Reliability and ratio in exponentiated complementary power function distribution

Yeung-Gil Moon[1] · Chang-Soo Lee[2] · Se-Gi Ryu[3]

[1]Department of Tourism Quality Management, Kangwon Tourism College
[2]Department of of Mobile Engineering, Kyungwoon University
[3]Gyeong-An Girl's Information High School

Abstract

As we shall define an exponentiated complementary power function distribution, we shall consider moments, hazard rate, and inference for parameter in the distribution. And we shall consider an inference of the reliability and distributions for the quotient and the ratio in two independent exponentiated complementary power function random variables.

Keywords: Exponentiated complementary distribution, hazard rate, power function distribution.

1. Introduction

An example of some importance is the use of a power function distribution to fit the distribution of certain likelihood ratios in statistical tests. If the likelihood ratio is based on n-independent distributed random variables, it's often found that a useful good fit can be obtained by supporting $(likelihood ratio)^{2/n}$ to have a power function distribution. For independent random variables X and Y, and a real number c, the probability $P(X < cY)$ is as given in Woo (2006): (i) it is the reliability when $c = 1$, (ii) it is distribution of ratio $X/(X + Y)$ when $c = t/(1 - t)$ for $0 < t < 1$.

Rider (1964) derived distributions of product and quotient of maximum values in samples from a population with a power function distribution and studied problems of estimating its parameter. Moments of order statistics for a power function distribution were considered by Lewin (1972), and Arnold and Press (1983) studied Bayes estimation for the scale parameter of the Pareto distribution using power function prior distribution. Moothathu (1984) studied characterizations of Lorenz curve in the power function distribution. Bowman and Shenton (1998) studied the ratio of the gamma variables with the unit shape parameter. Woo (2008)

studied reliability and ratio in two independent different variates. Moon and Lee (2009) considered an inference on the reliability in two independent gamma random variables.

We define an exponentiated complementary power function distribution based on definitions of Gupta and Kundu (2001), and we then consider k-th moment, hazard rate, and inference for parameter in the exponentiated complementary power function distribution. And we shall consider an inference of the reliability $P(Y < X)$ and distributions of the quotient X/Y and the ratio $X/(X + Y)$ when X and Y are independent exponentiated complementary power function random variables.

2. Exponentiated complementary power function

Let a continuous random variable X have the density $F'(x) = f(x)$, whose the cdf $F(x)$ is strictly increasing function and its support is (0,1).
Assume that random variable Z has a cdf

$$G(x) \equiv [F^{-1}(x)]^{\alpha}, \ 0 < x < 1, \ \alpha > 0. \tag{2.1}$$

Then the random variable Z is called to have an exponentiated complementary distribution in Gupta and Kundu (2001).

For given power function random variable

$$X \sim f(x; \theta) = \theta \cdot x^{\theta-1}, \ 0 < x < 1, \ 1 \leq \theta,$$

especially if $\theta = 1$, the density is a uniform over $(0, 1)$. From definition (2.1) and a power function distribution in Malik (1967), an exponentiated complementary power function random variable Z has the cdf and pdf :

$$G(x) = x^{\alpha/\theta}, \ \ 0 < x < 1,$$

and

$$G'(x) = g(x) = \alpha x^{\alpha/\theta-1}/\theta, \ \ 0 < x < 1, \ \ \alpha \geq \theta \geq 1 \tag{2.2}$$

The k-th moment of Z is given by

$$E(Z^k) = \frac{\alpha}{\alpha + \theta \cdot k}, \ \ k = 1, 2, \cdots.$$

and from formula 3.383(1) in Gradshteyn and Ryzhik (1965), the moment generating function $m_Z(t)$ of Z is obtained as:

$$m_Z(t) = F(\alpha/\theta; 1 + \alpha/\theta; t),$$

where $F(a; b; t)$ is a confluent hypergeometric function.

From a property of increasing (or decreasing) hazard rate in Saunders (2007), we obtain the following:

Fact 2.1. The density (2.2) has increasing hazard rate when $\alpha > \theta$, but it has decreasing hazard rate when $\alpha < \theta$.

