

A NOTE ON PRÜFER SEMISTAR MULTIPLICATION DOMAINS

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ABSTRACT. In this note we give a new generalization of the notions of Prüfer domain and PvMD which uses quasi semistar invertibility, the “quasi P \star MD”, and compare them with the P \star MD. We show in particular that the problem of when a quasi P \star MD is a P \star MD is strictly related to the problem of the descent to subrings of the P \star MD property and we give necessary and sufficient conditions.

1. Introduction

In [12], Houston, Malik and Mott defined the notion of \star -multiplication domain, in order to generalize the notions of Prüfer domain and PvMD to arbitrary star operations. After the introduction of semistar operations by Okabe and Matsuda [14], Fontana, Jara and Santos [5] have defined Prüfer semistar multiplication domains (P \star MD's), as a further generalization to the semistar context of these concepts. To define P \star MD's they used a notion of \star -invertibility analogous to the one already used for star operations. In [7] we have introduced the notion of quasi semistar invertibility, which is more natural in the semistar context. In this note we want to give a new generalization of the notions of Prüfer domain and PvMD which uses quasi semistar invertibility, the “quasi P \star MD”, and compare them with the P \star MD. We show in particular that the problem of when a quasi P \star MD is a P \star MD is strictly related to the problem of the descent to subrings of the P \star MD property and we give necessary and sufficient conditions. An interesting case is when \star is the b -operation (which associates to each ideal its integral closure). In fact, it turns out that QP b MD's are exactly the domains with Prüfer integral closure.

At the end of the paper, we briefly address the problem of giving a similar generalization of the notion of semistar Dedekind domain introduced in [2] (the *quasi \star -Dedekind domain*) but we show that some of the good properties of semistar Dedekind domains are not preserved. Both in the case of QP \star MD's

Received January 24, 2008.

2000 *Mathematics Subject Classification.* 13A15.

Key words and phrases. semistar operation, star operation.

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and quasi- \star -Dedekind domains we show that when $\star = \tilde{\star}$ we obtain the equivalence with $P\star MD$'s and \star -Dedekind domains, and these are the cases in which things work well. So, even if the notions of quasi Prüfer semistar multiplication domain and quasi semistar Dedekind domain are probably more natural in the semistar context it turns out that the notions of $P\star MD$ and \star -Dedekind domain are the ones which better generalize the classical case.

Let D be an integral domain. We denote by $\overline{\mathbf{F}}(D)$ the set of D -submodules of the quotient field K of D , by $\mathbf{F}(D)$ the set of nonzero fractional ideals of D and by $\mathbf{f}(D)$ the set of nonzero finitely generated fractional ideals of D .

A semistar operation on D is a map $\star : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathbf{F}}(D)$, the following properties hold:

- (\star_1) $(xE)^\star = xE^\star$;
- (\star_2) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
- (\star_3) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

A (semi)star operation \star on an integral domain D is a semistar operation that restricted to the set of fractional ideals is a star operation. It is easy to see that a semistar operation \star is a (semi)star operation if and only if $D^\star = D$.

If \star is a semistar operation on D , then we can consider a map $\star_f : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$ defined for each $E \in \overline{\mathbf{F}}(D)$ as follows: $E^{\star_f} := \bigcup \{F^\star \mid F \in \mathbf{f}(D) \text{ and } F \subseteq E\}$. It is easy to see that \star_f is a semistar operation on D , called *the semistar operation of finite type associated to \star* . Note that, for each $F \in \mathbf{f}(D)$, $F^\star = F^{\star_f}$. A semistar operation \star is called a *semistar operation of finite type* if $\star = \star_f$. It is easy to see that $(\star_f)_f = \star_f$ (that is, \star_f is of finite type).

As usual, we denote by v (or v_D) the (semi)star operation on D defined by $E^v = (D : (D : E))$ for all $E \in \overline{\mathbf{F}}(D)$ and by t (or t_D) the (semi)star operation of finite type associated to v . If T is an overring of D , the map $E \mapsto E^{\star^T} = ET$ for all $E \in \overline{\mathbf{F}}(D)$, is a semistar operation of finite type, which is not a (semi)star operation when $T \supsetneq D$.

A *quasi- \star -ideal* of D is an ideal I such that $I^\star \cap D = I$. A *quasi- \star -prime* is a prime ideal that is also a quasi- \star -ideal. A *quasi- \star -maximal* ideal is a (proper) ideal that is maximal in the set of all quasi- \star -ideals of D . If \star is a semistar operation of finite type, each quasi- \star -ideal of D is contained in a quasi- \star -maximal ideal. Moreover, each quasi- \star -maximal ideal of D is prime [4, Lemma 4.20].

If \star_1 and \star_2 are two semistar operations on D , we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\mathbf{F}}(D)$.

We say that a semistar operation is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for each $E, F \in \overline{\mathbf{F}}(D)$.

If \star is a semistar operation on D , we denote by $\tilde{\star}$ the semistar operation $\star_{\mathcal{M}(\star_f)}$ induced by the set $\mathcal{M}(\star_f)$ of the quasi- \star_f -maximal ideals of D , that is, $E^{\tilde{\star}} = \bigcap_{P \in \mathcal{M}(\star_f)} ED_P$ for each $E \in \overline{\mathbf{F}}(D)$. The semistar operation $\tilde{\star}$ is stable

and of finite type and $\tilde{\star} \leq \star_f$. The operation \tilde{v} is usually denoted by w , since it coincides with the w -operation studied in [19].

