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An Investigation of Two Seventh Graders'Modification of their Multiplicative Reasoning for Solving Combinatorial Problems and their Reciprocal Interactions with Represented Symbols

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This study presents data from a year-long teaching experiment which illustrate how two seventh graders modified their multiplicative thinking and interacted with their representing symbols in the context of combinatorial problem situations. Damon was at the process of construction of recursively multiplicative thinking by modifying his multiplicative reasoning, but Carol appeared to remain at the stage of a binary multiplicative scheme. The two students'struggles with their representing symbols or represented symbols by the teacher show that even well-organized symbolic systems from teachers'perspective do not necessarily help students advance their mathematical capacity.

I. Introduction

Symbols acquire the power to open or activate pathways to specific re-presentations without, however, obliging the proficient symbol user to produce the re-presentations there and then (von Glasersfeld, 1991, p. 53)

"Mathematical knowledge is essentially symbolic in nature" (Steffe & Olive, 1996, p. 113) in that mathematical knowledge requires students'mental operations to achieve a level of generalization that frees them from a specific figurative input and such interiorized mental operations can stand for any particular implementation on appropriate figurative input by using other forms of representation system, which might show totally different appearance from the figurative input (Steffe & Olive, 1996). For a word to function as a symbol, von Glasersfeld (1991) asserts that "the word/symbol must be associated with a conceptual structure that was abstracted from experience and, at least to some extent, generalized" (p. 52).However, a symbolic aspect of mathematical knowledge does not allow us to fix a specific form of students' externally represented symbols directly to their particular mathematical knowledge. Rather, mathematical symbols should be viewed as dynamic in nature in that a static symbol can stand for the result of students'actions and operations for construction of such a symbolic representation in a certain context. For example, making a rectangle in Tools for Interactive Mathematical Activity (TIMA): Bars (Olive & Steffe, 1994) can be accomplished by dragging a rectangle and releasing the mouse button. The 'rectangular object'on the screen is not simply interpreted as

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the static result of drawing the rectangle. Rather, the actions of the student are mentally recorded and become a constitutive part of the concept, 'rectangle.' Furthermore, this sweeping motion is a constitutive property of the concept, 'rectangle'that, when abstracted and measured, becomes the area of the swept region (Steffe & Olive, 1996). They also claim that symbolizing is a constructive mathematical activity because students create their own medium for action and achieve an independence from their experiential world in their mathematical activity. Therefore, "students'symbolizing activity introduces an element of conscious control into their mathematical activity" (Steffe & Olive, 1996, p. 134) just as Skemp (1987a) points out that we can achieve voluntary control over our thoughts by the use of symbols.

Skemp (1987b) also introduces the functions of symbols under ten categories: 1) communication, 2) recording knowledge, 3) communication of new concepts, 4) making multiple classification straightforward, 5) explanations, 6) making possible reflective activity, 7) helping to show structure, 8) making routine manipulations automatic, 9) recovering information and understanding, and 10) creative mental activity. In terms of types of symbols, Pimm (2002) differentiates symbols as a sign from as a counterpart. Whereas a sign names or points to something else, and has no necessary relation to the thing named, a counterpart stands for something else, but does not mean or point to it. Rather, a counterpart has an actual relation, a resemblance or connection between the object and its counterpart. In this regard, Pimm clarifies that "counterparts offer visible or tangible substitutes which are then available for 'manipulation', for acting on as if they were the object itself" (p. 264). Similarly, based on Goodman's (1968/1972) distinction for types of notation systems, Kaput, Blanton, and Moreno (2008) point out the difference between (a)analogue and iconic notation and (b) character-based notation. In former case, the notation embodies physical features that can be mapped directly onto its presumed referent. On the other hand, character-based notation has a different kind of notation-referent relationship. When the notation involves an arbitrary visual shape as with alphanumeric characters, the relation between the configuration of the notation and the phenomenon is far less direct and in need of much more interpretation.

As with the binary distinction for types of symbols, many researchers have been emphasizing a dual aspect of symbols or a reciprocal relationship between mental operations and represented symbols in mathematics learning. Basically, it has been argued that mathematical symbols facilitate "a compression of knowledge developing the ability to pivot between mental concepts to think about problems and time-dependent processes to do mathematical operations to produce solutions" (Tall, Gray, Ali, Crowley, DeMarois, McGowen, Oitta, Pinto, Thomas, & Yusof, 2001, p. 81). Similarly, Kaput et al. (2008) point out that the generalization expressed in symbols should be compacted and crystallized for those symbolizations to be meaningful or possible. Skemp (1987b) also asserts that a symbol is empty and meaningless without an idea attached. "Once the connection is established, its meaning is projected on to the symbol, and the two are perceived as a unity" (Skemp, 1987b, p. 47). Once some symbolization of a generalization is established, then the symbols themselves can help further the reasoning process, which corresponds to a function of symbols introduced by Skemp (1987b): making possible reflective and creative mental activity. While distinguishing types of symbols above, Pimm (2002) also proclaims that "one practice that is the hallmark of mathematics occurs when symbols start being used as if they were the objects themselves, namely as counterparts" (p. 265) and this argument has to be considered really serious in mathematics education. Traditional algorithms that have been learned only as procedures in a symbol system without active symbolization lack the referential connection to the reference field, and such deficiency results in not being able to support the kinds of attentional coordination afforded by a notation system (Kaput, Blanton, & Moreno, 2008). That is, students' being able to reason reversibly with their symbol systems presupposes that the symbol systems themselves become 'object of thought' (Steffe & Olive, 1996).

