THREE SOLUTIONS TO A CLASS OF NEUMANN DOUBLY EIGENVALUE ELLIPTIC SYSTEMS DRIVEN BY A \((p_1, \ldots, p_n)\)-LAPLACIAN

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Abstract. In this paper we establish the existence of at least three weak solutions for Neumann doubly eigenvalue elliptic systems driven by a \((p_1, \ldots, p_n)\)-Laplacian. Our main tool is a recent three critical points theorem of B. Ricceri.

1. Introduction

Here and in the sequel, \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a non-empty bounded open set with a boundary \(\partial \Omega\) of class \(C^1\), \(p_i \geq 2\) and \(a_i \in L^\infty(\Omega)\) with \(\text{ess inf}_{\Omega} a_i > 0\) for \(1 \leq i \leq n\).

In this paper we are interested in multiplicity results for the following Neumann elliptic system

\[
\begin{align*}
-\Delta_{p_1} u_1 + a_1(x)|u_1|^{p_1-2}u_1 &= \lambda F_{u_1}(x, u_1, \ldots, u_n) + \mu G_{u_1}(x, u_1, \ldots, u_n) \quad \text{in } \Omega, \\
-\Delta_{p_2} u_2 + a_2(x)|u_2|^{p_2-2}u_2 &= \lambda F_{u_2}(x, u_1, \ldots, u_n) + \mu G_{u_2}(x, u_1, \ldots, u_n) \quad \text{in } \Omega, \\
&\vdots \\
-\Delta_{p_n} u_n + a_n(x)|u_n|^{p_n-2}u_n &= \lambda F_{u_n}(x, u_1, \ldots, u_n) + \mu G_{u_n}(x, u_1, \ldots, u_n) \quad \text{in } \Omega, \\
\frac{\partial u_i}{\partial \nu} &= 0 \quad \text{for } 1 \leq i \leq n \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\Delta_{p_i} u_i = \text{div}(\nabla u_i |^{p_i-2} \nabla u_i)\) is the \(p_i\)-Laplacian operator, \(p_i > N\) for \(1 \leq i \leq n\), \(\lambda, \mu > 0\), \(F : \Omega \times \mathbb{R}^n \to \mathbb{R}\) is a function such that \(F(\cdot, t_1, \ldots, t_n)\) is continuous in \(\Omega\) for all \((t_1, \ldots, t_n) \in \mathbb{R}^n\) and \(F(x, \cdot, \ldots, \cdot)\) is \(C^1\) in \(\mathbb{R}^n\) for almost every \(x \in \Omega\), \(G : \Omega \times \mathbb{R}^n \to \mathbb{R}\) is a function such that \(G(\cdot, t_1, \ldots, t_n)\) is measurable in \(\Omega\) for all \((t_1, \ldots, t_n) \in \mathbb{R}^n\) and \(G(x, \cdot, \ldots, \cdot)\) is \(C^1\) in \(\mathbb{R}^n\) for almost every \(x \in \Omega\).

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almost every \( x \in \Omega \). \( F_{u_i} \) and \( G_{u_i} \) denote the partial derivative of \( F \) and \( G \) with respect to \( u_i \), respectively, and \( v \) is the outward unit normal to \( \partial \Omega \).

A weak solution of (1) is any \( u = (u_1, u_2, \ldots, u_n) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \times \cdots \times W^{1,p_n}(\Omega) \) such that

\[
\int_{\Omega} \sum_{i=1}^{n} |\nabla u_i(x)|^{p_i} - 2 \nabla u_i(x) \nabla v_i(x) \, dx + \int_{\Omega} \sum_{i=1}^{n} a_i(x)|u_i(x)|^{p_i} - 2 u_i(x)v_i(x) \, dx \\
- \left( \lambda \int_{\Omega} \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x) \, dx + \mu \int_{\Omega} \sum_{i=1}^{n} G_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x) \, dx \right) = 0
\]

for every \( (v_1, v_2, \ldots, v_n) \in W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \times \cdots \times W^{1,p_n}(\Omega) \).

In the literature many papers [5, 10, 12-14, 16, 17, 22, 23] discuss quasilinear elliptic systems. For example in [12] the authors studied a class of quasilinear elliptic systems involving a pair of \((p, q)\)-Laplacian operators. In [16], A. Kristály using an abstract critical point result of B. Ricci established the existence of an interval \( \Lambda \subseteq [0, +\infty) \) such that for each \( \lambda \in \Lambda \) the quasilinear elliptic system

\[
\begin{align*}
-\Delta_p u &= \lambda F_u(x, u, v) \quad \text{in } \Omega, \\
-\Delta_q v &= \lambda F_v(x, u, v) \quad \text{in } \Omega, \\
u = v = 0 \quad &\text{on } \partial \Omega,
\end{align*}
\]

