mapping $T : X \to Y$ such that T satisfies the equation (1) and the inequality

$$\|f(x) - (a+3)f(0) - T(x)\| \le \frac{1}{2}\Phi(2x, 0) \tag{11}$$

for all $x \in X$*, where* $\|(a+2)f(0)\| \le \varphi(0,0)$*. The mapping* T *is defined by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$*.*

Proof. The inequality (10) with $x = 0$, $y = 0$ implies

$$\|-(a+2)f(0)\| \le \varphi(0,0). \tag{12}$$

If we put $y := 0$ in (10), we have

$$\left\|af(x) + 2f\left(\tfrac{x}{2}\right) + 2f(x) - (a+3)f(x) - (a+3)f(0)\right\| \le \varphi(x,0),$$

$$\left\|2g\left(\tfrac{x}{2}\right) - g(x)\right\| \le \varphi(x,0) \tag{13}$$

for all $x \in X$, where $g(x) := f(x) - (a+3)f(0)$. Replacing x by $2x$ in the inequality (13), we obtain

$$\left\|\frac{g(2x)}{2} - g(x)\right\| \le \frac{1}{2}\varphi(2x,0) \tag{14}$$

for all $x \in X$.

Putting x by $2x$, inequality (14) gives

$$\|2^{-1}g(2^2x) - g(2x)\| \le \frac{1}{2}\varphi(2^2x,0) \tag{15}$$

for all $x \in X$. From (14) and (15) it follows that

$$\begin{aligned}
&\|2^{-2}g(2^2x) - g(x)\| \\
&\le \|2^{-2}g(2^2x) - 2^{-1}g(2x)\| + \|2^{-1}g(2x) - g(x)\| \\
&= 2^{-1}\|2^{-1}g(2^2x) - g(2x)\| + \|2^{-1}g(2x) - g(x)\| \\
&\le \frac{1}{2^2}\varphi(2^2x,0) + \frac{1}{2}\varphi(2x,0)
\end{aligned}$$

for all $x \in X$.

Applying an induction argument to n, we obtain

$$\left\|\frac{g(2^n x)}{2^n} - g(x)\right\| \le \frac{1}{2}\sum_{k=0}^{n-1} \frac{\varphi(2^{k+1}x,0)}{2^k} \tag{16}$$

for all $x \in X$.

Now, we claim that the sequence $\{\frac{g(2^n x)}{2^n}\}$ is a cauchy sequence in the Banach space Y. Indeed, for any integers n, m with $n > m \geq 0$ we have

$$\begin{aligned}\left\|\frac{g(2^n x)}{2^n} - \frac{g(2^m x)}{2^m}\right\| &= \frac{1}{2^m}\left\|\frac{g(2^{n-m}2^m x)}{2^{n-m}} - g(2^m x)\right\| \\ &\leq \frac{1}{2^m}\frac{1}{2}\sum_{k=0}^{n-m-1}\frac{\varphi(2^{k+m+1}x, 0)}{2^k} \\ &= \frac{1}{2}\sum_{p=m}^{n-1}\frac{\varphi(2^{p+1}x, 0)}{2^p}\end{aligned} \tag{17}$$

for all $x \in X$. Since the right hand side of (17) tends to zero as $m \to \infty$, we obtain the sequence $\{\frac{g(2^n x)}{2^n}\}$ is Cauchy for all $x \in X$. Because of the fact that Y is a Banach space it follows that the sequence $\{\frac{g(2^n x)}{2^n}\}$ converges in Y. Therefore we can define a function $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty}\frac{g(2^n x)}{2^n} = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}, \qquad x \in X.$$

Then $T(0) = 0$.

Now, replacing x, y by $2^n x$, $2^n y$ in inequality (10), respectively, and dividing both sides by 2^n, and after then taking the limit in the resulting inequality, we have

$$aT(x+y) + 2T\left(\frac{x}{2} + y\right) + 2T\left(x + \frac{y}{2}\right) - (a+3)T(x) - (a+3)T(y) = 0$$

for all $x, y \in X$. By Lemma 2.2 and Lemma 2.3, the mapping T is additive.

Taking the limit in (16) as $n \to \infty$, we obtain that

$$\|f(x) - (a+3)f(0) - T(x)\| \leq \frac{1}{2}\Phi(2x, 0)$$

for all $x \in X$.

