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Mathematics Teachers' Abstraction Levels and Multiple Approaches: The Case of Multiplicative and Divisibility Structure of Numbers

Unal, Hasan
Mathematics Education, Florida State University, Tallahassee, FL 32306, USA;
Email: huu1932@fsu.edu

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The purpose of this study was to investigate middle and high school mathematics teachers' levels and multiple approaches in United States practicing their abstraction levels and, different strategies and method of solutions towards given number theory problems.
The mathematics teachers taking part in this study are consisted of 25 members of online graduate and undergraduate course (MAE 5641 and MAE 4813) delivered through Online Learning System called as the Blackboard (http://www.blackboard.com). Data collection methods include journal entries, written solutions to problems, the teachers' reflections on said problems, and post interviews. Data analysis was done based on [Hazzan, O. & Zazkis, R. (2005). Reducing abstraction: The case of school mathematics. *Educ. Stud. Math.* **58(1)**, 101–119]. Analysis of students' written solutions revealed that transitions among the solution methods have major effect on abstraction levels. Elevation and reducing abstraction is a dynamic process.

Keywords: reducing abstraction, number system, mathematics teachers
ZDM Classification: C30
MSC2000 Classification: 97C30

1. INTRODUCTION

Abstraction is essential/fundamental part of mathematics. To improve the students' as well as teachers' abstract thinking skills we need to examine their abstraction process, more specifically reducing and/or elevation of abstraction levels of students. According to Mitchelmore and White (2005) "the abstraction term has different meanings in relation to mathematics and the learning of mathematics" (p. 335). Two different form of abstraction were discussed at the Research Forum on abstraction held at the 26th conference of PME.

(Boero *et al*, 2002). The concept of abstraction is not unitary; however, there are some agreements on some aspects of abstractions (Hassan & Mithcelmore, 2005; Hazzan, 1999; 2001; Eraslan, 2008). Glasersfeld (1991) clarifies the core of notion with the help of John Locke (1632–1704)'s definition because it is simple and widely accepted definition of process:

> "This is called Abstraction, whereby ideas taken from particular beings become general representations of all the same kind; and their names general names, applicable to whatever exists conformable to such abstract ideas" (Cited in von Glasersfeld, 1991).

Dienes (1963) describes abstraction as ... the extraction of what is common to a number of different situations. It is just another word for the formation of a class, the end-point being the realization of the attribute or attributes which make elements eligible or not for membership of the class. (Ozmantar & Monaghan, 2007, p. 90).

This study builds on the study in which a rich collection and cases presented by Hazzan & Zazkis (2005)'s reducing the abstraction level on properties of divisibility and multiplicative structure of numbers. Level of abstraction has been investigated through a number of problems related to prime factorizations and decision making. The problems have potential for examining students' mental process related to both empirical and theoretical abstractions. Empirical abstraction (Skemp, 1986) concept is based on the experience focusing on the identification of underlying features. On the other hand, theoretical abstraction (in mathematics Recognizing, Building-with, and Constructing, Hershkowitz, Schwarz & Dreyfus, 2001) based on epistemic actions. Hershkowitz et al. (2001)'s RBC model of abstraction involves production of final structure by reorganizing structures and making connections with primitive and elementary form (Ozmantar & Monaghan, 2007). The empirical abstraction comes from examining several worked exercises, while theoretical abstraction roots in deep analysis of one problem in order to identify its' essential variables and relationships (Mitchelmore & White, 2007). Participants were members of online (delivered through blackboard system) Number System Course (MAE 5641 and MAE 4813). When the choice was given, students have reduced the abstraction levels. Furthermore, students used elevations in abstraction to reduce abstraction. Abstraction is a dynamic process. Students go back and forth between higher level of abstractions and reduced ones.

2. LITERATURE REVIEW

The concept of reducing abstraction was first introduced by Hazzan (1999) as a result of examining students' understanding of abstract algebra concepts. Hazzan's (1999) model of reducing abstraction is based on interpretations of abstraction levels; retreating on familiar mathematical structures, using a canonical procedure, and adopting a local perspective. According to Hazzan (2001), "The mental process of reducing abstraction level indicates that students find ways to cope with new concepts they learn. They make these concepts mentally accessible, so that they would be able to think with them and handle them cognitively" (pp. 165–166). The concept of reducing abstraction recently has gained attention of Mathematics Education community (*i.e.*, MERJ, 2007 volume 19 issue 2 all articles). The reducing abstraction concepts were used to analyze students' mental process not only in braches of mathematics (*i.e.*, Differential equations, quadratic functions, etc.) but also in computer programming (Hazzan, 2003a; Aharoni, 1999). In one of the latest studies, Eraslan (2008) examined two tenth grade honor's students' thinking from the perspective of reducing abstraction model on given tasks related quadratic functions. Zazkis & Campbell (1996) investigated pre-service teachers' understanding of natural numbers, focusing on the properties of divisibility and multiplicative structure. When researcher asked in one of the interviews:

$M = 3^3 \times 5^2 \times 7$ whether or not "M" is divisible by 7.

Students responded by finding first the M=1575 and divided it by 7, then found 225 without a remainder (using Calculator) and concluded that M is divisible by 7. Instead of investigating structure of given number, students' preference to calculation is discussed in detail in Zazkis & Campbell (1996).