(2)

where \( \Omega \) is a strip-like domain and \( \lambda > 0 \) is a parameter, has at least two distinct nontrivial solutions. In [23], the authors studied the Nehari manifold for a class of quasilinear elliptic systems involving a pair of \((p, q)\)-Laplacian operators and a parameter, and proved the existence of a nonnegative solution for the system by discussing properties of the Nehari manifold, and they obtained global bifurcation results. We also refer the reader to [1, 2, 4, 6-9, 11, 17-19] where the three critical points theorem of B. Ricceri [20] is used. Chun Li and Chun-Lei Tang in [17] established the existence of an interval \( \Lambda \subseteq [0, +\infty) \) and a positive real number \( \rho \) such that for each \( \lambda \in \Lambda \) problem (2) admits at least three weak solutions whose norms in \( W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \) are less than \( \rho \). In [1] a similar result was obtained for the quasilinear elliptic system

\[
\begin{align*}
\Delta_{p_1} u_1 + \lambda F_{u_1}(x, u_1, u_2, \ldots, u_n) &= 0 \quad \text{in } \Omega, \\
\Delta_{p_2} u_2 + \lambda F_{u_2}(x, u_1, u_2, \ldots, u_n) &= 0 \quad \text{in } \Omega, \\
\cdots \\
\Delta_{p_n} u_n + \lambda F_{u_n}(x, u_1, u_2, \ldots, u_n) &= 0 \quad \text{in } \Omega, \\
u_i &= 0 \quad \text{for } 1 \leq i \leq n \quad \text{on } \partial \Omega.
\end{align*}
\]

(3)
In [8], G. Bonanno and P. Candito using Ricceri’s three critical points theorem,
proved the existence of an interval \( \Lambda \subseteq [0, +\infty] \) and a positive real number \( q \)
such that for each \( \lambda \in \Lambda \) the problem

\[
\begin{cases}
-\Delta_p u + a_1(x) |u|^{p-2}u = \lambda f(x, u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subseteq \mathbb{R}^N (N \geq 1) \) is a nonempty bounded open set with a boundary \( \partial \Omega \)
of class \( C^1 \), \( a \in L^\infty(\Omega) \) with \( \inf_{\Omega} a > 0 \), \( p > N \), \( \lambda > 0 \) and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a continuous function, admits at least three weak solutions whose norms in \( W^{1,p}(\Omega) \) are less than \( q \). Finally, B. Ricceri in [21] extended and revisited the
three critical point theorem he obtained in [20], and in [15] the results were
used to study the problem

\[
\begin{cases}
u'' + (\lambda f(t, u) + g(u))h(t, u') = \mu p(t, u)h(t, u') & \text{in } (a, b), \\
n(a) = n(b) = 0,
\end{cases}
\]

where \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a continuous function, \( g : \mathbb{R} \to \mathbb{R} \) is a Lipschitz
continuous function, \( \lambda > 0 \), \( h : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a bounded function, \( \mu > 0 \) and \( p : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a \( L^1 \)-Carathéodory function for every \( (t, u) \in [a, b] \times \mathbb{R} \).

2. Preliminaries

First we recall the three critical points theorem [21].

**Theorem 1.** Let \( X \) be a reflexive real Banach space, \( I \subseteq \mathbb{R} \) an interval, \( \Phi : X \to \mathbb{R} \) a sequentially weakly lower semicontinuous \( C^1 \) functional bounded on each bounded subset of \( X \) whose derivative admits a continuous inverse on \( X^* \) and \( J : X \to \mathbb{R} \) a \( C^1 \) functional with compact derivative. Assume that

\[
\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda J(x)) = +\infty
\]

for all \( \lambda \in I \), and that there exists \( \rho \in \mathbb{R} \) such that

\[
\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda (J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda (J(x) + \rho)).
\]

Then, there exist a non-empty open set interval \( \Lambda \subseteq I \) and a positive real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and every \( C^1 \) functional \( \Psi : X \to \mathbb{R} \) with compact derivative, there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the equation

\[
\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0
\]

has at least three solutions in \( X \) whose norms are less than \( q \).

In the proof of our main result we use the next result to verify the minimax
inequality in Theorem 1.
Proposition 1 ([Bonanno, 7]). Let $X$ be a non-empty set and $\Phi, J$ two real functions on $X$. Assume that $\Phi(x) \geq 0$ for every $x \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = J(u_0) = 0$. Further, assume that exist $u_1 \in X, r > 0$ such that

\begin{itemize}
\item[(n1)] $\Phi(u_1) > r$,
\item[(n2)] $\sup_{\Phi(x) < r} (-J(x)) < r \frac{\Phi(u_1)}{\Phi(u_1) - J(u_1)}$.
\end{itemize}

Then, for every $\nu > 1$ and for every $\rho \in R$ satisfying

$$\sup_{\Phi(x) < r} \left( -J(x) + \frac{\rho - \frac{\Phi(u_1)}{\Phi(u_1) - J(u_1)}}{\nu} \sup_{\Phi(x) < r} (-J(x)) \right) < \rho < r \frac{\Phi(u_1)}{\Phi(u_1) - J(u_1)},$$

one has

$$\sup_{\lambda \in R} \inf_{x \in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in [0, \sigma]} (\Phi(x) + \lambda(J(x) + \rho)),$$

where

$$\sigma = \frac{\rho r}{\frac{\Phi(u_1)}{\Phi(u_1) - J(u_1)} - \sup_{\Phi(x) < r} (-J(x))}.$$

In order to apply Theorem 1 to our problem, let $X$ be the Cartesian product of the $n$ Sobolev spaces $W^{1,p_1}(\Omega), W^{1,p_2}(\Omega), \ldots, W^{1,p_n}(\Omega)$, i.e., $X = W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \times \cdots \times W^{1,p_n}(\Omega)$ with the norm

$$\| (u_1, u_2, \ldots, u_n) \| = \| u_1 \| + \| u_2 \| + \cdots + \| u_n \|,$$

where $\| u_i \| = \left( \int_{\Omega} (|\nabla u_i(x)|^{p_i} + a_i(x)|u_i(x)|^{p_i}) dx \right)^{1/p_i}$ for $1 \leq i \leq n$.