A semistar operation \star is *a.b.* (resp. *e.a.b.*), if for all $E \in \mathbf{f}(D)$ and $F, G \in \mathbf{F}(D)$ (resp. $F, G \in \mathbf{f}(D)$), $(EF)^\star \subseteq (EG)^\star$ implies $F^\star \subseteq G^\star$.

The *Kronecker functions ring* $\text{Kr}(D, \star)$ of an integral domain D with respect to a semistar operation \star is the ring

$$\text{Kr}(D, \star) := \{f/g \mid f, g \in D[X] \setminus \{0\} \text{ there exists } h \in D[X] \setminus \{0\} \text{ such that } (c(f)c(h))^\star \subseteq (c(g)c(h))^\star\} \cup \{0\}.$$

(We denote by $c(h)$ the content of a polynomial h .)

The *Nagata ring* $\text{Na}(D, \star)$ of D with respect to \star is the ring

$$\text{Na}(D, \star) := \left\{ \frac{f}{g}, f, g \in D[X], g \neq 0, (c(g))^\star = D^\star \right\}.$$

For more results on semistar operations and in particular on Nagata rings and Kronecker function rings, see for example [6].

Finally, recall that if \star is a semistar operation on D , T an overring of D and ι the canonical embedding of D in T , then \star induces a semistar operation \star_ι by restriction to $\overline{\mathbf{F}}(T)$, that is, $E^{\star_\iota} = E^\star$ for all $E \in \overline{\mathbf{F}}(T)$. Conversely, if \star is a semistar operation on T , we can define a semistar operation \star^ι on D , setting $E^{\star^\iota} = (ET)^\star$ for all $E \in \overline{\mathbf{F}}(D)$ (For more results on these operations, see [15]).

2. QP \star MD's and the descent of the P \star MD property

As we mentioned in the introduction, in [5], M. Fontana, P. Jara, and E. Santos have introduced the notion of a P \star MD to generalize the concepts of Prüfer domains, PvMD's and \star -multiplication domains to the context of semistar operations.

Let D be an integral domain and \star a semistar operation on D . We say that D is a *Prüfer \star -multiplication domain* (P \star MD for short) if each finitely generated nonzero ideal I of D is \star_f -invertible (recall that an ideal I is \star -invertible if $(II^{-1})^\star = D^\star$).

Since \star_f -invertibility and $\tilde{\star}$ -invertibility coincide [7, Proposition 2.18], we have that D is a P \star MD if and only if each finitely generated nonzero ideal of D is $\tilde{\star}$ -invertible.

Clearly D is a P \star MD if and only if D is a P \star_f MD, if and only if D is a P $\tilde{\star}$ MD.

Let $\star_1 \leq \star_2$ be two semistar operations on D . It is easy to see that, if D is a P \star_1 MD, then D is a P \star_2 MD.

The following characterization of P \star MD's is due to M. Fontana, P. Jara, and E. Santos [5, Theorem 3.1, Remark 3.1] and generalizes several known results about PvMD's (cf. M. Griffin [10, Theorem 5], R. Gilmer [8, Theorem 2.5], J. Arnold and J. Brewer [1, Theorem 3], J. Querré [18, Théorème 3, p. 279], B. G. Kang [13, Theorem 3.5, Theorem 3.7] and M. Zafrullah [20, Theorem 11]).

Proposition 2.1. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) D is a $P\star MD$.
- (ii) D_Q is a valuation domain for each $Q \in \mathcal{M}(\star_f)$.
- (iii) $\text{Na}(D, \star)$ is a Prüfer domain.
- (iv) $\text{Na}(D, \star) = \text{Kr}(D, \star)$.
- (v) $\tilde{\star}$ is an e.a.b. semistar operation.
- (vi) \star_f is stable and e.a.b.

In particular, in a $P\star MD$, $\tilde{\star} = \star_f$.

In [7], we have introduced the notion of quasi- \star -invertibility, as a generalization, “typical” of the semistar context, of the notion of \star -invertibility: an ideal I of D is *quasi- \star -invertible* if there exists $H \in \overline{\mathbf{F}}(D)$ such that $(IH)^\star = D^\star$. This is equivalent to saying that I^\star is \star_ι -invertible in D^\star (where ι is the canonical embedding of D in D^\star). It is clear that a \star -invertible ideal is quasi- \star -invertible, but not all quasi- \star -invertible ideals are \star -invertible ([7, Example 2.9]).

We add three other equivalent conditions to Proposition 2.1, relating the notion of $P\star MD$ with quasi- \star -invertibility.

Proposition 2.2. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) D is a $P\star MD$.
- (ii) Each finitely generated nonzero ideal of D is quasi- $\tilde{\star}$ -invertible.
- (iii) $D^\star = D^{\tilde{\star}}$ and each finitely generated nonzero ideal of D is quasi- \star_f -invertible.
- (iv) \star_f is stable and a.b.

Proof. (i) \Leftrightarrow (ii) We have noticed that D is a $P\star MD$ if and only if each finitely generated nonzero ideal of D is $\tilde{\star}$ -invertible. By [7, Corollary 2.17(2)], we deduce immediately that, since $\tilde{\star}$ is a stable semistar operation, $\tilde{\star}$ -invertibility and quasi- $\tilde{\star}$ -invertibility coincide for finitely generated ideals. Hence the assertion is proved.