From the dual perspective of process–object, these students reduced the level of abstraction by considering the process of divisibility which attained a whole number result in division, rather than by analyzing the object of divisibility that is a property of whole numbers which could be considered independently of any specific implementation of division.

Following the excerpt of Zazkis & Campbell (1996)'s study, it also illustrates reducing abstraction. When researcher asked in one of the interviews: Under the condition that there is a number between 12358 and 12368 and that is divisible by 7, students were asked to find a number divisible by 7 between the two given numbers in order to reach a decision. Students had to consider 10 numbers. Then, they checked the divisibility of each number separately, rather than a more complex object, a set or interval of numbers.

As they take dialectic approach into account, Ozmantar & Monaghan (2007) explored the validity of RBC model of abstraction, considering scaffolding and peer interactions with Turkish students on tasks related to the absolute value of linear functions. Researchers found that learning absolute value functions is better first into RBC model of abstraction than empirical abstraction. Ozmantar & Monaghan (2007) states that

> "...abstraction did not arise simply from recognition of commonalities but, rather, from creating meaning by relating new observations to what they already knew. This, to us, is

an ascent to the concrete—a process of making meaning by establishing inter-connections amongst elements of the whole — and this is dialectics." (p. 110)

3. THE STUDY

During an online course, focusing on topics in Number Theory, middle and high school practicing mathematics teachers were given the following problems. Teachers were presented the problem, and provided a week to work on it with a goal of finding solution by using any methods. Due to the online attribute of the course, the solution presentations were collected through the course management system.

What are the students' abstraction levels on structured versus semi-structured problems?

How students attempt to reduce abstraction in solving number theory problems?

To what extent, if any, do the students' multiple solution methods and multiple representations take a role on their level of abstractions?

Problem 1: *How many integers between 1 and 100(including these numbers) have 2, 3, or 5 as factors?*

Due to the nature of the problem, multiple representations (such as table, Venn diagram) could be used to solve the problem. We can see if they make any connections among the solutions/representations which would help them for higher level of abstraction. By analyzing students' solution methods and to see if students make any connections among the solutions/representations help them for higher level of abstraction.

Problem 2: *Choose three numbers a, b, c, which are not relatively prime. Determine how many numbers from 1 to n, where n = abc, have a, b, or c as factors.*

In the second problem, students are free to choose their numbers. Different from the first problem, however, they need to select numbers which are not relatively prime.

Problem 3: *Think about the numbers between 1 to 100 (including these numbers) that have 2, 3, or 5 as factors. There are some numbers divisible by all these three numbers. There are some numbers divisible by (2 and 3), or (3 and 5), or (2 and 5). And finally there are some numbers that only divisible by 2 or 3 or 5. What can you tell about the final case? What makes those numbers unique that they ONLY divisible by 2 or 3 or 5?*

In Problem , students have asked directly to make abstractions.

In Problem 1, we can apply different strategies to find the solution such as creating Table, creating Sets, Drawing Venn diagrams, General Case: Formula: Cardinality of sets. Polya's last stage looking back involves generalization which is fundamental part of mathematical thinking, though often neglected in teaching. Generalization largely depends on the students' abstraction levels.

Solution to the First Problem: This problem could be solved by using multiple methods:

1- Set Solution
2- Table Solution
3- Venn Diagram
4- Using the Formula for union of three sets.

Solution 1: Creating Sets and finding the FINAL set.

Set 1: Create elements of a Set consisted of all the numbers that is divisible by 2 between 1 and 100 including these two numbers. Denoted by S_2

$$S_2 = \{2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44,46,48,50, 52,54,56,58,60,62,64,66,68,70,72,74,76,78,80,82,84,86,88,90,92,94,96,98,100\}$$

Set 2: All the numbers that is divisible by 3 between 1 and 100. In other words, all the integers between 1 and 100 have 3 as factor. Denoted by S_3

$$S_3 = \{3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48,51,54,57,60,63,66,69,72, 75,78,81,84,87,90,93,96,99\}$$

Set 3: All the numbers that is divisible by 5 between 1 and 100. In other words, all the integers between 1 and 100 have 5 as factor. Denoted by S_5

$$S_5 = \{5,10,15,20,25,30,35,40,45,50,55,60,65,70,75,80,85,90,95,100\}$$

Final Set:

$$\text{Final Set} = \{2,3,4,5,6,8,9,10,12,14,15,16,18,20,21,22,24,25,26,27,28,30,32,33,34,35, 36,38,39,40,42,44,45,46,48,50,51,52,54,55,56,57,58,60,62,63,64,65,66, 68,69,70,72,74,75,76,77,78,80,81,82,84,85,86,87,88,90,92,93,94,95,96,98, 99,100\}$$

Solution 2: Table Solution

Table solution if basically creating table and eliminating numbers that are divisible by 2, 3 and 5. An example of table solution is illustrated at Figure 4.