Put

$$c = \max \left\{ \sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \Omega} |u_i(x)|^{p_i}}{\| u_i \|^{p_i}} ; \ 1 \leq i \leq n \right\}.$$

Since $p_i > N$ for $1 \leq i \leq n$, one has $c < +\infty$. It follows from Proposition 4.1 of [3] that

$$\sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \Omega} |u_i(x)|^{p_i}}{\| u_i \|^{p_i}} > \frac{1}{\| u_i \|_1^{p_i}} ; \ 1 \leq i \leq n,$$

where $\| u_i \|_1 = \int_{\Omega} |a_i(x)| dx$ for $1 \leq i \leq n$, and so $\frac{1}{\| a_i \|_1} \leq c$ for $1 \leq i \leq n$. In addition, if $\Omega$ is convex, it is known [3] that

$$\sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \Omega} |u_i(x)|}{\| u_i \|} \leq 2 \frac{p_i - 1}{p_i} \max \left\{ \left( \frac{1}{\| a_i \|_1} \right)^{\frac{p_i}{p_i - 1}} ; \ \frac{\text{diam}(\Omega)}{N} \left( \frac{p_i - 1}{p_i - N} \right)^{\frac{p_i - 1}{p_i}} \| a_i \|_\infty \right\}$$

for $1 \leq i \leq n$, where $m(\Omega)$ is the Lebesgue measure of the set $\Omega$, and equality occurs when $\Omega$ is a ball.
3. Main results

We state our main result:

**Theorem 2.** Let \( F : \Omega \times \mathbb{R}^n \to \mathbb{R} \) be a function such that \( F(\cdot, t_1, \ldots, t_n) \) is continuous in \( \Omega \) for all \( (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( F(x, \cdot, \ldots, \cdot) \) is \( C^1 \) in \( \mathbb{R}^n \) for almost every \( x \in \Omega \). Assume that there exist a positive constant \( r \) and a function \( u^* = (u^*_1, \ldots, u^*_n) \in X \) such that

\[
(j_1) \sum_{i=1}^n \frac{|u^*_i|_{p_i}}{r_i} > r,
\]

\( (j_2) \) there is a positive constant \( \eta \) with

\[
\frac{cm(\Omega)}{\limsup_{|t_1|,\ldots,|t_n| \to (+\infty,\ldots,\infty)} \sum_{i=1}^n \frac{|u^*_i|_{p_i}}{r_i} \frac{F(x, t_1, \ldots, t_n)}{t_n^{p_n}} = \frac{1}{r\eta}} \]

uniformly with respect to \( x \in \Omega \) and satisfying

\[
\frac{\int_{\Omega} F(x, u^*_1(x), \ldots, u^*_n(x))dx}{\sum_{i=1}^n c_i|u^*_i|_{p_i}} - \frac{\int_{\Omega} \max_{(t_1, \ldots, t_n) \in K_1} F(x, t_1, \ldots, t_n)dx}{\eta} > \frac{1}{\eta},
\]

where \( K_1 = \{(t_1, \ldots, t_n) \sum_{i=1}^n \frac{|t_i|_{p_i}}{r_i} \leq cr\} \),

\( (j_3) \) \( F(x, 0, \ldots, 0) = 0 \) for almost every \( x \in \Omega \).

Then, there exist a non-empty open interval \( A \subseteq [0, r\eta] \) and a positive real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and an arbitrary function \( G : \Omega \times \mathbb{R}^n \to \mathbb{R} \) measurable in \( \Omega \) for all \( (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( C^1 \) in \( \mathbb{R}^n \) for almost every \( x \in \Omega \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \) problem (1) admits at least three weak solutions in \( X \) whose norms are less than \( q \).

Let us first give a consequence of Theorem 2 for a fixed test function \( u^* \).

**Corollary 1.** Let \( F : \Omega \times \mathbb{R}^n \to \mathbb{R} \) be a function such that \( F(\cdot, t_1, \ldots, t_n) \) is continuous in \( \Omega \) for all \( (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( F(x, \cdot, \ldots, \cdot) \) is \( C^1 \) in \( \mathbb{R}^n \) for almost every \( x \in \Omega \) such that Assumption \( (j_3) \) in Theorem 2 holds. Assume that there exist \( n+2 \) positive constants \( \theta_i, \tau \) and \( \eta \) with \( \theta_i < \tau \) for \( 1 \leq i \leq n \) such that

\[
(j_4) \quad cm(\Omega) \sum_{i=1}^n \frac{|u^*_i|_{p_i}}{r_i} \frac{F(x, t_1, \ldots, t_n)}{t_n^{p_n}} < \frac{c}{\eta} \sum_{i=1}^n \frac{\theta_i^{p_i}}{r_i},
\]

uniformly with respect to \( x \in \Omega \) and

\[
\frac{1}{c} \sum_{i=1}^n \prod_{j=1, j \neq i}^n p_j^{p_j} \prod_{j=1}^n p_j \tau^{p_j} \int_{\Omega} \frac{F(x, \tau, \ldots, \tau)}{t_1^{p_1} \cdots t_{n}^{p_n}} dx = -\frac{\int_{\Omega} \max_{(t_1, \ldots, t_n) \in K_2} F(x, t_1, \ldots, t_n)dx}{\eta} > \frac{1}{\eta},
\]

where \( K_2 = \{(t_1, \ldots, t_n) \sum_{i=1}^n \frac{|t_i|_{p_i}}{r_i} \leq \sum_{i=1}^n \frac{\theta_i^{p_i}}{r_i}\} \).