3. Estimations

3.1. Estimation of parameter

Let $X_1, X_2, \ldots, X_m$ be a random sample from the density (2.2). Let's first introduce well-known results as follows:

Lemma 1. (a) $-\sum_{i=1}^{m} lnX_i$ follows a gamma distribution with the shape parameter m and the scale parameter θ/α.

(b) If Y follows a gamma distribution with shape parameter p and the scale parameter b, then $E(1/Y^k) = \Gamma(p-k)/(\Gamma(p) \cdot b^k)$, if $p > k$, where $\Gamma(a)$ is the gamma function.

For given power function distribution (i.e. θ is known), MLE $\widehat{\alpha}$ of α is given as:

$$\widehat{\alpha} = m\theta/(-\sum_{i=1}^{m} lnX_i). \tag{3.1}$$

From Lemma 1, mean and variance of MLE $\widehat{\alpha}$ of α are obtained by:

$$E(\widehat{\alpha}) = m\alpha/(m-1)$$

and

$$Var(\widehat{\alpha}) = m^2\alpha^2/[(m-1)^2(m-2)], \quad m > 2. \tag{3.2}$$

From (3.2), an unbiased estimator $\widetilde{\alpha}$ of α is defined by:

$$\widetilde{\alpha} = (m-1)\theta/(-\sum_{i=1}^{m} lnX_i). \tag{3.3}$$

From Lemma 1, variance of an unbiased estimator $\widetilde{\alpha}$ of α is given as:

$$Var(\widetilde{\alpha}) = \alpha^2/(m-2), \quad m > 2. \tag{3.4}$$

From variances of estimators $\widehat{\alpha}$ and $\widetilde{\alpha}$ in (3.2) and (3.4), we obtain the following:

Fact 3.1. The unbiased estimator $\widetilde{\alpha}$ is more efficient in a sense of MSE than the MLE $\widehat{\alpha}$.

3.2. Estimation of reliability

For given power function distribution (i.e. θ is known), we consider estimation of reliability $P(Y < X)$ when X and Y are exponentiated complementary power function random variables each with parameters α_x and α_y, respectively.

Reliability $P(Y < X)$ is obtained by:

$$\gamma \equiv P(Y < X) = \frac{1}{1 + \alpha_y/\alpha_x} = \frac{1}{1 + \eta}, \quad \eta \equiv \alpha_y/\alpha_x, \tag{3.5}$$

which reliability is a monotone function of η.

Because γ is a monotone function of η, inference on γ is equivalent to inference on η (see McCool, 1991). We then consider inference on η.

Assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from the density (2.2) each with parameters α_x and α_y, respectively. Then

$$\widehat{\alpha}_x = \theta m / (-\sum_{i=1}^{m} ln X_i), \quad \widehat{\alpha}_y = \theta n / (-\sum_{i=1}^{n} ln Y_i).$$

And hence, MLE $\widehat{\eta}$ of η is $\widehat{\eta} = \widehat{\alpha}_y / \widehat{\alpha}_x$.

From Lemma 1, we obtain mean and variance of MLE $\widehat{\eta}$:

Fact 3.2. $E(\widehat{\eta}) = \dfrac{n}{n-1}\eta$ and $Var(\widehat{\eta}) = \dfrac{n^2(m+n-1)}{m(n-1)^2(n-2)}\eta^2, \quad n > 2$.

From unbiased estimator (3.3), we define another estimator $\widetilde{\eta}$ of η as:

$$\widetilde{\eta} = \widetilde{\alpha}_y / \widetilde{\alpha}_x.$$

From Lemma 1, we obtain the following mean and variance of $\widetilde{\eta}$:

Fact 3.3. $E(\widetilde{\eta}) = \dfrac{m}{m-1}\eta$ and $Var(\widetilde{\eta}) = \dfrac{m^2+mn-m}{(m-1)^2(n-2)}\eta^2, \quad m > 1, \quad n > 2$.

From Facts 3.2 and 3.3, we obtain the following:

Fact 3.4. (a) When m>n, the proposed estimator $\widetilde{\eta}$ is more efficient in a sense of mean squared error than MLE $\widehat{\eta}$, and vice versa when n>m.

(b) When m=n, the proposed estimator $\widetilde{\eta}$ and MLE $\widehat{\eta}$ have the same mean squared errors.