Solution 3: Venn Diagram

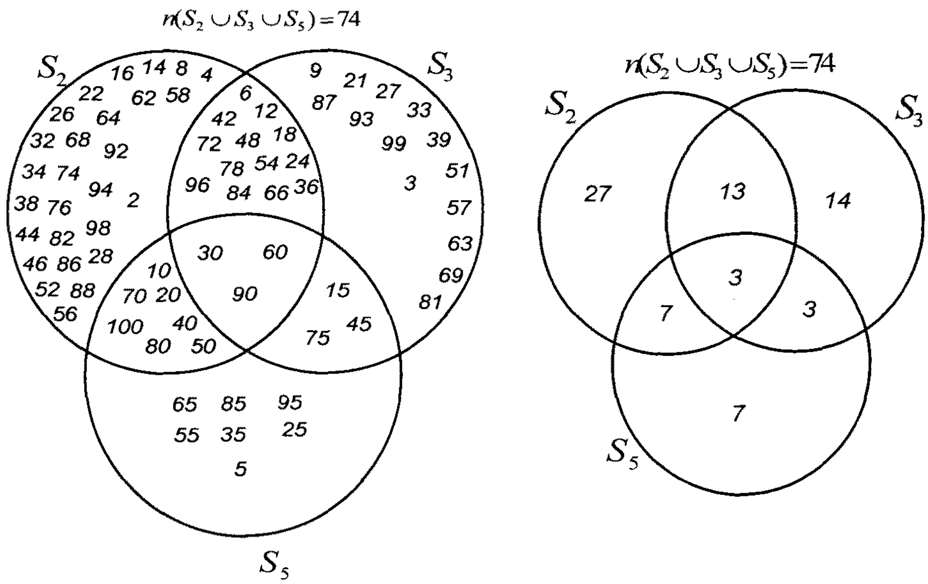

Figure 1. Venn Diagram Solutions

Solution 4: Union of Three Sets Formula

$$n(S_2 \cup S_3 \cup S_5) = n(S_2) + n(S_3) + n(S_5) - n(S_2 \cap S_3) - n(S_2 \cap S_5) - n(S_3 \cap S_5) + n(S_2 \cap S_3 \cap S_5)$$

$$n(S_2 \cap S_3 \cap S_5) = \frac{N}{LCM(2,3,5)} = \frac{100}{30} = 3.33$$

$$n(S_3 \cap S_5) = \frac{N}{LCM(3,5)} = \frac{100}{15} = 6.667$$

$$n(S_2 \cap S_3) = \frac{N}{LCM(2,3)} = \frac{100}{6} = 16.667$$

$$n(S_2 \cap S_5) = \frac{N}{LCM(2,5)} = \frac{100}{10} = 10$$

$$n(S_2) = \frac{N}{2} = \frac{100}{2} = 50$$

$$n(S_3) = \frac{N}{3} = \frac{100}{3} = 33.333$$

$$n(S_5) = \frac{N}{5} = \frac{100}{5} = 20$$

$$n(S_2 \cup S_3 \cup S_5) = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74 \quad n(S_2 \cup S_3 \cup S_5) = 74$$

4. RESULTS

Problem 1 was asking to find the number of integers [1,100] that have 2, 3, or 5 as factors.

Table 1. Summary of Solution Methods of Problem 1

Set Solution	Table	Venn Diagram	Cardinality	No Solution
9	11	1	5	5

Reducing Abstraction Levels

Set solution and table solution methods were the main solution methods used by participants. Either solution method does not require higher order thinking because both are just applications of procedure. Skemp(1987)'s category of conceptual versus procedural understanding, of which MacDonald (1977) named as "learned response", is not different finding area of triangle with given base and height. Students reduced the abstraction level (Hazzan & Zazkis, 2005), listing each element of sets and finding the final set counting once the repeated elements in each set.

S(2): Numbers from 1 to 100 have 2 as a factor (100/2=50)

S(2)={2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44,46,48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100}

S(3): Numbers from 1 to 100 have 3 as a factor (100/3=33.3; 99/3=33)

S(3)={3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90, 93, 96, 99}

S(5): Numbers from 1 to 100 have 5 as factor (100/5=20)

S(5)={5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100}

Union of these three sets gives the answer.

S(2) U S(3) U S(5) =

{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35 36, 38, 39, 40, 42, 44, 45, 46, 48, 50, 51, 52, 54, 55, 56, 57, 58, 60, 62, 63, 64, 65, 66, 68 69, 70, 72, 74, 75, 76, 78, 80, 81, 82, 84, 85, 86, 87, 88, 90, 92, 93, 94, 95, 96, 98, 99, 100}

74 numbers from 1-100 have 2, 3, or 5 as a factor.

Figure 2. Elevation at reduced abstraction level

Some of elevations in terms of abstractions have been observed, however. In Figure 2, students used a short cut/formula to find the number of elements in each set. When solving a problem, creating control mechanism/control points is important. By doing this, problem solver can monitor his/her actions if he/she is on the right track. According to Corey & Unal (2004), the use of meta-cognitive functions results in better problem solving.