Then, there exist a non-empty open interval \( A \subseteq [0, \frac{\eta}{c} \sum_{i=1}^n \frac{\theta_i^{p_i}}{r_i}] \) and a positive
real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and an arbitrary function \( G : \Omega \times \mathbb{R}^n \to \mathbb{R} \) measurable in \( \Omega \) for all \( (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( C^1 \) in \( \mathbb{R}^n \) for almost every \( x \in \Omega \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta) \) problem (1) admits at least three weak solutions in \( X \) whose norms are less than \( q \).

Proof. Choose \( u^*(x) = (u_1^*(x), u_2^*(x), \ldots, u_n^*(x)) = (\tau, \tau, \ldots, \tau) \) for every \( x \in \Omega \). Then we have
\[
\sum_{i=1}^n \frac{|u_i^*(x)|^{p_i}}{p_i} = \sum_{i=1}^n \frac{r_i}{p_i} |a_i|_1 |a_i|_1.
\]
Now since \( \theta_i < r \), bearing in mind that \( \frac{1}{|a_i|_1} \leq c \) for \( 1 \leq i \leq n \), one has
\[
\sum_{i=1}^n \frac{|u_i^*(x)|^{p_i}}{p_i} > r \quad \text{which is (j1).}
\]
Also, since
\[
r \int_{\Omega} F(x, u_1^*(x), \ldots, u_n^*(x)) \, dx
\]
\[
= \frac{1}{c} \sum_{i=1}^n \frac{\theta_i r_i}{p_i} \int_{\Omega} F(x, u_1^*(x), \ldots, u_n^*(x)) \, dx
\]
\[
= \frac{1}{c} \prod_{j=1}^n p_j \sum_{i=1}^n \frac{\theta_i r_i}{p_i} \int_{\Omega} F(x, u_1^*(x), \ldots, u_n^*(x)) \, dx
\]
\[
= \frac{1}{c} \prod_{j=1}^n p_j \sum_{i=1}^n \frac{\theta_i r_i}{p_i} \int_{\Omega} F(x, \tau, \ldots, \tau) \, dx,
\]
(j4) guarantees (j2).

Example 1. Consider the problem
\[
\begin{cases}
- \Delta_3 u_1 + \frac{2(x^2 + y^2)}{x} |u_1| u_1 = \lambda (x^2 + y^2) e^{-u_1} u_1^{14} (15 - u_1) & \text{in } \Omega, \\
- \Delta_3 u_2 + \frac{2(x^2 + y^2)}{x} |u_2| u_2 = \lambda (x^2 + y^2) e^{-u_2} u_2^{14} (15 - u_2) & \text{in } \Omega, \\
- \Delta_3 u_3 + \frac{2(x^2 + y^2)}{x} |u_3| u_3 = \lambda (x^2 + y^2) e^{-u_3} u_3^{14} (15 - u_3) & \text{in } \Omega,
\end{cases}
\]
(6)

where \( \Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 9\} \) and \( G : \Omega \times \mathbb{R}^3 \to \mathbb{R} \) is measurable in \( \Omega \) for all \( (t_1, t_2, t_3) \in \mathbb{R}^3 \) and is \( C^1 \) in \( \mathbb{R}^3 \) for almost every \( x \in \Omega \). Note that
\[
c = \frac{15 \pi}{\pi}
\]
and we choose \( \theta_i = 1, \tau_i = 10, a_i(x, y) = \frac{2(x^2 + y^2)}{x} \) for \( 1 \leq i \leq 3 \) and
\[
F(x, u_1, u_2, u_3) = (x^2 + y^2) \sum_{i=1}^3 e^{-u_i} u_i^{15}
\]
for each \((x, y) \in \Omega\) and \((u_1, u_2, u_3) \in \mathbb{R}^3\). We see that

\[
\max_{u_1^2 + u_2^2 + u_3^2 \leq 3} \sum_{i=1}^{3} e^{-u_i} u_i^{15} \leq 3 \max_{u_i^2 \leq 3} (e^{-u_i} u_i^{15}) \quad \text{for } 1 \leq i \leq 3
\]

and so

\[
\frac{1}{c} \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} p_j \theta_{c_i} \int_{\Omega} F(x, \tau, \tau) dx
\]

\[
= \int_{\Omega} \max_{(t_1, t_2, t_3) \in K_2} F(x, y, t_1, t_2, t_3) dx
\]

\[
= \frac{\pi^2}{1024} 10^{12} e^{-10} - \frac{81 \pi}{2} \max_{u_1^2 + u_2^2 + u_3^2 \leq 3} \sum_{i=1}^{3} e^{-u_i} u_i^{15}
\]

\[
\geq \frac{\pi^2}{1024} 10^{12} e^{-10} - \frac{243 \pi}{2} \max_{u_1^2 \leq 3} (e^{-u_i} u_i^{15}) \quad \text{for } 1 \leq i \leq 3
\]

\[
= \frac{\pi^2}{1024} 10^{12} e^{-10} - \frac{243 \pi}{2} e^{-\frac{3}{2}\Theta^3} > 0,
\]

and

\[
\limsup_{(|t_1|, |t_2|, |t_3|) \to (+\infty, +\infty, +\infty)} \frac{1}{p_1 |t_1|^{p_1} + \frac{1}{p_2} |t_2|^{p_2} + \frac{1}{p_3} |t_3|^{p_3}} = 0
\]

since \(p_i = 3\) for \(1 \leq i \leq 3\). We can now apply Corollary 1 to problem (6) for every

\[
\eta > \frac{\pi^2}{1024} 10^{12} e^{-10} - \frac{81 \pi}{2} \max_{u_1^2 + u_2^2 + u_3^2 \leq 3} \sum_{i=1}^{3} e^{-u_i} u_i^{15}.
\]

We now point out a special situation of Corollary 1, in which the function \(F\) has separated variables.