(i) \Rightarrow (iii) By definition of $P\star MD$, each finitely generated nonzero ideal of D is \star_f -invertible and then quasi- \star_f -invertible. Moreover, $\star_f = \tilde{\star}$ (Proposition 2.1). Then, $D^\star = D^{\star_f} = D^{\tilde{\star}}$.

(iii) \Rightarrow (ii) Since $D^\star = D^{\tilde{\star}}$, a quasi- \star_f -invertible ideal is also quasi- $\tilde{\star}$ -invertible ([7, Proposition 2.21]).

(i) \Rightarrow (iv) We know already (Proposition 2.1(i) \Rightarrow (vi)) that \star_f is stable. Let $E \in \mathbf{f}(D)$, $F, G \in \overline{\mathbf{F}}(D)$, such that $(EF)^{\star_f} = (EG)^{\star_f}$. From the definition of $P\star MD$, $(EE^{-1})^{\star_f} = D^\star$. We have that $(E^{-1}(EF)^{\star_f})^{\star_f} = (E^{-1}(EG)^{\star_f})^{\star_f}$. Thus, $F^{\star_f} = ((E^{-1}E)^{\star_f}F)^{\star_f} = ((E^{-1}E)^{\star_f}G)^{\star_f} = G^{\star_f}$.

(iv) \Rightarrow (i) It is a straightforward consequence of Proposition 2.1(vi) \Rightarrow (i) since an a.b. semistar operation is e.a.b. \square

The next example shows that the requirement that $D^\star = D^{\tilde{\star}}$ cannot be removed from condition (iii).

Example 2.3. Let D be a pseudovaluation non valuation domain, that is, a local domain with maximal ideal M , such that $M^{-1} = V$ is a valuation domain with maximal ideal M ([11]). Let $\star := \star_{\{V\}}$ (note that $\star = \star_f$). Since M is the only quasi- \star_f -maximal, $\tilde{\star} = d$, so in particular $\tilde{\star} \neq \star_f$ and D is not a $P\star MD$. But it is easy to see that each finitely generated nonzero ideal I of D is quasi- \star_f -invertible (since IV is invertible in V). Note that in this case $D = D^{\tilde{\star}} \neq V = D^{\star_f}$.

So, a domain in which each finitely generated nonzero ideal is quasi- \star_f -invertible is not in general a $P\star MD$. It is natural to investigate the structure of the integral domains having this property, as a generalization of $P\star MD$'s.

Let D be an integral domain and \star a semistar operation on D . We say that D is a $QP\star MD$ (*quasi Prüfer semistar multiplication domain*) if each finitely generated nonzero ideal of D is quasi- \star_f -invertible.

Since a \star_f -invertible ideal is quasi- \star_f -invertible, a $P\star MD$ is always a $QP\star MD$, but the converse is not always true, as Example 2.3 shows.

As we have already noticed, the notions of $P\star MD$, $P\star_f MD$ and $P\tilde{\star} MD$ coincide. Clearly D is a $QP\star MD$ if and only if D is a $QP\star_f MD$, while there exist $QP\star MD$'s that are not $QP\tilde{\star} MD$. In fact, from Proposition 2.2 it follows that a domain D is a $QP\tilde{\star} MD$ if and only if D is a $P\star MD$. So, in particular, Example 2.3 gives an example of a $QP\star MD$ which is not a $QP\tilde{\star} MD$.

It is easy to see that if $\star_1 \leq \star_2$ are two semistar operations on D and D is a $QP\star_1 MD$, then D is also a $QP\star_2 MD$, so, in particular, a $QP\tilde{\star} MD$ is always a $QP\star MD$.

Finally, we remark that for a (semi)star operation \star , the notions of $P\star MD$ and $QP\star MD$ coincide, since in this case $D^\star = D$ and so the notions of \star -invertibility and quasi- \star -invertibility coincide ([7, Corollary 2.17]).

The next lemma relates the notions of $QP\star MD$ and of $PvMD$, showing that a $QP\star MD$ is in fact a particular subring of a $PvMD$. This implies that the study of a $QP\star MD$ can be done studying a particular overring which is a $PvMD$.

Lemma 2.4. *Let D be an integral domain and \star a semistar operation on D . Let ι be the canonical inclusion of D in D^\star . The following are equivalent:*

- (i) D is a $QP\star MD$.
- (ii) D^\star is a $P\star_\iota MD$.
- (iii) D^\star is a $PvMD$ and $\star_f = t(D^\star)(:= (t_D^\star)^\iota)$, the descent to D of the t -operation of D^\star .

Proof. (i) \Rightarrow (ii) Let $I = (a_1, a_2, \dots, a_n)D^\star$ be a finitely generated ideal of D^\star . Consider $I_0 = (a_1, a_2, \dots, a_n)D$. Since D is a $QP\star MD$, I_0 is quasi- \star_f -invertible. So, I_0^\star is $(\star_f)_\iota$ -invertible, that is, $(\star_\iota)_f$ -invertible (since in this case $(\star_f)_\iota = (\star_\iota)_f$ [16, Proposition 2.8]). But $(I_0)^\star = ((a_1, a_2, \dots, a_n)D)^\star =$

$((a_1, a_2, \dots, a_n)D^*)^* = I^*$. Thus, I^* (and consequently I) is $(\star_\iota)_f$ -invertible. Hence D^* is a $P\star_\iota$ MD.