S(2) = {2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34,36,38,40,42,44,46,48,50,52,54,
56,58,60,62,64,66,68,70,72,74,76,78,80,82,84,86,88,90,92,94,96,98,100}
S(3) = {3,6,9,~~12~~,15,~~18~~,21,~~24~~,27,~~30~~,33,~~36~~,39,~~42~~,45,~~48~~,51,~~54~~,57,~~60~~,63,~~66~~,69,~~72~~,75,~~78~~,81,
~~84~~,87,~~90~~,93,~~96~~,99}
S(5) = {5,~~10~~,~~15~~,~~20~~,25,~~30~~,35,~~40~~,~~45~~,~~50~~,55,~~60~~,65,~~70~~,~~75~~,~~80~~,85,~~90~~,95,~~100~~}

S(2) U S(3) U S(5) = {2,3,4,5,6,8,9,10,12,14,15,16,18,20,21,22,24,25,26,27,28,30,32,33,
34,35,36,38,39,40,42,44,45,46,48,50,51,52,54,55,56,57,58,60,62,
63,64,65,66,68,69,70,72,74,75,76,78,80,81,82,84,85,86,87,88,90,9
2,93,94,95,96,98,99,100} **A TOTAL OF 74 NUMBERS**

Figure 3. Set Solution

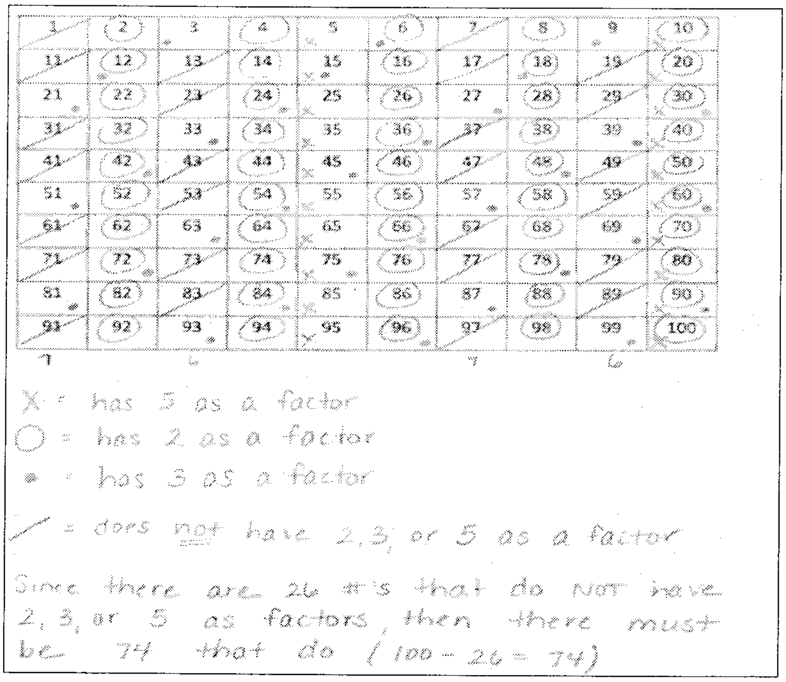

Figure 4. Table Solution 1

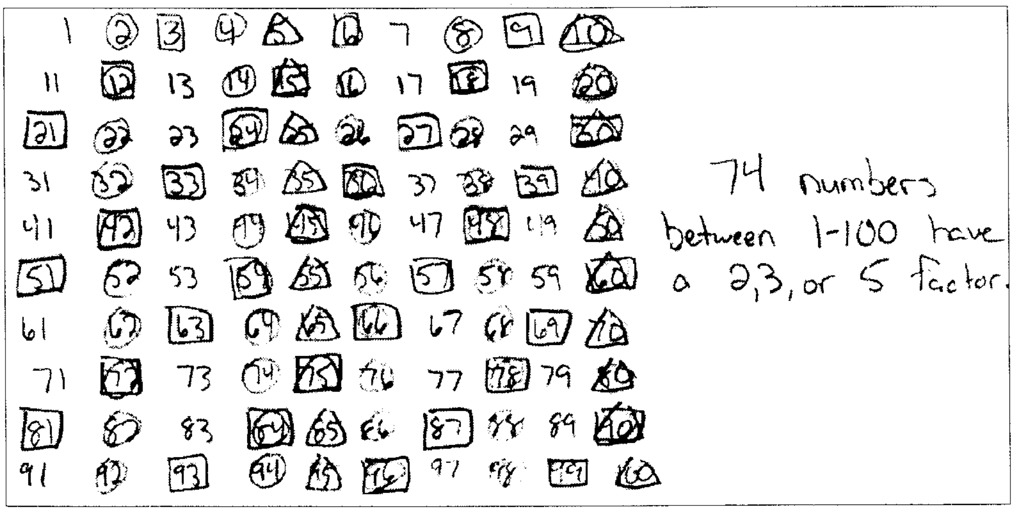

Figure 5. Table Solution 2

Higher Level of Reasoning

How many between 1- 100 have 2, 3, or 5 as a factor?

Let S(2)=Factors of 2
S(3)=Factors of 3
S(5)=Factors of 5

There are 50 elements in S(2). (100/2=50)
There are 33 elements in S(3). (100/3=33 R1)
There are 20 elements in S(5). (100/5=20)

The intersection of S(2) and S(3) has 16 elements. (LCM=6 and 100/6=16 R4).
The intersection of S(2) and S(5) has 10 elements. (LCM=10 and 100/10=10).
The intersection of S(3) and S(5) has 6 elements. (LCM=15 and 100/15=6)

The intersection of S(2), S(3), and S(5) has 3 elements. (LCM=30 and 100/30=3 R10)

The union of the three sets is S(2)+S(3)+S(5)-S(2,3)-S(2,5)-S(3,5)+S(2,3,5).
=50+33+20-16-10-6+3=74.