**Corollary 2.** Let \(f\) be a continuous function in \(\Omega\) and \(\tilde{f}_i\) for \(1 \leq i \leq n\) be a function in \(C^1\) in \(\mathbb{R}^n\) for almost every \(x \in \Omega\). Assume that there exist \(n + 2\) positive constants \(\theta_i, \tau\) and \(\eta\) for \(1 \leq i \leq n\) with \(\theta_i < \tau\) for \(1 \leq i \leq n\) such that

\[
\left< \min_{(t_1, \ldots, t_n) \to (\infty, \ldots, \infty)} \prod_{i=1}^{n} \tilde{f}_i(t_i) \right>_{\mathbb{R}^n}
\]

\[
\leq \frac{c}{\eta \sum_{i=1}^{n} \frac{\theta_i}{p_i}}
\]

uniformly with respect to \(x \in \Omega\) and

\[
\int_{\Omega} f(x) dx \left[ \frac{1}{c} \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} p_j \theta_{c_i} \prod_{j=1}^{n} \tilde{f}_i(t_j) - \max_{(t_1, \ldots, t_n) \in K_2} \prod_{i=1}^{n} \tilde{f}_i(t_j) \right] > \frac{1}{\eta},
\]

where \(K_2 = \{(t_1, \ldots, t_n) | \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \leq \sum_{i=1}^{n} \frac{\theta_i}{p_i}\}\),

\((j_0) \tilde{f}_i(0) = 0 \quad \text{for } 1 \leq i \leq n\).

Then, there exist a non-empty open interval \(\Lambda \subseteq [0, \frac{2}{c} \sum_{i=1}^{n} \frac{\theta_i}{p_i}]\) and a positive
real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and an arbitrary function \( G : \Omega \times \mathbb{R}^n \to \mathbb{R} \) measurable in \( \Omega \) for all \( (t_1, \ldots, t_n) \in \mathbb{R}^n \) and \( C^1 \) in \( \mathbb{R}^n \) for almost every \( x \in \Omega \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \) problem

\[
\begin{align*}
-\Delta_p u_1 + a_1(x)|u_1|^{p_1-2}u_1 &= \lambda f(x)\bar{f}_1(u_1) + \mu G_{u_1}(x, u_1, \ldots, u_n) \quad \text{in } \Omega, \\
-\Delta_p u_2 + a_2(x)|u_2|^{p_2-2}u_2 &= \lambda f(x)\bar{f}_2(u_2) + \mu G_{u_2}(x, u_1, \ldots, u_n) \quad \text{in } \Omega, \\
& \vdots \\
-\Delta_p u_n + a_n(x)|u_n|^{p_n-2}u_n &= \lambda f(x)\bar{f}_n(u_n) + \mu G_{u_n}(x, u_1, \ldots, u_n) \quad \text{in } \Omega, \\
\frac{\partial u_i}{\partial \nu} &= 0 \quad \text{for } 1 \leq i \leq n \quad \text{on } \partial \Omega.
\end{align*}
\]

admits at least three weak solutions in \( X \) whose norms are less than \( q \).

**Proof.**

Set

\[ F(x, u_1, \ldots, u_n) = f(x)\left( \prod_{i=1}^{n} \bar{f}_i(u_i) \right) \]

for each \( (x, u_1, \ldots, u_n) \in \Omega \times \mathbb{R}^n \), and note that

\[
\int_{\Omega} \max_{(t_1, \ldots, t_n) \in \mathcal{K}_2} F(x, t_1, \ldots, t_n)dx = \max_{(t_1, \ldots, t_n) \in \mathcal{K}_2} \prod_{i=1}^{n} \bar{f}_i(t_i) \int_{\Omega} f(x)dx
\]

and

\[
\int_{\Omega} F(x, \tau, \ldots, \tau)dx = \prod_{i=1}^{n} \bar{f}_i(\tau) \int_{\Omega} f(x)dx,
\]

and from (j5) and (j6), it is easy to verify that all the assumptions of Corollary 1 are satisfied. Hence, the proof is complete. \( \square \)

**Corollary 3.** Let \( F : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function and assume that there exist \( n+2 \) positive constants \( \theta_i, \tau \) and \( \eta \) for \( 1 \leq i \leq n \) with \( \theta_i < \tau \) for \( 1 \leq i \leq n \) such that

\[ (j7) \]

\[
\operatorname{cmn}(\Omega) \limsup_{(t_1, \ldots, t_n) \to (\pm \infty, \ldots, \pm \infty)} \frac{F(t_1, \ldots, t_n)}{\frac{1}{p_1}|t_1|^{p_1} + \cdots + \frac{1}{p_n}|t_n|^{p_n}} \leq \frac{c}{\eta \sum_{i=1}^{n} \frac{\theta_i^p}{p_i}}.
\]
and

\[ m(\Omega) \left( \frac{1}{c} \frac{\sum_{i=1}^{n} \prod_{j=1,j \neq i}^{n} p_j^{\theta_i}}{\sum_{i=1}^{n} \prod_{j=1,j \neq i}^{n} p_j^{\theta_i}} ||u||_1 F(\tau, \ldots, \tau) - \max_{(t_1, \ldots, t_n) \in K_2} F(t_1, \ldots, t_n) \right) > \frac{1}{\eta}, \]

where \( K_2 = \{(t_1, \ldots, t_n)| \sum_{i=1}^{n} \frac{|t_i|}{p_i} \leq \sum_{i=1}^{n} \frac{\theta_i}{p_i} \} \),

\( (j_8) \) \( F(0, \ldots, 0) = 0 \).