In sum, in order for the use of symbols to be investigated in the context of students' mathematical learning, it is crucial not only to identify the independence between students' representing symbols and their mental, mathematical operations, but also to examine the reciprocal interactions between students' mental constructs and their external representations. Therefore, in this study, based on the suggested two important aspects (independence and reciprocal interactions) the analyses of several episodes with two seventh-grade students are presented. Specially, for this research, students' mathematical activities in the problems related to enumerative combinatorics are the focus. Enumerative combinatorial problems are counting problems. That is, students should count the number of ways that certain patterns can be formed. However, comparing with a simple counting problem such as "How many apples are on the table?", we could notice that there is a huge difference between those two counting situations because enumerative combinatorial problems require more than students'whole number concepts and direct sensory-motor information. That is, for a combinatorial problem, students should, first of all, attend to units of the indefinite, but should-be-counted quantity based on the assimilated problem situation and further be able to be aware of when to stop counting them. For example, consider the outfit problem: If you have two shirts and three pairs of pants, how many outfits can you make? For this problem, students should construct units that are to be counted, which means that one shirts and a pair of pants should be combined as a countable unit. Steffe (1992), in his teaching experiment, observed students'construction of such mathematical operations and called them as lexicographic units-coordinating operations. Thus, the construction of lexicographic units-coordinating operations and symbolizing actions for those operations open pathways for students' further mathematical activities, which might lead them to construct combinatorial reasoning. With this reason, the research questions for this study of our teaching experiment data were

- Investigate the independence of students' mathematical operations with theirr representing symbols - How did two students in the teacher experiment construct their unit-coordinating operations and further symbolize their mathematical constructs in the context of enumerative combinatorial problems?
- Identify the mutual influences between the modification of students'mental constructs and their externally representative forms - How did the students' externally represented symbolic system reciprocally help them modify their mathematical operations in those problem situations?

II. Methodology

1. Teaching experiment methodology

In this section, we present the teaching experiment (Steffe & Thompson, 2000) as a methodology for conducting scientific research on mathematics learning. A primary purpose for using a teaching experiment methodology is for researchers to experience students' mathematical learning and reasoning. In other words, the assumption is that there would be no ground for understanding the mathematical concepts and operations that students construct without the experiences benefitted by teaching. Researchers who do not engage in teaching of children run the risk that their models might be biased to reflect their own mathematical knowledge (Cobb & Steffe, 1983). The teaching experiment methodology is deeply rooted in radical constructivism in the sense that researchers in teaching experiments attribute mathematical realities to students that are independent of their own mathematical realities and, therefore, a primary goal of the teacher in a teaching experiment is to establish living models of students' mathematics. Steffe and Thompson (2000) argue that the goal of establishing living models is sensible only when the idea of teaching is predicated on an understanding of human beings as self-organizing and self-regulating. According to Steffe and Thompson (2000), teaching experiment methodology is based on the necessity of providing an ontogenetic justification of mathematics and that kind of justification is different from the impersonal, universal, and ahistorical justification. In other words, mathematics should be regarded as a product of the function of human intelligence (Piaget, 1980, as cited in Steffe and Thompson, 2000) rather than as a product of impersonal, universal, and ahistorical reason.

A teaching experiment consists of a sequence of teaching episodes. A teaching episode includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what happens during each episode (Steffe & Thompson, 2000). The important duty of the teacher-researcher in the teaching experiment is to attempt to put aside his or her own concepts and operations and not to insist that the students learn what he or she knows. The research hypotheses one formulates prior to a teaching experiment usually guide the initial selection of the students and the researchers' overall general intentions, but also new hypotheses are to be generated and tested during the teaching episodes. However, it should be noted that the researchers do their best to 'forget' these hypotheses during the course of the teaching episodes and to remain aware of the students' contributions to the trajectory of teaching interactions because students'unanticipated ways and means of operating force the teacher-researcher to abandon prior hypotheses while interacting with the students and to create new hypotheses and situations on the spot. Through generating and testing hypotheses, boundaries of the students' ways and means of operating can be formulated (Steffe & Thompson, 2000).

2. Teaching experiment for the present study

The data for this study were collected from a year-long constructivist teaching experiment, in which a pair of seventh-grade students was taught at a rural middle school in north Georgia from October 2007 to May 2008.The experiment is part of the larger, longitudinal study called the Ontogenesis of Algebraic Knowing (OAK), whose purpose is to bring forth and understand middle school students'algebraic reasoning. Carol and Damon, the two participants of this study, had been chosen on October of 2007 and paired after an individual selection interview. The criterion for selection of the two students was the ability to use composite units as iterable units, which is an indicator of their multiplicative thinking. During the teaching experiment, we met once or twice a week in about 40-minute teaching episodes in which the primary author participated mostly as a teacher-researcher, or sometimes as a witness-researcher. All teaching episodes were videotaped with two cameras for on-going and retrospective analysis. One camera usually captured the whole picture of interactions among the pair of students and the teacher/researcher, and the other camera followed the students'written or computer work with the aid of two witness-researchers. The role of the witness-researcher was not only assisting in video recording but also providing other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and the retrospective analysis of the videotapes.