(ii) \Rightarrow (iii) Since \star_ι is a (semi)star operation on D^* , we have that D^* is a PvMD and $(\star_f)_\iota = (\star_\iota)_f = t_{D^*}$, the t -operation of D^* ([5, Proposition 3.4]). Thus, $\star_f = ((\star_f)_\iota)^\iota = (t_{D^*})^\iota = t(D^*)$.

(iii) \Rightarrow (i) Let I be a finitely generated nonzero ideal of D . Since $(\star_f)_\iota = (t(D^*))_\iota = t_{D^*}$, we have $(I(D^* : I))^* = (ID^*(D^* : ID^*))^* = ((ID^*)^{t_{D^*}}(D^* : I))^{t_{D^*}} = D^*$, since D^* is a PvMD. \square

So, we can restate the analogue of Proposition 2.1 for QP \star MD's:

Proposition 2.5. *Let D be an integral domain and \star a semistar operation on D . Let ι be the canonical embedding of D in D^* . The following are equivalent:*

- (i) D is a QP \star MD.
- (ii) $(D^*)_Q$ is a valuation domain for each $Q \in \mathcal{M}((\star_f)_\iota)$.
- (iii) $\text{Na}(D^*, \star_\iota)$ is a Prüfer domain.
- (iv) $\text{Na}(D^*, \star_\iota) = \text{Kr}(D^*, \star_\iota) (= \text{Kr}(D, \star))$.
- (v) $(\tilde{\star})_\iota$ is an e.a.b. semistar operation.
- (vi) \star_f is e.a.b. and $(\star_f)_\iota$ is stable.

In stating condition (iv) we have used the fact that $\text{Kr}(D, \star) = \text{Kr}(D^*, \star_\iota)$. For condition (vi) we have used that \star_f is e.a.b. if and only if $(\star_f)_\iota$ is e.a.b. (it is a straightforward consequence of Propositions 3.1, 3.2 and 3.5 of [15]).

Example 2.6. The operation which associates to each $E \in \overline{F}(D)$ its integral closure is denoted by b (i.e., $E^b = \bigcap EV$, where V varies over the set of all valuation overrings of D). A domain D with Prüfer integral closure is always a QP b MD. Indeed, it is easy to see that, if ι is the canonical embedding of D in D^b , the integral closure of D , then b_ι is the b -operation of D^b , which is the identity, since D^b is a Prüfer domain ([9, Theorem 24.7]). So, in particular, D^b is a Pb_ι MD and D is a QP b MD, by Lemma 2.4.

Note that if D is not itself a Prüfer domain, D is not a Pb MD. Indeed, $\tilde{b} = d$ ([17, Corollary 2.16]), and so D is a Pb MD if and only if D is a Prüfer domain.

Conversely, if D is a QP b MD, then D^b is a Pb_ι MD (Lemma 2.4), where, as we have noticed, b_ι is the b -operation of D^b . So, again by [17, Corollary 2.16], D^b , the integral closure of D , is a Prüfer domain.

So we have shown that:

A domain D is a QP b MD if and only if D has Prüfer integral closure.

(Note that in Gilmer's book [9, Section 32] the notation D^b is used to indicate the Kronecker function ring with respect to the b -operation, that is, the ring $\text{Kr}(D, b)$. For us, D^b is simply the b -closure of D , that is, its integral closure.)

Remark 2.7. Note that in the situation of Example 2.6 the b -operation of D is the extension to the integral closure of D . In fact, we have the following result:

A domain D has Prüfer integral closure if and only if the b -operation of D coincides with the extension to the integral closure of D .

If D has Prüfer integral closure D' , as we have noticed, b_ι is the identity, and so, since $b = (b_\iota)^\iota$ ([15, Proposition 3.5]), $b = \star_{D'}$. Conversely, if $b = \star_{D'}$, then $b_\iota = d_{D'}$, the identity operation of D' . But we have already noticed that $b_\iota = b_{D'}$ is the b -operation of D' . Thus $b_{D'} = d_{D'}$ and D' is a Prüfer domain by [9, Theorem 24.7].

The next corollary gives a way to construct all the QP \star MD's:

Corollary 2.8. *Let T be a PvMD and let D be a subring of T . Let $\star := t(T)$, the descent to D of the t -operation of T . Then, D is a QP \star MD. (Note that, by Lemma 2.4, all the QP \star MD's can be constructed in this way.)*

Proof. It is similar to Lemma 2.4 (iii) \Rightarrow (i). \square

Remark 2.9. By Proposition 2.5, the semistar operation $\tilde{\star}$ on D is a.b. if and only if D is a QP $\tilde{\star}$ MD (that is, a P \star MD). This is not true for an arbitrary semistar operation of finite type, that is, if \star_f is a.b., D is not necessarily a QP \star MD. For example, if D is a v -domain which is not a PvMD (recall that a v -domain is a domain in which the v -operation is e.a.b., or equivalently, a domain in which each finitely generated ideal is v -invertible), we have that the t -operation is a.b. (it is easy to see that t is e.a.b. if and only if v is e.a.b. and t is e.a.b. if and only if t is a.b. [9, Exercise 2, p. 406]), but D is not a QPtMD (since a QPtMD is precisely a PvMD).