Then, there exist a non-empty open interval \( \Lambda \subseteq [0, \frac{2}{\eta} \sum_{i=1}^{n} \theta_i] \) and a positive real number \( G : \mathbb{R}^n \to \mathbb{R} \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \) problem

\[ \begin{cases} -\Delta_p u_1 + a_1(x) |u_1|^{p_1-2} u_1 = \lambda F_{u_1}(u_1, \ldots, u_n) + \mu G_{u_1}(u_1, \ldots, u_n) \quad \text{in} \ \Omega, \\ -\Delta_p u_2 + a_2(x) |u_2|^{p_2-2} u_2 = \lambda F_{u_2}(u_1, \ldots, u_n) + \mu G_{u_2}(u_1, \ldots, u_n) \quad \text{in} \ \Omega, \\ \vdots \\ -\Delta_p u_n + a_n(x) |u_n|^{p_n-2} u_n = \lambda F_{u_n}(u_1, \ldots, u_n) + \mu G_{u_n}(u_1, \ldots, u_n) \quad \text{in} \ \Omega, \end{cases} \]

admits at least three solutions in \( X \) whose norms are less than \( q \).

We now look at a consequence of Corollary 3 in the ordinary case \( p_i = 2 \) for \( 1 \leq i \leq n \). For simplicity, we fix \( \Omega = (0, 1) \). Note in this situation we have \( c = 2 \max\{||a||_1^{-1}, ||a||_\infty^2||a||_1^{-2}; \quad 1 \leq i \leq n\} \).

**Corollary 4.** Let \( F : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function and assume that there exist \( n + 2 \) positive constants \( \theta_i \), \( \tau \) and \( \eta \) for \( 1 \leq i \leq n \) with \( \theta_i < \tau \) for \( 1 \leq i \leq n \) such that

\( (j_9) \)

\[ \begin{align*} 
\text{cm}(\Omega) \limsup_{(|t_1|, \ldots, |t_n|) \to (+\infty, \ldots, +\infty)} & \quad \frac{F(t_1, \ldots, t_n)}{\sum_{i=1}^{n} \theta_i t_i^2} < \frac{4\tilde{c}}{\eta \sum_{i=1}^{n} \theta_i^2}, \\
\text{where} \quad \tilde{c} = \max\{||a||_1^{-1}, ||a||_\infty^2||a||_1^{-2}; \quad 1 \leq i \leq n\} \quad \text{and} \\
\frac{\sum_{i=1}^{n} \theta_i^2}{2\tilde{c}(\tau^2 \sum_{i=1}^{n} ||a||_1)} F(\tau, \ldots, \tau) - \max_{(t_1, \ldots, t_n) \in K_3} F(t_1, \ldots, t_n) > \frac{1}{\eta}, \\
\text{where} \quad K_3 = \{(t_1, \ldots, t_n)| \sum_{i=1}^{n} t_i^2 \leq \sum_{i=1}^{n} \theta_i^2 \}, \\
(\text{jr}) \quad F(0, \ldots, 0) = 0. 
\end{align*} \]

Then, there exist a non-empty open interval \( \Lambda \subseteq [0, \frac{2}{\eta} \sum_{i=1}^{n} \theta_i] \) and a positive real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and an arbitrary \( C^1 \) function \( G : \mathbb{R}^n \to \mathbb{R} \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \) such that,
\begin{equation}
\begin{aligned}
\begin{cases}
-u''_1 + a_1(x)u_1 &= \lambda F_{u_1}(u_1, \ldots, u_n) + \mu G_{u_1}(u_1, \ldots, u_n) \quad \text{in (0, 1)}, \\
-u''_2 + a_2(x)u_2 &= \lambda F_{u_2}(u_1, \ldots, u_n) + \mu G_{u_2}(u_1, \ldots, u_n) \quad \text{in (0, 1)}, \\
& \quad \vdots \\
-u''_n + a_n(x)u_n &= \lambda F_{u_n}(u_1, \ldots, u_n) + \mu G_{u_n}(u_1, \ldots, u_n) \quad \text{in (0, 1)}, \\
\frac{\partial u_i}{\partial n}(0) &= \frac{\partial u_i}{\partial n}(1) = 0 \quad \text{for } 1 \leq i \leq n
\end{cases}
\end{aligned}
\end{equation}

admits at least three weak solutions in $X$ whose norms are less than $q$.