Figure 6. Higher level abstractions 1

How many between 1- 100 have 2, 3, or 5 as a factor?

100/2 = 50 numbers that have 2 as a factor

100/3 = 33 1/3; 33 numbers have 3 as a factor

100/5 = 20 numbers that have 5 as a factor

If the numbers for each factor were added together, this would double-count numbers that share two factors, and triple-count numbers that have all three for factors.

50 + 33 + 20 = 103

100/6 = 16 2/3; 16 numbers have 2 and 3 as a factor

100/10 = 10 numbers that have 2 and 5 as a factor

100/15 = 6 2/3; 6 numbers that have 3 and five as a factor

If the numbers for each pair of factors were subtracted from the first total, all the numbers that were double-counted would now be counted once. But numbers that have all three factors would have been triple-counted, and then triple-eliminated, as they would be included in each of the three pairs of factors.

103 – (16 + 10 + 6) = 103 – 32 = 71

100/30 = 3 1/3; 3 numbers have 2, 3 and 5 as a factor

Finally the numbers that have all three factors are added back in to the total, so that any number that has 2, 3, or 5 as a factor would only be counted once.

71 + 3 = 74.

Figure 7. Higher level abstractions 2

A student was able to solve the problem by using logic. His solution is presented in Figure 8.

There are 74 numbers between 1 and 100 that have 2, 3, or 5 as factors. Every even number from 1-100 is a multiple of 2. There are 50 even numbers between 1 and 100. There are 33 multiples of 3 between 1 and 100; however, all the even were already mentioned as multiples of 2. This leaves 17 odd multiples. There are 20 multiples of 5 between 1 and 100. All even are eliminated leaving 10. In addition, all odd multiples 1 are already covered as they are multiples of 3 as well. This leaves 7.

50 + 17 + 7 = 74

Figure 8. The use of logic

Some elevation is observed, which is creating another control point by using the color code. Students in Figure 9 used color code to avoid listing common elements more than once in the Final Set.

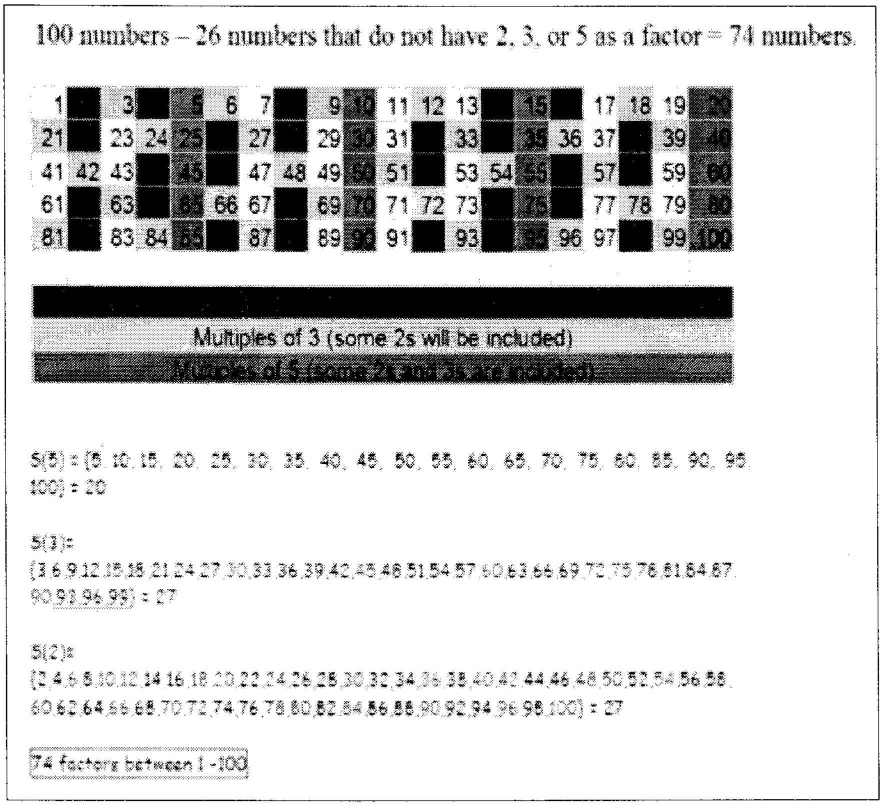

Figure 9. Color Coded Solution

Table 2. Summary of Problem 2

Sets	Table	Subsets	Cardinality	No Solution
4	4	8	2	12

Reducing Abstraction: Adaptation from complex to manageable

Problem 2: Choose three numbers a, b, and c which are not relatively prime. Determine how many numbers from 1 to *n*, where $n = abc$, have *a, b*, or *c* as factors.

For this problem participants were free to choose their numbers. Different from the first problem, these numbers are not relatively primes.