Moreover, if T is a v -domain which is not a PvMD and D a subring of T , the semistar operation $t(T)$ on D is a.b., since it is the descent of the t -operation of T which is a.b., but clearly D is not a QPt(T)MD, since $D^{t(T)} = T$ is not a PvMD.

In [5, Proposition 3.1], the authors show that the property of being a P \star MD ascends to overrings, that is, if T is an overring of a P \star MD D , then T is a P \star_ι MD (where ι is the canonical embedding of D in T). On the contrary, they show ([5, Example 3.2]) that the P \star MD property does not descend always to subrings.

We show that the property of being a QP \star MD both ascends and descends.

Proposition 2.10. *Let D be an integral domain, T an overring of D and ι the canonical embedding of D in T . Then*

- (1) *If \star is a semistar operation on D such that D is a QP \star MD, then T is a QP \star_ι MD.*
- (2) *If \star is a semistar operation on T such that T is a QP \star MD, then D is a QP \star^ι MD.*

Proof. (1) Let I be a finitely generated nonzero ideal of T . Then $I = (a_1, a_2, \dots, a_n)T$ for some $a_1, a_2, \dots, a_n \in T$. Let $I_0 = (a_1, a_2, \dots, a_n)D$, the fractional ideal of D generated by the a_i 's. Since D is a QP \star MD, we have $(I_0(D^\star : I_0))^\star = D^\star$. Since $T^\star \supseteq D^\star$, we have $T^\star = (I_0(D^\star : I_0))^\star T^\star \subseteq ((I_0(D^\star : I_0))^\star T^\star)^\star \subseteq (I_0 T (T^\star : I_0 T))^\star = (I(T^{\star^\iota} : I))^\iota$ and so I is quasi- \star_ι -invertible.

(2) Note that $D^{*\iota} = T^*$. Let I be a finitely generated nonzero ideal of D . Since IT is a finitely generated ideal of T and T is a QP*MD, there exists $H \in \overline{F}(T) \subseteq \overline{F}(D)$ such that $(ITH)^* = T^*$ and so $(IH)^{*\iota} = D^{*\iota}$, that is, I is quasi- $^{*\iota}$ -invertible. \square

We note that as a consequence of Lemma 2.4 (i) \Leftrightarrow (ii), the problem of studying when a QP*MD is a P*MD is equivalent to the study of when the fact that D^* is a P_{ι}^* MD (where \star is a semistar operation on D and ι is the canonical embedding of D in D^*) implies that D is a P*MD.

So, the next goal is to study when a QP*MD is a P*MD and deduce results on the descent of the P*MD property.

We have already shown that when $\star = \tilde{\star}$ and when \star is a (semi)star operation a QP*MD is always a P*MD.

In [5, Proposition 3.2], the authors prove that the flatness of T over D is a sufficient condition for the P*MD property to descend from T to D . So, analogously, the flatness of D^* over D is sufficient for a QP*MD D to be a P*MD.

We want to improve this result, giving necessary and sufficient conditions.

While $\text{Kr}(D, \star) = \text{Kr}(D^*, \star_{\iota})$ always, it is not always true that $\text{Na}(D, \star) = \text{Na}(D^*, \star_{\iota})$ (this is true for example when $\star = \tilde{\star}$). It turns out that this condition on the Nagata ring is the key for the problem we are addressing, as the next proposition shows.

Proposition 2.11. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^* . The following are equivalent:*

- (i) D is a P*MD.
- (ii) D is a QP*MD and $\text{Na}(D, \star) = \text{Na}(D^*, \star_{\iota})$.

Proof. It follows immediately from Proposition 2.5(iv) and Proposition 2.1(iv). \square

We show that the condition on the Nagata ring in (ii) of Proposition 2.11, is strictly related to the notion of semistar flatness, as studied in [3].

We start recalling some results on quasi- \star -ideals and on semistar flatness.

Lemma 2.12. [6, Lemma 2.3(3)(4)] *Let D be an integral domain, \star a semistar operation on D and ι the embedding of D in D^* .*

- (1) *If Q is a quasi- \star_f -maximal ideal of D , then there exists a quasi- $(\star_f)_{\iota}$ -maximal ideal M of D^* such that $Q = M \cap D$.*
- (2) *If N is a quasi- $(\star_f)_{\iota}$ -prime ideal of D^* , then $N \cap D$ is a quasi- \star_f -prime ideal of D .*

Let D be an integral domain and T an overring of D . Let \star be a semistar operation on D and let \star' be a semistar operation on T . We say [3, Section 3] that T is (\star, \star') -linked to D if, for each finitely generated nonzero integral ideal F of D , we have that $F^{\star} = D^{\star}$ implies $(FT)^{\star'} = T^{\star'}$. We say [3, Section 4]

that T is (\star, \star') -flat over D if T is (\star, \star') -linked to D and for each $Q \in \mathcal{M}(\star'_f)$, $D_{Q \cap D} = T_Q$.