4. Existence results in the case $n = 1$

Consider the following Neumann elliptic problem
\begin{equation}
\begin{aligned}
\begin{cases}
-\Delta_p u + a(x)|u|^{p-2}u &= \lambda f(x, u) + \mu g(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\end{equation}

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a nonempty bounded open set with a boundary $\partial \Omega$ of class $C^1$, $a \in L^\infty(\Omega)$ with $\text{ess inf}_\Omega a > 0$, $p > N$, $\lambda$, $\mu > 0$ and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two $L^1$-Carathéodory functions. Let $X$ be the Sobolev space $W^{1,p}(\Omega)$ equipped with the norm
\[
||u|| = \left( \int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p)dx \right)^{1/p}.
\]

Corresponding to $f$ and $g$ we introduce the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively as follows
\[
F(x, t) = \int_0^t f(x, \xi)d\xi
\]
and
\[
G(x, t) = \int_0^t g(x, \xi)d\xi
\]
for each $(x, t) \in \Omega \times \mathbb{R}$. Put
\[
k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\max_{x \in \Omega} |u(x)|}{||u||}.
\]

Since $p > N$, one has $k < +\infty$. It follows from Proposition 4.1 of [3] that $k \geq \frac{1}{||a||_1^p}$, where $||a||_1 = \int_\Omega |a(x)|dx$.

Now, we present the following result as an application of Corollary 1. Note that $F(x, 0) = 0$ for all $x \in \Omega$.

**Corollary 5.** Assume that there exist three positive constants $\theta$, $\tau$ and $\eta$ with $\theta < \tau$ such that
Then, there exist a non-empty open interval \( A \subseteq [0, \frac{\alpha(\frac{\theta}{\xi})^p}{p}] \) and a positive real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and an arbitrary \( L^1 \)-Carathéodory function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), problem (10) admits at least three weak solutions in \( X \) whose norms are less than \( q \).

We conclude this section by giving a simple consequence of Corollary 5 when \( N = 1 \) and \( p = 2 \). For simplicity, we fix \( \Omega = (a, b) \). In this case, we have

\[
k = \sqrt{2} \max\{||a||_{1}^{-\frac{1}{2}}, (a + b)(b - a)^{-\frac{1}{2}}||a||_{1}||a||_{1}^{-1}\}.
\]

**Corollary 6.** Assume that there exist three positive constants \( \theta, \tau \) and \( \eta \) with \( \theta < \tau \) such that

\[
\limsup_{|t| \to +\infty} \frac{F(x, t)}{|t|^p} < \frac{1}{\eta^p} \quad \text{uniformly with respect to } x \in \Omega \text{ and } t \in [a, b]
\]

Then, there exist a non-empty open interval \( A \subseteq [0, \frac{\alpha(\frac{\theta}{\xi})^p}{p}] \) and a positive real number \( q \) with the following property: for every \( \lambda \in \Lambda \) and an arbitrary \( L^1 \)-Carathéodory function \( g : [a, b] \times \mathbb{R} \to \mathbb{R} \), there is \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \) problem

\[
\begin{align*}
-u'' + a(x)u &= \lambda f(x, u) + \mu g(x, u), \\
u'(a) &= u'(b) = 0
\end{align*}
\]

admits at least three weak solutions in \( X \) whose norms are less than \( q \).

### 5. Proof of Theorem 1

We introduce the functionals \( \Phi, J : X \to \mathbb{R} \) for each \( u \in X \), as follows

\[
\Phi(u) = \sum_{i=1}^{n} \frac{|u_i|^p_i}{p_i},
\]

and

\[
J(u) = -\int_{\Omega} F(x, u_1(x), \ldots, u_n(x))dx.
\]

Note \( \Phi \) is bounded on each bounded subset of \( X \), it is continuously differentiable and sequentially weakly lower semicontinuous functional, its differential admits a continuous inverse on \( X^* \) and since \( p_i > N \) for \( 1 \leq i \leq n \), \( J \) is a continuously
differentiable functional with compact derivative. In particular, for each \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in X \) one has

\[
\Phi'(u)(v) = \int_{\Omega} \sum_{i=1}^{n} \left( |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x) + a_i(x)|u_i(x)|^{p_i-2}u_i(x)v_i(x) \right) dx,
\]

and

\[
J'(u)(v) = -\int_{\Omega} \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx.
\]

Furthermore from \((j_2)\) there exist two constants \( \alpha, \beta \in R \) with \( 0 < \alpha < \frac{1}{r_\eta} \) such that

\[
\text{cm}(\Omega) F(x, t_1, \ldots, t_n) \leq \alpha \left( \frac{1}{p_1} |t_1|^{p_1} + \cdots + \frac{1}{p_n} |t_n|^{p_n} \right) + \beta
\]

for a.e. \( x \in \Omega \) and all \( t \in \mathbb{R} \). Fix \( u = (u_1, \ldots, u_n) \in X \). Then

\[
F(x, u_1(x), \ldots, u_n(x)) \leq \frac{\alpha}{\text{cm}(\Omega)} \left( \frac{1}{p_1} |u_1(x)|^{p_1} + \cdots + \frac{1}{p_n} |u_n(x)|^{p_n} \right) \left( 1 - \rho \eta \right) \text{cm}(\Omega)
\]

for a.e. \( x \in \Omega \).