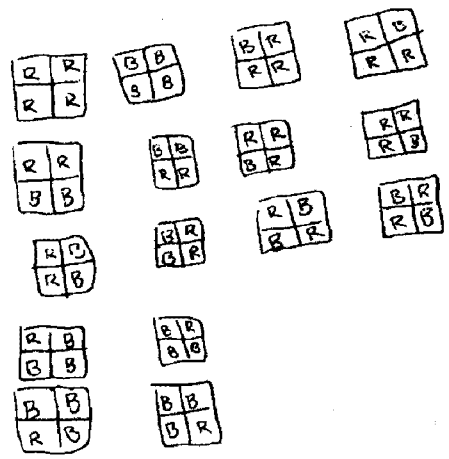
In terms of data analysis, the first type of analysis was ongoing analysis that occurred by watching videos of the teaching episodes and debating and planning future episodes. For the most part, the resources from two cameras were mixed for a single, digitalized video file on the day of each teaching episode. In this way, we created a restored view of the teaching experiment. "A restored view is a wider view of activity than can typically be captured with an individual camera, but is still a selective view that reflects the researchers' perspective of the recorded lesson" (Olive & Vomvoridi, 2006, p. 21). Then a sequence of summaries for the teaching episodes were created in consecutive time, each of which provided not only a written description of students' mathematical activities and interactions with the teacher, but also emerging key points in students'thinking and learning that were taken into account for the next teaching episode. The second type of analysis, which was conducted after finishing the process of data collection, is retrospective analysis. The purpose of the retrospective analysis of the sequence of teaching episodes is to make models of students' ways of operating mathematically through conceptual analysis of students' mathematical activities. First of all, a researcher is involved in understanding what the students'actions are and hypothesizing why the students acted in such ways. Then researcher's construction of the mathematical constructs attributed to the students can be drawn from the process. After construction of such constructs, the researcher revisits the recorded teaching episodes and consciously goes back and forth over the records to see how those constructs are related to the other mathematical constructs and structures.

III. Analysis

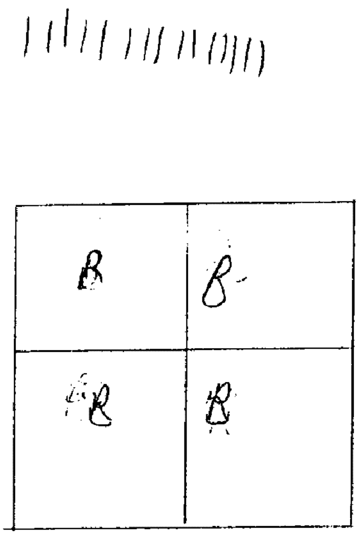
1. Episode 1 Coloring-window problem

(February 4): Find how many ways to paint four windows with two colors.

Carol and Damon, two seventh-grade students in the teaching experiment, were given a picture of a window consisting of four sub-windows. Shin and Steffe (2009) reported that both of students executed additive enumeration. Carol tried to draw pictorial symbols for all possible windows and counted them all whereas Damon repeated writing and erasing the letter 'r' (for color red) and 'b'(for color black) on a given window while adding tally marks above the window for executed counting (see [Figure 1] and 2). Finally, Damon found sixteen cases whereas Carol got fourteen, missing two cases.