Proposition 2.13 ([3, Lemma 3.1(e), Proposition 3.2, Corollary 5.4, Theorems 4.5 and 5.7]). *Let D be an integral domain and \star a semistar operation on D . Let T be an overring of D and \star' a semistar operation on T .*

- (1) *T is (\star, \star_ι) -linked to D (where ι is the canonical embedding of D in T).*
- (2) *T is (\star, \star') -linked to D if and only if $(N \cap D)^{\star_f} \neq D^\star$ for each quasi- \star'_f -maximal ideal N .*
- (3) *T is (\star, \star') -flat over D if and only if $\text{Na}(T, \star')$ is a flat overring of $\text{Na}(D, \star)$.*
- (4) *If D is a $P\star MD$ and T is (\star, \star') -linked to D , then T is a $P\star MD$.*
- (5) *D is a $P\star MD$ if and only if for each overring R of D and for each semistar operation \star' on R such that R is (\star, \star') -linked to D then R is (\star, \star') -flat over D .*

We can now show that our condition on the Nagata ring is in fact equivalent to the (\star, \star_ι) -flatness of D^\star over D .

Proposition 2.14. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^\star . Then, the following are equivalent:*

- (i) $\text{Na}(D, \star) = \text{Na}(D^\star, \star_\iota)$.
- (ii) D^\star is (\star, \star_ι) -flat over D .
- (iii) $(D^\star)_P = D_{P \cap D}$ for each $P \in \mathcal{M}((\star_\iota)_f)$.
- (iv) $D^\star = D^{\tilde{\star}}$ and $(\star_\iota) = (\tilde{\star})_\iota$.

Proof. (i) \Rightarrow (ii) It follows immediately from Proposition 2.13(3).

(ii) \Rightarrow (iii) It is trivial.

(iii) \Rightarrow (iv) First we note that, in this case, $(\star_f)_\iota = (\star_\iota)_f$ [16, Proposition 2.8]. Now, let $E \in \overline{\mathbf{F}}(D^\star)$. Then,

$$E^{\tilde{\star}} = \bigcap_{Q \in \mathcal{M}(\star_f)} ED_Q \text{ and } E^{\tilde{\star}_\iota} = \bigcap_{M \in \mathcal{M}((\star_f)_\iota)} E(D^\star)_M.$$

For each $Q \in \mathcal{M}(\star_f)$ there exists $M \in \mathcal{M}((\star_f)_\iota)$ such that $Q = M \cap D$ (Lemma 2.12(1)). Since, by the hypothesis, $D_Q = (D^\star)_M$, we have that $E^{\tilde{\star}_\iota} \subseteq E^{\tilde{\star}}$. Conversely, if M is a quasi- $(\star_f)_\iota$ -maximal ideal, then $M \cap D$ is a quasi- \star_f -prime ideal of D (Lemma 2.12(2)). So, $M \cap D \subseteq Q$ for some $Q \in \mathcal{M}(\star_f)$. It follows that $D_Q \subseteq D_{M \cap D} = (D^\star)_M$. Thus $E^{\tilde{\star}} \subseteq E^{\tilde{\star}_\iota}$ and so $E^{\tilde{\star}} = E^{\tilde{\star}_\iota}$. Moreover $D^\star = D^{\tilde{\star}}$. Indeed $D^\star \subseteq \bigcap_{M \in \mathcal{M}((\star_f)_\iota)} (D^\star)_M = \bigcap_{Q \in \mathcal{M}(\star_f)} D_Q = D^{\tilde{\star}} \subseteq D^\star$.

(iv) \Rightarrow (i) By the hypothesis, $D^\star = D^{\tilde{\star}}$ and so, ι is also the canonical embedding of D in $D^{\tilde{\star}}$. Thus we have $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star}) = \text{Na}(D^{\tilde{\star}}, (\tilde{\star})_\iota) = \text{Na}(D^{\tilde{\star}}, (\star_\iota)) = \text{Na}(D^\star, \star_\iota)$ (recalling that $\text{Na}(D, \tilde{\star}) = \text{Na}(D^{\tilde{\star}}, \tilde{\star}_\iota)$ [6, Corollary 3.5]). \square

Corollary 2.15. *Let D be an integral domain and T an overring of D . Let ι be the canonical embedding of D in T and let $*$ be a semistar operation on T . Then, the following are equivalent:*

- (i) $\text{Na}(D, *^\iota) = \text{Na}(T, *)$.
- (ii) T is $(*^\iota, *)$ -flat over D .
- (iii) $T_P = D_{P \cap D}$ for each $P \in \mathcal{M}(*_f)$.
- (iv) $T = D^{\widetilde{*}^\iota}$ and $\widetilde{*} = (\widetilde{*}^\iota)_\iota$.

Proof. Apply Proposition 2.14 to $\star := *^\iota$, recalling that $*$ = $(*^\iota)_\iota$ (see for example [14, Lemma 45]). \square

We have the following result for QP \star MDs and a corollary for the descent of the P \star MD property.

Proposition 2.16. *Let D be an integral domain and \star a semistar operation on D . Assume that D is a QP \star MD. Then, D is a P \star MD if and only if the equivalent conditions of Proposition 2.14 hold.*

Proof. It is a consequence of Proposition 2.11. \square

Corollary 2.17. *Let D be an integral domain, T an overring of D . Let ι be the canonical embedding of D in T and $*$ a semistar operation on T . Assume that T is a P $*$ MD. Then, D is a P $*^\iota$ MD if and only if the equivalent conditions of Corollary 2.15 hold.*

Proof. It follows from Proposition 2.16, since T is a P $*$ MD if and only if D is a QP $*^\iota$ MD. \square

As a corollary we recover the results proven in [5, Proposition 3.2 and Proposition 3.3]. First we prove an easy lemma.