Then, for any fixed \( \lambda \in [0, r_\eta] \), since

\[
\text{sup}_{x \in \Omega} |u_i(x)|^{p_i} \leq c ||u_i||^{p_i}
\]

for \( 1 \leq i \leq n \), from \((11), (12)\) and \((13)\), we have

\[
\Phi(u) + \lambda J(u) = \sum_{i=1}^{n} \frac{||u_i||^{p_i}}{p_i} - \lambda \int_{\Omega} F(x, u_1(x), \ldots, u_n(x)) dx \geq \sum_{i=1}^{n} \frac{||u_i||^{p_i}}{p_i}
\]

\[
- \alpha \frac{\lambda}{\text{cm}(\Omega)} \left( \frac{1}{p_1} \int_{\Omega} |u_1(x)|^{p_1} dx + \cdots + \frac{1}{p_n} \int_{\Omega} |u_n(x)|^{p_n} dx \right) - \frac{\lambda \beta}{c}
\]

\[
\geq (1 - \alpha r \eta) \sum_{i=1}^{n} \frac{||u_i||^{p_i}}{p_i} - \frac{r \eta \beta}{c},
\]

and so

\[
\lim_{||u|| \to +\infty} (\Phi(u) + \lambda J(u)) = +\infty.
\]

To check the other assumption in Theorem 1 we use Proposition 1. From \((j_1)\) and \((11)\) we obtain \((\kappa_1)\). Moreover, from \((14)\) we have

\[
\text{sup}_{x \in \Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq c \sum_{i=1}^{n} \frac{||u_i||^{p_i}}{p_i}
\]
for each \( u = (u_1, \ldots, u_n) \in X \), and so for each \( r > 0 \)

\[
\Phi^{-1}([-\infty, r]) = \{ u = (u_1, u_2, \ldots, u_n) \in X \mid \sum_{i=1}^{n} \frac{|u_i|^{p_i}}{p_i} < r \}
\]

\( \subseteq \{ u = (u_1, u_2, \ldots, u_n) \in X \mid \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq cr \text{ for each } x \in \Omega \} \).

Now since

\[
r \int_{\Omega} F(x, u_1^{*}(x), \ldots, u_n^{*}(x))dx
\]

\[
\frac{\sum_{i=1}^{n} |u_i|^{p_i}}{p_i} \geq \int_{\Omega} \max_{(t_1, \ldots, t_n) \in K_1} F(x, t_1, \ldots, t_n)dx > \frac{1}{\eta} > 0
\]

we have

\[
\int_{\Omega} \max_{(t_1, \ldots, t_n) \in K_1} F(x, t_1, \ldots, t_n)dx < r \int_{\Omega} F(x, u_1^{*}(x), \ldots, u_n^{*}(x))dx
\]

and we obtain (see (14))

\[
\sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) = \sup_{\sum_{i=1}^{n} |u_i|^{p_i} < r} \int_{\Omega} F(x, u_1(x), \ldots, u_n(x))dx
\]

\[
\leq \int_{\Omega} \max_{(t_1, \ldots, t_n) \in K_1} F(x, t_1, \ldots, t_n)dx
\]

\[
< r \int_{\Omega} F(x, u_1^{*}(x), \ldots, u_n^{*}(x))dx
\]

\[
= r \int_{\Omega} F(x, u_1^{*}(x), \ldots, u_n^{*}(x))dx
\]

where \( K_1 = \{(t_1, \ldots, t_n) \mid \sum_{i=1}^{n} |t_i|^{p_i} \leq cr \} \), so

\[
\sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) < r \frac{-J(u^{*})}{\Phi(u^{*})},
\]

which means that \((\kappa_2)\) is fulfilled. Next recall from \((j_2)\) that

\[
\eta > \frac{1}{r \frac{-J(u^{*})}{\Phi(u^{*})} - \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u))}.
\]

Choose

\[
\nu = \eta \left( r \frac{-J(u^{*})}{\Phi(u^{*})} - \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) \right),
\]

and note \( \nu > 1 \). Also, since

\[
\eta > \frac{1}{r \frac{-J(u^{*})}{\Phi(u^{*})} - \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u))},
\]
we have
\[ \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) + \frac{1}{\eta} < r \frac{-J(u^*)}{\Phi(u^*)}, \]
and so with our choice of \( \nu \) we have
\[ \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) + \frac{r - J(u^*)}{\Phi(u^*)} \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) < r \frac{-J(u^*)}{\Phi(u^*)}. \]
Now from Proposition 1 (with \( u_0 = 0 \) and \( u_1 = u^* \)) for every \( \rho \in \mathbb{R} \) satisfying
\[ \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) + \frac{r - J(u^*)}{\Phi(u^*)} \sup_{u \in \Phi^{-1}([-\infty, r])} (-J(u)) < r \frac{-J(u^*)}{\Phi(u^*)}, \]
we have (note \( \sigma = r \eta \))
\[ \sup_{\lambda \in \mathbb{R}, \ u \in X} \inf_{\eta \in [0, r \eta]} (\Phi(u) + \lambda J(u) + \rho \lambda) \leq \inf_{\eta \in [0, r \eta]} \sup_{u \in X} (\Phi(u) + \lambda J(u) + \rho \lambda). \]
For any fixed function \( G : \Omega \times \mathbb{R}^n \to \mathbb{R} \) as in the statement of the theorem, set
\[ \Psi(u) = - \int_\Omega G(x, u_1(x), \ldots, u_n(x)) \, dx. \]
It is well known that \( \Psi \) is a continuously differentiable functional whose differential \( \Psi'(u) \in X^* \), at \( u \in X \) is given by
\[ \Psi'(u)v = - \int_\Omega \sum_{i=1}^n G_{ui}(x, u_1(x), \ldots, u_n(x)) v_i(x) \, dx \quad \text{for every} \ (v_1, \ldots, v_n) \in X, \]
such that \( \Psi' : X \to X^* \) is a compact operator. Now, all the assumptions of Theorem 1, are satisfied. Hence, applying Theorem 1, and taking into account that the critical points of the functional \( \Phi + \lambda J + \mu \Psi \) are exactly the weak solutions of the problem (1), we have the conclusion.

References

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