[Figure-1] Carol's pictorial representation

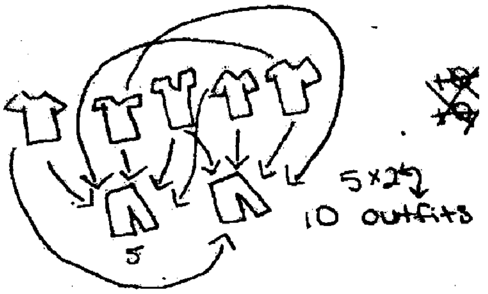


[Figure-2] Damon's representation using tally marks

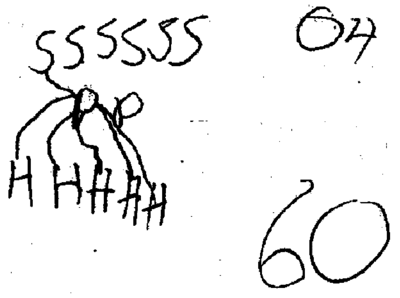
Carol symbolized units of her unit-coordinating operations by drawing identical-looking windows like a given window, whereas Damon symbolized his results of units-coordinating operations using tally marks with an aid of writing possible initial letters for each window. Their graphic representations could be regarded mathematically equivalent in that both symbols (pictures and tally marks) played a role as counterparts for their coordinated units. In other words, there is a resemblance between each Carol's drawn window and each Damon's tally mark in that they were constructed as an individual countable unit for the indefinite (but should-be-counted) quantities in this problem situation. However, two students' continuing activities revealed that a crucial difference existed in their ways of symbolic representation for further development of combinatorial reasoning. After checking out his answer with the teacher, Damon exclaimed that "Look at it. There is an easier way!" Then, his following explanation indicated that he constructed a multiplicative structure for his counted windows. That is, he argued that the answer could be easily found by two to the fourth because we could use two colors for four squares (sub-windows). Carol, however, did not even seem to get a clue for reflecting her answer multiplicatively by Damon's explanation although she finally drew her missing two windows and got sixteen as an answer. At this moment, however, the use of such a multiplicative structure as given for further recursive operations can not be attributed to Damon because he could not provide satisfactory justification for why two to the fourth worked for his problem (Shin & Steffe, 2009). Nevertheless, it could be assumed that his way of symbolic representation opened pathways for his construction of combinatorial reasoning because through repeating to write and erase possible letters on a given sub-windows, each sub-window might be able to imply to him a multiplicative factor of the should-be-counted possible windows (in this case, the factor, two of the quantity, sixteen). To construct a multiplicative structure for an indefinite quantity is a crucial mathematical modification because such a structure convinces a student to close her attention and terminate the counting activity, and further helps her interiorize conducted actions and operations so that she could use them as a priori for the construction of more abstracted combinatorial reasoning. Damon's graphical symbols and the way of representation, although he might not intend to, seemed to emphasize the positioning for variables, which led him to reorganize his counted quantity as having a multiplicative structure. In sum, two students' mathematical activity for this coloring-window problem confirms that Steffe's and Olive's (1996) view about reciprocal interactions between mental operation and symbolic representation. That is, the fact that "students' re-presentation constituted a reenactment of prior symbolic mental operational activity is really crucial because by reenacting those symbolic mental operations using the productive activities of making drawing or writing numerals, the symbolic mental operations are coordinated with elements in the current productive activity and become recorded and connected to these elements" (p. 125).

2. **Episode 2 Card-deck problem** (February 4): There is a deck of cards. If you draw the first card and put it back to the deck, and draw the second card from the deck of cards, how many combinations of a pair of cards can you make?

When the card-deck problem was posed following the coloring-window problem, almost two minutes after teacher's asking of the problem, Carol wrote down '104' by adding '52' vertically to '52' on her worksheet and Damon got '1352' by the written calculation of '52' times '52' divided by '2.' Observing the students' struggling with the problem, the teacher turned to a different problem situation and asked two different outfit problems - for Carol, "If you have five different shirts and two different pairs of pants, how many different outfits can you make?" and for Damon, "If you have six shirts, two pairs of pants and five hats, how many outfits can you make?"The difference of difficulty between two problems for Carol and Damon was the improvised response of the teacher based on two students' mathematical behaviors shown in the previous problem. Carol solved her outfit problem by drawing pictorial symbols for five shirts and two pairs of pants and matching each shirt with each pair of pants by drawing lines between them (see Figure 3). On the other hand, Damon got '60' by writing six 's'es, two 'p's, and five 'h'es and drawing some lines among them (see Figure 4). He explained that "one shirt goes with that pair of pants and multiply by those hats, so that's five combinations, and so there are six total shirts, five times six equals to thirty and then do the same thing with that pair of pants, which is also thirty and thirty plus thirty is sixty"



[Figure-3] Carol's symbols for outfit problem



[Figure-4] Damon's symbols for outfit problem

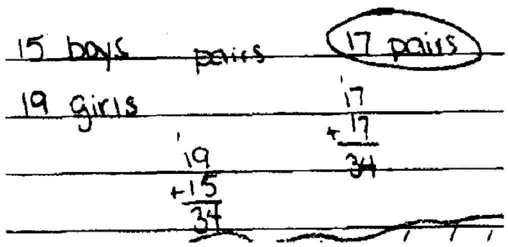
When the teacher increased the number of shirts and pairs of pants to fifty two, which was the same number of objects (cards) in the card-deck problem, both students immediately multiplied fifty two by fifty two. Carol justified her answer that "fifty two shirts can go with one pair of pants and do that for each pair of pants" However, when the teacher asked Damon the card-deck problem immediately after Carol's explanation about the outfit problem with fifty two shirts and fifty two pair of pants, his written answer was '26×26', changed from '(52×52)/2' even though he also did have right calculation for the outfit problem with fifty two shirts and fifty two pairs of pants.