Lemma 2.18. *Let D be an integral domain, \star a semistar operation on D and ι the canonical embedding of D in D^\star . Then D^\star is $(\tilde{\star}, (\tilde{\star})_\iota)$ -flat over D .*

Proof. It is immediate by Proposition 2.13, since $\text{Na}(D, \star) = \text{Na}(D^\star, (\tilde{\star})_\iota)$ ([6, Corollary 3.5]). \square

Corollary 2.19 (cf. [5, Proposition 3.2 and Proposition 3.3]). *Let D be an integral domain.*

- (1) *Let T be a flat overring of D and let $*$ be a semistar operation on T . Let $\iota := \iota_T$ be the canonical embedding of D in T . Assume that T is a P $*$ MD. Then D is a P $*^\iota$ MD.*
- (2) *Let \star be a semistar operation on D . Let ι be the canonical embedding of D in D^\star . Then D is a P \star MD if and only if D^\star is a P $(\tilde{\star})_\iota$ MD.*

Proof. (1) It is immediate by Corollary 2.17.

(2) It is immediate by Lemma 2.18 and Proposition 2.16. \square

By using Corollary 2.8 and Corollary 2.17, we can characterize P \star MDs as a particular class of subrings of PvMDs.

Theorem 2.20. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) D is a $P\star MD$;
- (ii) *There exists an overring T of D such that T is a PvMD, $\star_f = t(T)$ and for each t_T -maximal ideal Q of T , $T_Q = D_{Q \cap D}$.*
- (iii) *There exists an overring T of D such that T is a PvMD, $\star_f = t(T)$ and T is $(t(T), t_T)$ -flat over D .*

Proof. (i) \Rightarrow (ii) Let $T := D^\star$. Thus T is a PvMD and $\star_f = t(T)$ (Lemma 2.4 (i) \Rightarrow (iii)). Moreover, by Proposition 2.1 $\star_f = \tilde{\star}$. Thus, by Lemma 2.18 D^\star is $(\star_f, (\star_f)_\iota)$ -flat over D (where ι is the canonical embedding of D in T). Since $t_T = (t(T))_\iota = (\star_f)_\iota$, the assertion is proven.

(ii) \Rightarrow (i) Recall that D is a $P\star MD$ if and only if it is a $P\star_f MD$. Let ι be the canonical embedding of D in T . Since T is a $Pv_T MD$, it is a $Pt_T MD$. By the hypothesis on the t_T -maximal ideals, applying Corollary 2.17, we have that D is a $P(t_T)^\iota MD$, that is, a $P\star_f MD$ (recalling that, by definition, $(t_T)^\iota = t(T)$).

(ii) \Leftrightarrow (iii) follows immediately by Corollary 2.15 (ii) \Leftrightarrow (iii), since by definition $t(T) = (t_T)^\iota$. \square

3. Quasi- \star -Dedekind domains

A domain D with a semistar operation \star is called \star -Noetherian if the ascending chain condition on quasi- \star -ideals holds (see [2] and [16]).

If one defines a \star -Dedekind domain (a \star -DD) to be a \star -Noetherian $P\star MD$, one recovers semistar versions of all the main characterizations and properties of Dedekind domains, using the semistar operation $\tilde{\star}$ ([2]). So, \star -Dedekind domains are a very good generalization of Dedekind domains (and of Krull domains, since Krull domains are exactly the v -Dedekind domains [2, Remark 4.2(1)]).

It is natural to ask whether the notion of $QP\star MD$ leads to a notion of “quasi- \star -Dedekind domain”, which preserves the same good characterizations for \star (or \star_f). To end this note, we study briefly this problem.

Let D be an integral domain and \star a semistar operation on D . We say that D is a *quasi- \star -Dedekind domain* (a $Q\star DD$) if D is a \star -Noetherian $QP\star MD$. It is clear that a \star -Dedekind domain is a quasi- \star -Dedekind domain and that D is a $Q\star DD$ if and only if D is a $Q\star_f DD$. Moreover, it is easily seen that if $\star_1 \leq \star_2$ are two semistar operations on D , if D is a $Q\star_1 DD$, then D is a $Q\star_2 DD$. Note also that obviously for a (semi)star operation \star the notions of $\star DD$ and $Q\star DD$ coincide.

Example 3.1. (1) An example of a $Q\star DD$ which is not a $\star DD$ is given by the ring in Example 2.3 if one takes D Noetherian. Indeed, in this case, D is \star -Noetherian (since it is Noetherian) and we have seen that it is a $QP\star MD$, so it is a $Q\star DD$, but it is not a $\star DD$ since it is not a $P\star MD$. Note that this is also an example of a $Q\star DD$ which is not a $Q\tilde{\star} DD$.

(2) Similarly to the case of $QP\star MD$ (Example 2.6), it is not difficult to show that a domain with Dedekind integral closure (e.g., a one dimensional Noetherian domain) is always a quasi- b -Dedekind domain and it is never a b -Dedekind domain if it is not Dedekind (again because $\tilde{b} = d$ and so b -Dedekind is equivalent to Dedekind).