To prove uniqueness, we assume that there exists an additive mapping $T' : X \to Y$ such that

$$\|f(x) - (a+3)f(0) - T'(x)\| \leq \frac{1}{2}\Phi(2x, 0)$$

for all $x \in X$. Then it is obvious that

$$T(2^n x) = 2^n T(x), \qquad T'(2^n x) = 2^n T'(x)$$

for all $x \in X$. Thus we have

$$\begin{aligned}&\|T(x) - T'(x)\| = \|2^{-n}T(2^n x) - 2^{-n}T'(2^n x)\| \\ &\leq 2^{-n}\|T(2^n x) - f(2^n x) + (a+3)f(0)\| + 2^{-n}\|f(2^n x) - (a+3)f(0) - T'(2^n x)\|\end{aligned}$$

$$\le 2^{-n}2\Big[\frac{1}{2}\Phi(2^{n+1}x,0)\Big] = 2^{-n}\sum_{k=0}^{\infty}\frac{\varphi(2^{k+n+1}x,0)}{2^k} = 2\sum_{p=n}^{\infty}\frac{\varphi(2^{p+1}x,0)}{2^p}$$

for all $x \in X$. Taking the limit as $n \to \infty$, we conclude that

$$T(x) = T'(x)$$

for all $x \in X$. This completes the proof. □

Further, we are going to establish another theorem about the Hyers-Ulam stability of the equation (1) as follows.

Theorem 3.2. *Let $f : X \to Y$ be a function such that*

$$\|Df_a(x,y)\| \le \varphi(x,y) \tag{18}$$

for all $x, y \in X$. If φ satisfies the condition (b), then there exists a unique additive mapping $T : X \to Y$ such that T satisfies the equation (1) and the inequality

$$\|f(x) - (a+3)f(0) - T(x)\| \le \Psi(x,0)$$

for all $x \in X$, where $f(0) = 0$ if $a \ne -2$. The mapping T is defined by

$$T(x) = \lim_{n\to\infty} 2^n\Big\{f(\frac{x}{2^n}) - (a+3)f(0)\Big\}$$

for all $x \in X$.

Proof. Substituting $x = y = 0$ in (18) yields

$$\| - (a+2)f(0)\| \le \varphi(0,0)$$

and so we have $f(0) = 0$ if $a \ne -2$, since $\sum_{k=1}^{\infty} 2^k\varphi(0,0) < \infty$ and so $\varphi(0,0) = 0$.

If we put $y := 0$ in (18), we have

$$\Big\|2g\Big(\frac{x}{2}\Big) - g(x)\Big\| \le \varphi(x,0)$$

for all $x \in X$, where $g(x) := f(x) - (a+3)f(0)$. The rest of the proof is similar to that of Theorem 3.1. □

We observe $T(0) = 0$ for any given integer a in the Theorem 3.2.

From the main Theorem 3.1 and 3.2, we obtain the following corollary concerning the stability of the equation (1).

Corollary 3.3. *Let X be a normed space and Y a Banach space. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df_a(x,y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{19}$$

for all $x, y \in X$, where $\epsilon \geq 0$, $0 < p < 1$ are constants. Then there exists a unique additive mapping $T : X \to Y$ which satisfies the equation (1) and the inequality

$$\|f(x) - (a+3)f(0) - T(x)\| \leq \frac{2^p}{2 - 2^p}\epsilon\|x\|^p$$

for all $x \in X$, where $f(0) = 0$ if $a \neq -2$. The mapping T is defined by

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$.

Proof. Letting $\varphi(x, y) := \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and then applying Theorem 3.1, we obtain easily the results. Here if $a \neq -2$, then $f(0) = 0$ since one gets that $\varphi(0,0)$=0 by putting $x = y = 0$ in (19). □

Corollary 3.4. *Let X be a normed space, Y a Banach space. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df_a(x, y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where $\epsilon \geq 0$ and $p > 1$ are constants. Then there exists a unique additive mapping $T : X \to Y$ which satisfies the equation (1) and the inequality

$$\|f(x) - (a+3)f(0) - T(x)\| \leq \frac{2^p}{2^p - 2}\epsilon\|x\|^p$$

for all $x \in X$, where $f(0) = 0$ if $a \neq -2$. The mapping T is defined by

$$T(x) = \lim_{n\to\infty} 2^n\Big\{f\Big(\frac{x}{2^n}\Big) - (a+3)f(0)\Big\}$$

for all $x \in X$.

Proof. We set $\varphi(x, y) := \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. By Theorem 3.2, we obtain the results. □

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.5. *Let X be a normed space, Y a Banach space. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df_a(x, y)\| \leq \theta$$

for all $x, y \in X$, where $\theta \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ satisfying the inequality

$$\|f(x) - (a+3)f(0) - T(x)\| \leq \theta$$

for all $x \in X$.

Proof. Letting $\varphi(x,y) := \theta$, we get immediately the result. □

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