Next, we consider to estimate a confidence interval of η.

We define the following random variables:

$$Z \equiv -\sum_{i=1}^{m} ln X_i, \quad W \equiv -\sum_{i=1}^{n} ln Y_i.$$

By formula 3.381(4) in Gradshteyn and Ryzhik (1965) and the quotient density of two independent random variables in Rohatgi (1976), the pdf of $U(\boldsymbol{X}, \boldsymbol{Y}) = Z/W$ is obtained by the following:

For $U \equiv U(\boldsymbol{X}, \boldsymbol{Y}) = (-\sum_{i=1}^{m} ln X_i)/(-\sum_{i=1}^{n} ln Y_i)$, $\boldsymbol{X} = (X_1, \ldots, X_m)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$,

$$f_U(u) = \frac{1}{B(m,n)\eta^m} u^{m-1} (1 + \frac{u}{\eta})^{-m-n}, \ u > 0.$$

Fact 3.5. Let $B \equiv U/(\eta + U)$. Then B follows a beta distribution with parameter m and n.

Based on pivot quantity B in Fact 3.5, for given $0 < p < 1$, an (1-p)100% confidence interval of η is given by:

$$(\frac{m}{n}\frac{1-b_{1-p/2}}{b_{1-p/2}}\widehat{\eta}, \ \frac{m}{n}\frac{1-b_{p/2}}{b_{p/2}}\widehat{\eta}), \tag{3.6}$$

where for $0 < p < 1$, there exists $0 < b_p < 1$ which

$$p = \int_0^{b_p} \frac{1}{B(m,n)} t^{m-1} (1-t)^{n-1} dt.$$

From Fact 3.2, the expected length of the confidence interval of η is given by

$$\frac{m}{n-1}\left(\frac{1}{b_{p/2}} - \frac{1}{b_{1-p/2}}\right)\eta.$$

Next, we want to test the hypothesis: $H_0; \alpha_x = \alpha_y$ $against$ $H_1; \alpha_x \neq \alpha_y$. If the null hypothesis is true, $\alpha_x = \alpha_y = \alpha_0$, then MLE of α_0 is given as:

$$\widehat{\alpha_0} = \theta(m+n)/\left(-\sum_{i=1}^{m} lnX_i - \sum_{i=1}^{n} lnY_i\right).$$

From the likelihood ratio test in Rohatgi (1976),

$$\Lambda(x,y) = \left(\frac{m+n}{m}\right)^m \left(\frac{m+n}{n}\right)^n \frac{1}{(1+1/U)^m} \frac{1}{(1+U)^n} < c \quad \Leftrightarrow \quad U < c_1 \ \ or \ U > c_2.$$

If H_0 is true, then from Fact 3.5 and an upper limit of integral (3.2), since $B_0 = U/(1+U)$ is a monotone function of U, " $U < c_1$ or $U > c_2$ " is equivalent to $U/(1+U) < b_{p/2}$ or $U/(1+U) > b_{1-p/2}$. And hence we obtain the following:

Fact 3.6. For a level $0 < p < 1$, the test

$$\phi(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 1, & \text{if } U/(1+U) < b_{p/2} \text{ or } U/(1+U) > b_{1-p/2} \\ 0, & \text{else} \end{cases},$$

is the likelihood ratio size p for testing $H_0; \alpha_x = \alpha_y$ $against$ $H_1; \alpha_x \neq \alpha_y$.

4. Distribution of ratio $X/(X+Y)$

For given α, $(\alpha > \theta \geq 1)$, let X and Y be two independent exponentiated complementary power function random variables each with parameters θ_x and θ_y, respectively. Then the density of quotient $W = Y/X$ is given by

$$f_W(w) = \begin{cases} \dfrac{\alpha}{\theta_x + \theta_y} w^{\alpha/\theta_y - 1}, & \text{if } 0 < w < 1 \\[2mm] \dfrac{\alpha}{\theta_x + \theta_y} w^{-\alpha/\theta_x - 1}, & \text{if } w \geq 1. \end{cases}$$

Hence we obtain the density of ratio $R = X/(X+Y)$:

$$f_R(r) = \begin{cases} \dfrac{\alpha}{\theta_x + \theta_y} \dfrac{(1-r)^{\alpha/\theta_y - 1}}{r^{\alpha/\theta_y + 1}}, & \text{if } \dfrac{1}{2} < r < 1 \\[3mm] \dfrac{\alpha}{\theta_x + \theta_y} \dfrac{r^{\alpha/\theta_x - 1}}{(1-r)^{\alpha/\theta_x + 1}}, & \text{if } 0 < r \leq \dfrac{1}{2} \end{cases}.$$

Especially if $\theta_x = \theta_y$, the density (4.1) is symmetric about $1/2$.