The type of reducing abstraction observed was mainly subset models of solutions. In this type of solutions, students come to realize that if they choose three numbers a, b, and c in a way that $a,\ b = na,\ c = ma \quad (m, n \in Z^{+})$, they do not need to do more complex thinking. One of the participants' solutions is presented in Figure 6.

$$N = 5 \bullet 10 \bullet 15 = 750$$

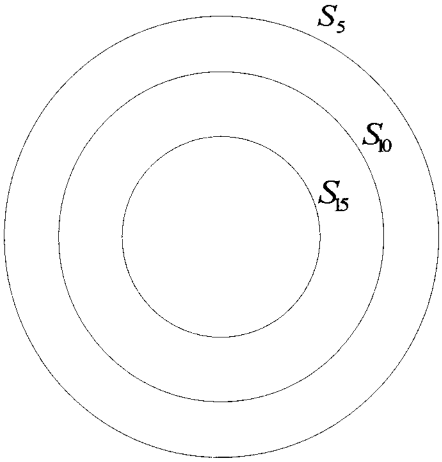

Figure 10. Reducing abstraction by subsets

$$S_{15} \subset S_{10} \subset S_5$$

$$S_5 = \{5,10,15,...,750\} \qquad S_{10} = \{10,20,30,...,750\} \qquad S_{15} = \{15,30,45,...,750\}$$

The complexity of mathematical concept reducing by choosing numbers creates subsets. In this type of solutions, fist, there is an elevation in terms of abstractions. Students can see relationships between the numbers and elements of each set created by these numbers. This elevation however, has been used to reduce abstractions.

The easiest way to do this is to choose numbers for a, b, & c such that b & c are multiples of a, for example:

a = 3 b = 6 c = 9 (3 is a common factor so they are not relatively prime.)
N = 3*6*9 = 162
A: How many numbers between 1 & 162 are divisible by 3?
A = 54 because every third number is a multiple of 3 giving 162÷3 = 54
B: How many numbers between 1 & 162 are divisible by 6?
B = 27 because every sixth number is a multiple of 6 giving 162÷6 = 27
C: How many numbers between 1 & 162 are divisible by 9?
C = 18 because every ninth number is a multiple of 9 giving 162÷9 = 18
How many numbers form 1 to 162 have 3, 6, or 9 as factors?
54 because the set of numbers that are divisible by 3 includes all numbers divisible by 6 and 9 also.

Now that seemed just too easy, so I worked it out with your numbers:

Figure 11. Reducing abstraction by subsets

In Figure 11, we can see also reducing abstraction by using subsets. However, we asked the participants to try out with different numbers. The given numbers were 6, 8, and 12 for *a*, *b*, and *c* respectively.

a = 6 b = 8 c = 12 (2 is a common factor/not relatively prime)
N = 6*8*12 = 576

A: 96 numbers between 1 & 576 are divisible by 6 because 576÷6 = 96.
B: 72 numbers between 1 & 576 are divisible by 8 because 576÷8 = 72.
C: 48 numbers between 1 & 576 are divisible by 12 because 576÷12 = 48.

We could list each set of multiples, or construct a "Sieve of Eratosthenes" type table to answer the fourth question.

#s divisible by 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144, 150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216, 222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354, 360, 366, 372, 378, 384, 390, 396, 402, 408, 414, 420, 426, 432, 438, 444, 450, 456, 462, 468, 474, 480, 486, 492, 498, 504, 510, 516, 522, 528, 534, 540, 546, 552, 558, 564, 570, 576

#s divisible by 8 (omitting those already listed that were also multiples of 6): 8, 16, 32, 40, 56, 64, 80, 88, 104, 112, 128, 136, 152, 160, 176, 184, 200, 208, 224, 232, 248, 256, 272, 280, 296, 304, 320, 328, 344, 352, 368, 376, 392, 400, 416, 424, 440, 448, 464, 472, 488, 496, 512, 520, 536, 544, 560, 568

#s divisible by 12 have all been listed (they are a subset of those divisible by 6)

Total number = 144 that are divisible by 6, 8, or 12.

Figure 12. First attempt of sSet solutions

I let "A∩B" and "A∩C" represent the overlap in numbers that are multiples of A & B or of A & C, respectively. I used the formula **A + B + C - [(A∩B) + (A∩C)]** to remove those duplicate multiples resulting from that overlapping.

After observing patterns in the lists of the multiples and the prime factorizations of "a" and "b", I found that **A∩B** is the product of B and the reciprocal of those prime factors of "a" that were not also factors of "b". If all factors of "a" are used as factors of "b", then use "1" as the reciprocal. Using the previous problem B = 72, a = 6 has prime factors 2*3, b = 8 has prime factors 2*2*2. "3" is the only prime factor of "a" that is not a prime factor of "b", so the reciprocal of 3 (which is 1/3) will be multiplied by "B". Therefore, A∩B = 72(1/3) = 24.

Doing likewise for **A∩C**, C = 48, a = 6 is factored to 2*3, and b = 12 is factored to 2*2*3. Since all factors of 6 are also factors of 12, use 48(1) for A∩C. You can also look at it as C is a subset of A and A∩C = C or 48.

Using the formula 96 + 72 + 48 - [(24) + (48)] = 216 - 72 = 144, which agrees with the answer I got listing out the members of the sets. With the formula, finding the answer is much quicker than making a table or listing out multiples, but middle school students would understand the longer method better.