Note that the notion of $Q\star DD$ does not ascend to overrings (this depends on the fact that the notion of \star -Noetherian domain does not ascend), as the next example shows:

Example 3.2. Let D be a Noetherian domain with a non-Noetherian valuation overring V , and ι the canonical embedding of D in V . Let $\star := \star_{\{V\}}$. It is easy to see that D is a quasi- \star -Dedekind domain. Since $\star_{\iota} = d_V$ is the identity semistar operation on V , we have that V is not a quasi- \star_{ι} -Dedekind domain, since it is not a Dedekind domain.

Proposition 3.3. *Let D be an integral domain, T an overring of D , ι the canonical embedding of D in T , \star a semistar operation on D and $*$ a semistar operation on T .*

- (1) *If T is a $Q\star DD$, then D is a $Q\star_{\iota} DD$. (In particular, if D^{\star} is a \star_{ι} -DD, then D is a $Q\star DD$.)*
- (2) *If \star is stable and D is a $Q\star DD$, then D^{\star} is a $Q\star DD$. (More precisely, D^{\star} is a Krull domain and $\star = v(D^{\star})$.)*

Proof. (1) It follows from Proposition 2.10 and [2, Lemma 3.1].

(2) D^{\star} is a $QP\star_{\iota} MD$ by Proposition 2.10. Moreover, since \star is stable, D^{\star} is \star_{ι} -Noetherian ([2, Proposition 3.4]). The parenthetical statement is a consequence of [2, Proposition 4.5], since \star_{ι} is a (semi)star operation on D^{\star} . \square

Proposition 3.4. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) *D is a $Q\star DD$.*
- (ii) *Each nonzero ideal of D is quasi- \star_f -invertible.*

Proof. (i) \Rightarrow (ii) Let I be a nonzero ideal of D . Since D is \star -Noetherian, $I^{\star} = J^{\star f}$ for a finitely generated ideal $J \subseteq I$ of D ([2, Lemma 3.3]). Since D is a $QP\star MD$, J is quasi- \star_f -invertible. So, $D^{\star} = (J(D^{\star} : J))^{\star f} = (J^{\star f}(D^{\star} : J^{\star f}))^{\star f} = (I^{\star}(D^{\star} : I^{\star}))^{\star} = (I(D^{\star} : I))^{\star}$.

(ii) \Rightarrow (i) It follows from [7, Lemma 2.14] and [2, Lemma 3.3]. \square

Corollary 3.5. *Let D be an integral domain and \star a semistar operation on D . The following are equivalent:*

- (i) *D is a \star -DD.*
- (ii) *D is a quasi- $\tilde{\star}$ -DD.*

Proof. It is immediate from Proposition 3.4 since a \star -DD is a domain in which each ideal is quasi- $\tilde{\star}$ -invertible [2, Proposition 4.1]. \square

Remark 3.6. A \star -Dedekind domain is characterized by the fact that each nonzero ideal is quasi- $\tilde{\star}$ -invertible, and not \star_f -invertible, as one would expect considering that $P\star MD$ s are defined using \star_f -invertibility. From this point of view, things work well for $QP\star MD$'s and $Q\star DD$'s, in fact, $QP\star MD$'s, which are defined using quasi- \star_f -invertibility, give rise to $Q\star DD$'s, which are characterized using quasi- \star_f -invertibility, as shown in Proposition 3.4. Note that this recovers also the case of a $P\star MD$: indeed, we have shown in Proposition 2.2 that a $P\star MD$ is a domain in which all finitely generated ideals are *quasi- $\tilde{\star}$ -invertible* (i.e., a $QP\tilde{\star}MD$), and so it is natural that they give rise to the \star -DD (or quasi- $\tilde{\star}$ -DD), which are characterized using quasi- $\tilde{\star}$ -invertibility.

To finish, we recall that Dedekind domains can be characterized as the Noetherian, integrally closed, one-dimensional domains. The analogous characterization for semistar Dedekind domains holds, using the notion of quasi- $\tilde{\star}$ -integral closure and $\tilde{\star}$ -dimension (see [2, Theorem 4.15]). (Recall that the \star -dimension of D is the supremum of the lengths of the chains of quasi- \star -prime ideals.)

It is easy to see that a $Q\star DD$ is quasi- \star -integrally closed (a domain D is quasi- \star -integrally closed if $(I^\star : I^\star) \subseteq D^\star$ for all finitely generated nonzero ideals I of D). This fact is an easy consequence of [2, Lemma 4.13(1)]. Moreover, by definition, $Q\star DD$ is \star -Noetherian.

With the next example, we show that a $Q\star DD$ can have \star -dimension higher than one, even when \star is a semistar operation of finite type. It follows that the most classical characterization of Dedekind domains cannot be transferred to the semistar case, when $\star \neq \tilde{\star}$, that is when D is a $Q\star DD$ that is not a \star -DD.

Example 3.7. Let D be a Noetherian 2-dimensional domain. Let $P_1 \subsetneq P_2$ be two primes of D . By [9, Corollary 19.7], there exists a valuation overring V of D with prime ideals $M_1 \subsetneq M_2$ contracting respectively to P_1 and P_2 . Let $\star := \star_{\{V\}}$ be the finite type semistar operation given by the extension to the overring V . Clearly D is a $Q\star DD$ because it is \star -Noetherian and a $QP\star MD$ (since $D^\star = V$ is Prüfer). But it is easily seen that P_1 and P_2 are quasi- \star -ideals. Since $P_1 \subsetneq P_2$ we have $\star\text{-dim}(D) = 2$.

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