The card-deck problem appears to have a different feature, which might be a crucial factor for the problem situation to be less accessible to two students. Contrast to the outfit problem where two or three independent sets of outfits for students'units-coordination activity were given, in the card-deck problem the students should have constructed, with one deck of cards, two pools of cards for the first and the second card to make a pair of cards to be counted. While assimilating the problem situation, the students might have difficulty in representing individual coordinated units for the situation. Such difficulty seemed to prevent the students from counting possible pairs of two cards, which resulted in Carol's first answer, one hundred and four, which was fifty two plus fifty two. The cognitive difference in assimilating the two problem situations also could be witnessed when the teacher suggested the outfit problem as an alternative. Carol solved the outfit problem by construction of counterparts for the objects (i. e., shirts and pairs of pants) and units-coordinating operations with them. Damon could even solve the outfit problem for triples as similarly as Carol although his symbols were literal, not pictorial. Furthermore, both of them could solve the outfit problem with fifty two shirts and fifty two pairs of pants by multiplying fifty two by fifty two. However, when going back to the card-deck problem situation, Damon failed to obtain fifty two times fifty two. In terms of symbolic representations, they were not able to construct any form of symbols, with which they could have counted possible pairs of two cards. That is, when they were solving the outfit problem with fifty two shirts and fifty two pairs of pants, they already had (actually constructed) a sequence of symbolic crutches inherited from the previous mathematical activity with a small number of shirts and pairs of pants. Their symbolizing activities and the results of such symbolization made possible for them to expand their symbolic procedure so that they could overcome the limitation of their graphical or literal representation system, that is, the limitation of not being able to draw all fifty two shirts and fifty two pairs of pants on their worksheets. However, in the case of the card-deck problem, they did not have or construct any form of symbols, which could play a role as a crutch. Although Carol drew two card shapes on her worksheet and Damon wrote down some numerical notations, it was not enough to believe that necessary units-coordinating operations were embodied in those symbols. Rather, those mental operations were not experientially accessible to them through their drawings and numerals. In sum, their failure for the solution implied that Damon, let alone Carol, had yet to reorganize their bounded indefinite quantity in the problem as a multiplicatively structured one. Although Damon wrote down two numerical symbols of '52' for multiplication and one '2'for division, it is inappropriate to conjecture that the two '52's in his answer were symbolizing the construction of a multiplicative structure as counterparts for the first and the second positions in a coordinated pair of two cards. Rather, he seemed to assimilate his number concept of fifty two to his problem situation and tried to modify it without units-coordinating operations. This hypothesis was corroborated by the following observation that he could not provide a clear explanation for his answer and even changed it into twenty six times twenty six.

3. **Episode 3 Boy & girl problem** (February 11): There are 15 boys and 19 girls in a classroom. How many pairs of a boy and a girl can you have? -

Although several outfit problems were dealt with a week before, Carol wrote down '15 boys' and '19 girls' on her worksheet and she added them up and divided it by two, which resulted in

'17 pairs' on her worksheet as an answer (see [Figure 5]).

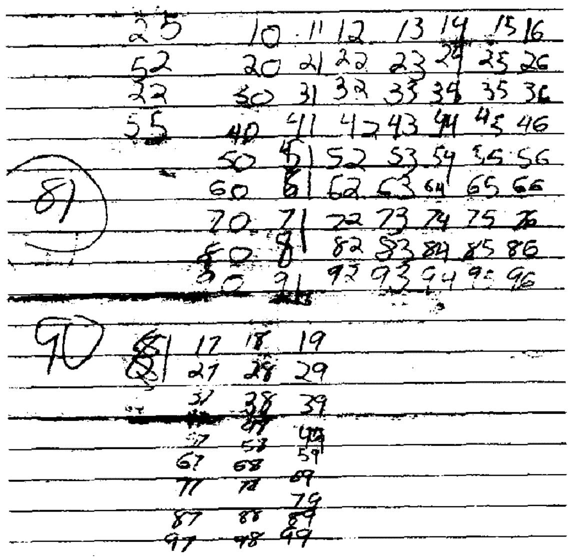


[Figure-5] Carol's solution for boy & girl problem

Her answer was to count how many pairs she could make with thirty four persons (fifteen boys and nineteen girls), rather than to count possible combined pairs between fifteen boys and nineteen girls. In other words, her written symbols stood in for her unit-segmenting operations with thirty four by two. Her symbolic use for the construction of the problem situation and the necessary combining activities for the solution were very different from what she showed in the previous outfit problems. The large numbers, fifteen boys and nineteen girls, compared with five shirts and two pairs of pants, seemed to lead her to represent the problem situation by writing numerical symbols '15' and '19', rather than drawing pictorial icons for all boys and girls as in the outfit problem and these symbols might prevent her from conducting pairing activities among boys and girls, which was possible in the outfit problem by drawing lines between each pictorial shirt and each pair of pants. Therefore, it could be argued that inter-dependent interactions and transformation happened between Carol's mental operations and her external symbolic representations while she was solving this problem. That is, she transformed her experiential construction of the problem situation into two numerical symbols, which appeared to be more compressed than pictorial symbols in that two numerals stood in for fifteen boys and nineteen girls, and then the two numeral symbols reflectively influenced and interacted with her mental operations for pairing 'something'The compactness of such symbolic representations might activate her unit-segmenting operations of thirty four by two, rather than trigger her lexicographic units-coordinating operations. On the other hand, Damon immediately wrote down numerical symbols '15' and '19' for multiplication of two numbers, and got '285' by numerical calculation on his worksheet. We claim that Damon's use of numerical symbols, '15' and '19', were totally different from Carol's use of those symbols because his numerical representations and its calculation activity possibly symbolized his lexicographic units-coordinating operations between fifteen boys and nineteen girls. In other words, unlike Carol's use of numerical symbols, Damon's construction of the problem situation and units-coordinating operations were transformed into his numerical representations. Steffe and Olive (1996) argue that "to be constituted as symbolic, the sensory material recorded in the operations would need to be recorded at the level of interiorization"(p. 132). Therefore, his writing of two numerals can be regarded as symbolic constitution in that the sensory materials recorded in the operations of matching boys with girls were interiorized and recorded in the two numerals. This argument was corroborated by his following explanation "I got '285'cause like, fifteen boys can be with one girl and then I multiplied by nineteen and I got two hundred eighty five". Carol also exclaimed that she understood it while Damon was explaining his answer. Although there was no further focus on this problem, Damon's verbal representation about the problem situation and his conducted operations for the solution seemed to activate her units-coordinating scheme.