Figure 13. Higher level abstraction 3

Student's first solution was set solution which was reduced abstraction level. Then, student now come to realize the relationship among the sets and try to elevate abstraction level, that is, he/she is trying to reason higher level of generalization or abstraction as given in Figure 11.

6, 8, 12 are NOT relative prime since 2 divide them all. They have a common factor, which is 2.

N= 6*8*12= 576

A: How many between 1-to-576 numbers are divisible by 6? 576/6 = 96

B: How many between 1-to-576 numbers are divisible by 8? 576/8 = 72

C: How many between 1-to-576 numbers are divisible by 12? 576/12 = 48

How many numbers from 1 to n, where n = abc, have a, b, or c as factors?

$$\frac{6}{12} = \frac{1}{2}(96) = 48$$
$$\frac{8}{12} = \frac{2}{2}(72) = 48$$
$$\frac{12}{12} = 1(48) = 48$$

48 + 48 + 48 = 144 numbers from 1 to 576, where n = 6 x 8 x 12 have 6, 8, or 12 as factors.

Each factor can have no more than the number of factors of the largest factors. All others will be repeats.

Figure 14. Unique Solution

Some students used Table and Set solutions at the expense of exhaustive amount of time and energy to get the correct answer. Sample of these solutions presented at Figure 15, 16, and 17.

61	62	63	64	65	66	67	68	69	70	71		73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107		109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140
141	142	143		145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200
201	202	203	204	205	206	207	208	209	210	211	212	213	214	215		217	218	219	220
221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250	251		253	254	255	256	257	258	259	260
261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287		289	290	291	292	293	294	295	296	297	298	299	300
301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320
321	322	323		325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340
341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	
361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380
381	382	383	384	385	386	387	388	389	390	391	392	393	394	395		397	398	399	400
401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420
421	422	423	424	425	426	427	428	429	430	431		433	434	435	436	437	438	439	440
441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460
461	462	463	464	465	466	467		469	470	471	472	473	474	475	476	477	478	479	480
481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500
501	502	503		505	506	507	508	509	510	511	512	513	514	515	516	517	518	519	520
521	522	523	524	525	526	527	528	529	530	531	532	533	534	535	536	537	538	539	
541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	556	557	558	559	560
561	562	563	564	565	566	567	568	569	570	571	572	573	574	575		577	578	579	580
581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599	600
601	602	603	604	605	606	607	608	609	610	611		613	614	615	616	617	618	619	620
621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639	640
641	642	643	644	645	646	647													

Divisible by 6 only	36
Divisible by 9 only	32
Divisible by 12 only	0
Divisible by 6 and 9 only	18
Divisible by 6 and 12 only	36
Divisible by 9 and 12 only	0
Divisible by 6, 9, and 12	18

Figure 15. Table Solutions

a, b, and c are the integers 6, 8, 12

1\. a) I listed the multiples of 6 less than or equal to 576.
6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72,
78, 84, 90, 96, 102, 108, 114, 120, 126, 132, 138, 144,
150, 156, 162, 168, 174, 180, 186, 192, 198, 204, 210, 216,
222, 228, 234, 240, 246, 252, 258, 264, 270, 276, 282, 288,
294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354, 360,
366, 372, 378, 384, 390, 396, 402, 408, 414, 420, 426, 432,
438, 444, 450, 456, 462, 468, 474, 480, 486, 492, 498, 504,
510, 516, 522, 528, 534, 540, 546, 552, 558, 564, 570, 576

There are 96 numbers between 1 and 576 that were divisible by 6.

b) I listed the multiples of 8 less than or equal to 576.
8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96,
104, 112, 120, 128, 136, 144, 152, 160, 168, 176, 184, 192,
200, 208, 216, 224, 232, 240, 248, 256, 264, 272, 280, 288,
296, 304, 312, 320, 328, 336, 344, 352, 360, 368, 376, 384,
392, 400, 408, 416, 424, 432, 440, 448, 456, 464, 472, 480,
488, 496, 504, 512, 520, 528, 536, 544, 552, 560, 568, 576

There are 72 numbers between 1 and 576 that were divisible by 8.

c) I listed the multiples of 12 less than or equal to 576.
12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144,
156, 168, 180, 192, 204, 216, 228, 240, 252, 264, 276, 288,
300, 312, 324, 336, 348, 360, 372, 384, 396, 408, 420, 432,
444, 456, 468, 480, 492, 504, 516, 528, 540, 552, 564, 576

There are 48 numbers between 1 and 576 that were divisible by 12.

Figure 16. Table Solutions

4,6,10

1) -	46)–	91)–	136) 4
2) -	47)–	92)4	137) –
3) -	48)4, 6	93)–	138) 6
4) 4	49)–	94)–	139) –
5) -	50)10	95)–	140) 4, 10
6) 6	51)–	96)4, 6	141) –
7) -	52)4	97)–	142) –
8) 4	53)–	98)–	143) –
9) -	54)6	99)–	144) 4, 6
10) 10	55)–	100) 4, 10	145) –
11)–	56)4	101) –	146) –
12)4, 6	57)–	102) 6	147) –
13)–	58)–	103) –	148) 4
14)–	59)–	104) 4	149) –
15)–	60)4, 6, 10	105) –	150) 6, 10
16)4	61)–	106) –	151) –
17)–	62)–	107) –	152) 4
18)6	63)–	108) 4, 6	153) –
19)–	64)4	109) –	154) –
20)4, 10	65)–	110) 10	155) –
21)–	66)6	111) –	156) 4, 6
22)–	67)–	112) 4	157) –

Figure 17. List Solutions

Summary of Problem 3:

Abstraction completed	Not able to generalize
5	25

The third problem was asking directly to generalize or reason at higher level abstractions. Among the 30 students, only 5 of them were able to generalize.