From the density (4.1) and binomial expansion, we obtain k-th moment of ratio R: For k=1, 2, 3, ..,

$$E(R^k) = \frac{\alpha}{\theta_x + \theta_y} \sum_{j=0}^{\infty} \frac{(-k)_j}{j!} \left[\frac{\theta_y}{\alpha + j \cdot \theta_y} + \frac{\theta_x}{\alpha + (k+j) \cdot \theta_x} \right] \tag{4.1}$$

where $(b)_j = b(b-1)(b-2)\ldots(b-j+1)$ and $(b)_0 \equiv 1$.

From k-th moment (4.2), Table 4.1 provides approximate numerical values of mean and variance of ratio when $\alpha =$ 2, 4, 8, 10 and $(\theta_x = 1, \theta_y = 2)$.

Table 4.1 Mean and variance of ratio $R = X/(X+Y)$ when $\theta_x = 1, \theta_y = 2$.

α	2	4	8	10
mean	0.59089	0.55556	0.53016	0.52443
variance	0.03727	0.01431	0.00440	0.00290

From Table 4.1, we observe the following when the density (2.2) has parameters α and θ :

Fact 4.1. For given α ($\alpha > \theta \geq 1$), let X and Y be two independent exponentiated complementary power function random variables each with parameters θ_x and θ_y, respectively. Then variance of ratio is decreasing when α is increasing, where $\alpha =$ 2, 4, 8, 10 and $\theta_x = 1$ and $\theta_y = 2$.

References

Arnold, B. C. and Press, S. J. (1983). Bayes inference for Pareto population. *Journal of Econometrics*, **21**, 287-306.

Bowman, K. O. and Shenton, L. R. (1998). Distribution of the ratio of gamma variates. *Communication in Statistics-Simulations* , **27**, 1-19.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Table of integrals, series, and product,* Academic Press, New York.

Gupta, R. C., Gupta, R. D. and Gupta, P. L. (1998). Modelling failure time data by Lehman alternative. *Communication in Statistics-Theory & Methods*, **27,** 887-904.

Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: An alternative to gamma and Weibull. *Biometrical Journal,* **43,** 117-130.

Lewin, T. (1972). Estimation of the tail of Pareto law. *Scandinavisk Aktuarietidskrift*, **55**, 170-178.

Malik, H. J. (1967). Exact moments of order statistics for a power function population. *Scandinavisk Aktuarietidskrift* , **50**, 64-69.

McCool, J. I. (1991). Inference on P(X<Y) in the Weibull case. *Communications in Statistics-Simulations*, **20**, 129-148.

Moon, Y. G. and Lee, C. S. (2009). Inference on the reliability P(Y<X) in the gamma case. *Journal of the Korean Data & Information Science Society,* **20**, 219-223.

Moothatu, T. S. (1984). Characterizations of power function and Pareto distribution in terms of their Lorenz curves. *Research Review*, **3**, 118-123.

Rider, P. R. (1964). Distribution of product and quotient of maximum values in samples from a power function population. *Journal of the American Statistical Association,* **59**, 877-880.

Rohatgi, V. K. (1976). *An introduction to probability theory and mathematical statistics*, John Wiley & Sons, New York.

Saunders, S. C. (2007). *Reliability, life testing, and prediction of service lives*, Springer, New York.

Woo, J. (2006). Reliability P(Y<X), ratio X/(X+Y) and a skew-symmetric distribution of two independent random variables. *Proceedings of the Korean Data & Information Science Society*, 37-42.

Woo, J. (2008). Estimating reliability and distribution of ratio in two independent different variates. *Journal of the Korean Data & Information Science Society*, **19**, 967-977.