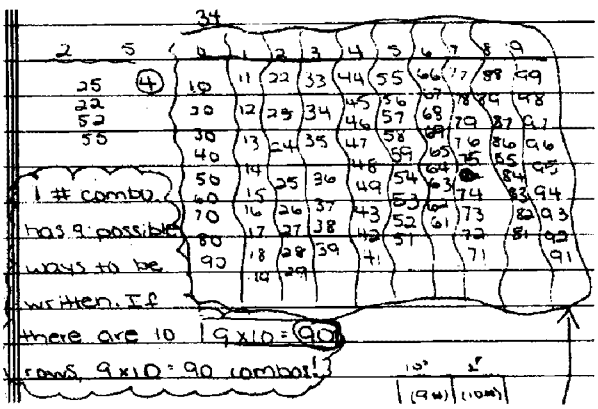
4. Episode 4-1 Two-digit number problem (February 11): How many two-digit numbers can you make?

Although Damon constructed a multiplicative structure for the solution of the boy & girl problem, at this time he started to write numerical symbols from ten to ninety by ten in a column and kept writing eleven, twenty one, and thirty one on the right of ten, twenty and thirty making another column and then stopped. After murmuring something, he wrote '81'little away from two columns as an answer, but resumed writing two-digit numbers for making sure of his answer because the teacher gave him a negative message for his answer "go ahead (keep writing) and check your answer" Then he finally corrected it as '90'for his answer (see [Figure 6]). He explained that zero in one-digit place can go with nine numbers in ten-digit place and so do the other numbers in one-digit place. However, he failed to find the right answer until he ended up with writing down tabular forms of numerals for all possible two-digit numbers (Shin & Steffe, 2009).



[Figure-6] Damon's answer for two-digit number problem

Although his explanation was similar to that for the boy & girl problem, his symbolic use for representing of the problem situation and necessary operations for the solution was quite dissimilar in that he had to transform all possible units-coordinating operations into symbolic counterparts each by each for him to be able to count them. This observation was quite in contrast to that of the boy & girl problem where only two numerical symbols and their calculation symbolized the problem situation and all required units-coordinating operations. His comment that he thought his table was going to be a nine by nine table in the middle of his problem solving process supports that he had to finish making tabular numerical representations for the completion of his problem solving activity. This also indicates that his construction of a multiplicative structure for the solution of basic enumerative combinatorial problems was not perfect in a sense that he failed to transform into a compressed form of symbols, for example '9×10=90' , for this problem situation structurally similar to the previous one. In other words, he could not construct and utilize the signs like '9×10=90' just as he did with his counterparts, the written tabular form of numerical arrays for this problem. Similarly, Carol made a table for arranging all possible two-digit numbers and after writing all possible two-digit numbers, she wrote '9×10=90' under the table (see [Figure 7])

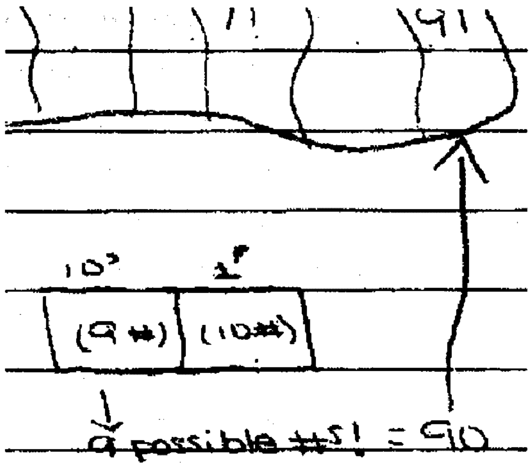


[Figure-7] Carol's answer for two-digit number problem

After getting the solution, she exclaimed "I know a faster way. I could have done that. Oh~" and as an explanation she said "nine different combinations (pointing out columns) in each way. There are ten rows (actually ten columns) and ten times nine is ninety···If I figured out that each row has nine, I could have just done that" Nevertheless, it is hard to believe that Carol's construction of the problem situation and her mental operations for the solution were transformed into a multiplicative structure like the numerical symbols, '9×10 = 90' prior to operating. Rather, as Skemp (1987b) illustrated as one of functions of symbols, Carol's numeral table for her solution made possible her reflective thinking for this problem and to reorganize the situation, which led her to represent her construction of the situation and operations for the answer by writing a compressed form of numerical notations (Shin & Steffe, 2009). At this moment, the use of a multiplicative structure as a given material for the solution can not attributed to her.