2	1	0.67	0.4	2*1
3	1.5	1.00	0.6	3*1
4	2	1.33	0.8	2*2
5	2.5	1.67	1	5*1
8	4	2.67	1.6	2*2*2
9	4.5	3.00	1.8	3*3
14	7	4.67	2.8	2*7
16	8	5.33	3.2	2*2*2*2
21	10.5	7.00	4.2	3*7
22	11	7.33	4.4	2*11
25	12.5	8.33	5	5*5
26	13	8.67	5.2	2*13
27	13.5	9.00	5.4	3*3*3
28	14	9.33	5.6	2*2*7
32	16	10.67	6.4	2*2*2*2*2
33	16.5	11.00	6.6	3*11
34	17	11.33	6.8	2*17
35	17.5	11.67	7	5*7
38	19	12.67	7.6	2*19
39	19.5	13.00	7.8	3*13
44	22	14.67	8.8	2*2*11
46	23	15.33	9.2	2*23

Numbers divisible by only 2 are powers of two times either one or a prime number.

Numbers divisible by only 3 are powers of three times either one or a prime number.

Numbers divisible by only 5 are powers of five times either one or a prime number.

Figure 18. Abstraction is a dynamic process

In Figure 17 students' thinking process revealed that they first list all the numbers and their prime factorizations. We have coded table solution as a reduced abstraction level. In this solution however, students used low level of abstraction to move on the next level of higher thinking.

Numbers that are only divisible by 2 do not end in zero and the sum of their digits is not equal to a number divisible by 3.

Numbers that are only divisible by 3 do not end with a five or an even number and have the sum of their digits equal to a number divisible by 3.

Numbers that are only divisible by 5 do not end with a zero and the sum of their digits is not equal to a number divisible by 3.

Figure 19. Reduced abstraction level

Figure 18 illustrates the example of reducing abstraction level. Student just wrote the divisibility rules for each number and tries to explain the uniqueness by excluding other two numbers' divisibility rules.

The numbers from 1 to 100 (including these numbers) that are divisible

- **Only by 2:** 2, 4, 8, 14, 16, 22, 26, 28, 32, 34, 38, 44, 46, 52, 56, 58, 62, 68, 74, 76, 82, 86, 88, 92, 94, and 98
- **Only by 3:** 3, 9, 21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, and 99
- **Only by 5:** 5, 25, 35, 55, 65, 85, and 95

What can I tell about this case? What makes these numbers unique that they are only divisible by 2 or 3 or 5?

1. The values 3 and 5 are each relatively prime to each of the values listed as numbers divisible by only 2. For instance, the g.c.f. between 3 and 14 are relatively prime just as the g.c.f. between 5 and 22 are relatively prime.
2. The values 2 and 5 are each relatively prime to each of the values listed as numbers divisible by only 3. For instance, the g.c.f. between 2 and 27 are relatively prime just as the g.c.f. between 5 and 39 are relatively prime.
3. The values 2 and 3 are each relatively prime to each of the values listed as numbers divisible by only 5. For instance, the g.c.f. between 2 and 25 are relatively prime just as the g.c.f. between 3 and 55 are relatively prime.

Figure 20. Abstraction is a dynamic process

Students first find the sets for each number and then try to make generalization or higher level of abstraction in Figure 20.

5. DISCUSSIONS AND RECOMMENDATIONS

Three problems and students' solutions to said problems are examined from the perspective of reducing abstraction model. The results of one descriptive study are insufficient to make generalization on students' abstraction process. But yet it might help us to understand the abstraction phenomena.

Researcher's intention with the third problem was to examine the students' theoretical abstraction levels. In theoretical abstraction, students should analyze the problem in order to identify the characteristics and structure of the problem. Only small parts (5 students) of participants were able to reason at higher level. Mitchelmore & White (2007) discuss the potential reasons by stating "students often learn abstract concepts in isolation, without engaging the abstraction process…" (p. 3).

The reducing abstraction model (Hazzan, 2001) suggests that students try to make mathematical concepts more accessible to them by changing from abstract to concrete. In this study, students' preference of set solution and table solution of the first problem was an example of reducing abstraction concepts.

According to Hazzan (2001) there is more than one way to reduce the level of abstraction. This was evident in this research; some students used the Table solution (see Figure 4, 9, and 15), some others used Set solution (see Figures 2 and 3) while others used just listing (Figure 16) the numbers. Their preferences were based on students' management of the complexity.

Furthermore, students used elevations in abstraction, higher level of abstract thinking, to reduce the level of abstractions seen in Subset solutions (Figure 10).

Speaking of it metaphorically, we want students to learn to swim, in the ocean, in the swimming pool, in the lake, and so forth. But they seem to prefer to practice swimming in shallow sides where they can feel safe yet.

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