5. Episode 4-2 Two (three)-digit number problem (February 11): How many two (three)-digit numbers can you make? (using drawn boxes)

At the moment, the teacher intervened and asked them to use two boxes for the same problem. The problem was restated as "in two connected boxes, how many two-digit numbers can you write down?"The intention of the teacher was to encourage them to use a more compressed form of symbols for the problem, which might help them construct a multiplicative structure for the indefinite quantity, that is, the number of possible two-digit numbers. There was some confusion in communication between the teacher and two students when the teacher was giving verbal explanations for the use of boxes, but after additional verbal interactions for clarification, both students easily solved the two-digit number problem using two connected boxes (see [Figure 8]). The explanations for their answers were similar - in ten-digit place nine numbers can go in and in one-digit place ten numbers can go in, therefore the answer is nine times ten, which is ninety.



[Figure-8] Carol's use of boxes for two-digit number problem

Excited with the students' being able to use box symbols, the teacher expanded the problem situation to making of three-digit numbers as "how many three-digit numbers can you make?" Although the teacher did not explicitly ask them to use box symbols for the three-digit number problem, they voluntarily started with three connected boxes. However, Carol's answer was eight hundred ten rather than nine hundred because she wrote down '9' in the box for the ten-digit place whereas Damon put '10' in the corresponding box. She explained that she just multiplied nine in the hundred-digit place by the obtained answer in the previous two-digit number problem, but she immediately corrected her answer with an aid of Damon's explanation pointing out that "zero can go in to the second spot since it's a three-digit number". However, it is questionable that they had constructed a recursive multiplicative structure as a priori for three-digit numbers. For students'operations to be recursive, they should be able to externalize the results of their scheme and operate on these results with operations external or internal to the scheme (Tillema, 2007). Instead, with an aid of such pictorial (box) symbols as a tool for their mental operations, they seemed to begin to construct the quantity of indefinite three-digit numbers as having a recursive multiplicative structure among three independent variables, that is, a one-digit place, a ten-digit place, and a hundred-digit place. We claim that this transformation from arrays of numerical symbols for all individual numbers into a more compressed form of connected box symbols might be really crucial for their construction of combinatorial reasoning because the latter can open pathways for students to overcome the limitation of dimensionality of the former representation system and construct recursive multiplicative thinking for enumerative combinatorial problems. In this sense, the teacher's introduction of box symbols was helpful for the development of their combinatorial reasoning.

6. Episode 5 Three-digit number problem with five different numbers (February 11): If you are given five numbers, 1,2,5,8, and 9, how many three-digit numbers can you make from those numbers? (You can use any number more than one time)

When the problem situation was changed a little as above, the students did not begin with box symbols. Damon started to write possible three-digit numbers from '125', but stopped when he had nine three-digit numbers. Then he drew three connected boxes and wrote down the numerical symbol '5'in each box. The ensuing justification for his answer indicated that he constructed a multiplicative structure for counting of thee-digit numbers with five different numerals. He said "All five numbers could go in the hundredth place, and all five numbers could go in the tenth place, and all five numbers could go in the first place. I multiplied five times five times five and I got one hundred twenty five" His explanation confirmed the claim that teacher's introduction of box symbols facilitated the construction of a multiplicative structure for this combinatorial problem. That is, box symbols played a role as 'helping to show structure'for the indefinite quantity (Skemp, 1987b) and further with an aid of box symbols he was able to modify his multiplicative reasoning for this problem situation. On the contrary, Carol misunderstood the problem as not allowing the repetition of any number, and wrote all possible three-digit numbers starting with '1', which resulted in twelve numbers, and then got sixty by multiplying five. Although she realized her misinterpretation of the problem, she failed to solve it regardless of the teacher's encouragement of the use of box symbols. At the moment, the other teacher (a witness-researcher) intervened and asked Damon to solve what Carol did, that is, not allowing the repetition of any number. Immediately, he drew three connected boxes and wrote down the numerical symbol '4' in each box. However, after seventy seconds later, he replaced '4' in the ten-digit box with '5' and got eighty by multiplying five times four times four. He explained that "the hundred-digit place can be five cases but the ten-digit and the one-digit can not because that breaks the rule" He did not seem to realize that his '4' in the one-digit place actually broke the rule. The fact that he failed to solve the last problem has a significant meaning because it indicates that students' sense of difficulty with the problem, in a certain degree, might depend on their representing symbolic system. In other words, Carol, who appeared to be left mathematically a little behind compared with Damon, had no problem with the arrangement of five numerals without repetition. She was able to solve the problem by writing down some possible numerals as counterparts for the combinations of three numerals for three-digit numbers. On the contrary, Damon relied on three connected boxes for the arrangement problem without repetition as he did for the problem with repetition. Unfortunately, as long as he was using three connected boxes for the former problem, he appeared to lose track of his thinking - there was a seventy-second gap until getting his final answer, eighty, which was wrong. We conjecture that his sense of difficulty might be due to his use of three connected boxes. The difference between the problem with repetition and the other without repetition is whether the range of numbers is restricted by the previous choice or not. That is, if a number is chosen for a hundred-digit place and repetition is allowed, the student could choose any number for a ten-digit place and for a one-digit place independent of the chosen number in the hundred-digit place. However, if repetition is not allowed, the student should take into account of the first choice in a hundred-digit place for the second choice in a ten-digit place and further keep in mind the first and the second choices for the third in a one-digit place. This dependency among three digit places inhibited Damon from constructing a multiplicative structure for the indefinite three-digit numbers. Arguably, he could have solved the problem if he had tried it by writing down each possible three-digit numbers using numerical symbols as Carol did. In arrays of numerical symbols, the dependency among three digits is implicit because the dependency is already embedded in each three-digit number. However, in order to use box symbols for the problem without repetition, students should be aware of not only the multiplicative structure of indefinite three-digit numbers, but also the dependency among those three digits.

IV. Conclusions

Shin and Steffe (2009), through a year of teaching experiments with Carol and Damon, showed that the students' enumerative combinatorial counting constructs could be based on their units-coordinating operations, and further identified three distinct levels of enumeration in the students' mathematical behavior: additive enumeration, multiplicative enumeration, and recursive multiplicative enumeration. In the developmental processes of their combinatorial thinking, various kinds of symbols were implemented by the students, or sometimes by the teacher/researcher, which resulted in a lot of unexpected mathematical responses of the students. That is, it was in evidence that such utilized symbols or symbolic systems played a critical role in the students' development of combinatorial reasoning. Thus, of particular important is to explore the students' ways of using symbols involved in their mathematical behaviors through several teaching episodes. At the time of the end of our teaching experiments, Damon was in the process of construction of a recursive multiplicative scheme for enumerative combinatorial problems by modifying his multiplicative reasoning, but Carol appeared to stay at the stage of construction of a binary multiplicative scheme, indicated by her juxtaposing of counterparts for all combined digit numbers in episode 5. These teaching episodes imply that an introduction of a different symbolic system, which might be viewed as efficient from a teacher's point of view, does not always guarantee students'expected use of the introduced symbolic system and mathematical activities with them just as Carol failed to utilize connected box symbols for digit number problems and had to go back to writing down individual three-digit numbers. In addition, the use of different symbols even for the same problem could reveal a different or a hidden aspect of the problem situation, which might disturb students'problem solving process. Once Damon started to see the problem situation as a digit-variable structure in connected boxes, the problem without repetition seems to turn out to be a harder one for him than that with repetition because of the revealed dependency among digit variables.

In sum, this study deals with the idea of students' re-presentation of their mental operations embedded in symbols and the ways of external formulation into symbolic systems. In mathematics classroom, if building a consensual domain of communications among the teacher and her students would be one of the most important goals, understanding students' ways of representation of their mathematical ideas to external forms of symbols should be viewed as really crucial because the link of a particular symbol or a symbolic system to a particular mathematical concept can not be predetermined. Through the analysis of our teaching episodes, we have discussed how two seventh graders modified their multiplicative reasoning and interacted with represented symbols in combinatorial problem solving processes. The observation of two students' struggles with represented symbolic systems tells us that even well-formed symbolic systems from teacher's point of view do not necessarily guarantee the facilitation of students' reasoning competencies. Skemp (1987a) declare that "symbols are magnificent servant, but bad masters, because by themselves they do not understand what they are doing" (p. 188). In addition, a view of mathematics learning as social and cultural activity in current mathematics education seems to emphasize the importance of teachers' understanding of students' representing symbols and their reciprocal interactions as dynamic media in classroom.

We claim that the drawn implications from this study must be brought into play in the wider context of the processes of development involved in learning algebra, calculus and beyond, because the more advanced mathematics students have to learn, the more important the role of symbols becomes in their learning process. However, as in Carol's and Damon's developmental trajectories when solving combinatorial counting problems, without teacher's attention of students'reciprocal interactions between their symbolic representation system and mathematical mental operations, those symbols will turn out to be bad masters, rather than magnificent servant. "Symbolic operations can become the focus of instruction once students have developed coherent and stable meaning that they may express symbolically" (Thompson & Saldanha, 2003, p. 109).

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중학교 1학년 학생들의 '경우의 수' 문제 해결과정에서 나타나는 표현기호와의 상호작용을 통한 곱셈추론 양식의 변화

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본 연구는 일 년간 행해진 두 명의 중학교 1학년 학생들과의 교육실험 자료로부터 '경우의 수'를 다룬 문제를 푸는 과정에서 어떻게 두 학생들이 자신들의 곱셈 추론양식을 수정, 변경해 나가며, 표현된 기호들과 상호작용을 해 나가는 지 보여주고 있다. Damon은 그의 곱셈적 추론방식을 수정하여 순환적 곱셈 추론의 구성단계에 있으며, Carol은 여전히 이원적 곱셈 추론 도식에 머물러 있는 것으로 보여 졌다. 이 연구는, 학생들이 그들 자신이 구성한 또는 교사에 의해 제시된 기호들을 다루는 과정에서 두 학생들이 겪는 어려움을 통해, 교사의 관점에서 잘 구성된 기호양식이라 하더라도, 그것이 반드시 학생들의 수학적 능력향상으로 연결되지 않을 수도 있음을 보여주고 있다.

* key words : multiplicative reasoning(곱셈 추론), combinatorial thinking(조합 추론), symbolic system(기호 체계)

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