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A Far Field Solution of the Slowly Varying Drift Force on the Offshore Structure in Bichromatic Waves-Three Dimensional Problems

Sang-Moo Lee*

Department of Guidance and Control, DongYang University, Kyungbuk, Korea

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ABSTRACT: *A far field approximate solution of the slowly varying force on a 3 dimensional offshore structure in gravity ocean waves is presented. The first order potential, or at least the far field form of the Kochin function, of each frequency wave is assumed to be known. The momentum flux of the fluid domain is formulated to find the time variant force acting on the floating body in bichromatic waves. The second order difference frequency force is identified and extracted from the time variant force. The final solution is expressed as the circular integration of the product of Kochin functions. The limiting form of the slowly varying force is identical to the mean drift force. It shows that the slowly varying force components caused by the body disturbance potential can be evaluated at the far field.*

1. Introduction

This paper presents a prediction method for the slowly varying drift force acting on ocean structures in gravity waves. The slowly varying drift force is known as a second order force excited at a frequency lower than that of the incident waves. The determination of the slowly varying drift force becomes of importance in the design of offshore structures, because this low frequency force can activate the resonance of a mooring system.

Analytic methods for the slowly varying force are more or less related to those for the mean drift force, which is also second order but non oscillating. While the mean drift forces can be deduced from the time mean of the wave force acting on the body in a monochrome wave, the slowly varying force requires the analysis of the second order time variant forces in bichromatic waves. Because of the difficulty in handling these forces, there have been different trends in the development of the analytic methods. Far field solutions (Maruo, 1960; Newman, 1967) for the mean drift forces were developed in the 1960s, and near and middle field solutions (Chen, 2007) followed. In the case of the slowly varying force, near field solutions (Pinkster, 1979) were first developed, and then the middle field solutions (Dai et al., 2005) followed.

Though the far field solutions are expected to have fast convergence and less complexity in computation, their mathematical derivation is so far unresolved. This paper presents a far field solution for the slowly varying forces on a three dimensional body. The theoretical derivation is an extension of the first paper by the same author (Lee, 2008) for two dimensional problems.

First order outgoing waves from the floating body are assumed to be evaluated using first order potentials. Then, the momentum conservation principle is applied to the far field outgoing bichromatic waves. The second order slowly varying force terms are then extracted from the momentum flux equation. It is also shown that when the difference between the two wave frequencies becomes small, the limiting case of the slowly varying force is identical to the mean drift force.

2. Mathematical Formulation

The conservation of momentum flux is expressed in terms of the linear velocity potential that describes the pressure field at the far field. Then, the bichromatic incident waves of different frequencies are introduced to define the potential field. By applying the bichromatic wave potential to the conservation of the momentum flux, the formula for the second order slowly varying force is derived.

2.1 Momentum flux

Take a Cartesian coordinate system with vertical z-axis and an xy plane coinciding with a calm water surface of infinite water depth. Consider a monochromatic plane sinusoidal wave progressing in a direction that makes an angle α with the x-axis. The velocity potential is expressed in the form

Correspondig author Sang-Moo Lee: #1 Kyochon Pungki Yeongju Kyungbuk, 054-630-1329 smlee@dyu.ac.kr

$$\Phi(x,\ y,\ z,\ t) = Re\phi(x,\ y,\ z)e^{i\sigma t} \tag{1}$$

where σ is the circular frequency. The free surface condition satisfied by ϕ at $z=0$ becomes

$$\frac{\partial\phi}{\partial z} - K\phi = 0 \tag{2}$$

where $K = \sigma^2/g$ is the wave number and g is the acceleration of gravity. The velocity potential of the incident wave is written as

$$\phi_I = ch\exp(Kz - iKx\cos\alpha - iKy\sin\alpha) \tag{3}$$

where $c = g/\sigma$ and h is the wave amplitude. The disturbance velocity potential due to the floating or fixed body in the incident wave is designated as ϕ_B. This first order disturbance potential is assumed to be the linear sum of the diffraction and radiation potential. Taking cylindrical coordinates as

$$x = R\cos\theta,\ y = R\sin\theta,\ z = z' \tag{4}$$

the asymptotic expression of the velocity potential at the infinity radius R is written as

$$\begin{aligned}\phi \rightarrow\ & ch\exp(Kz - iKR\cos(\theta - \alpha)) \\ & - i(K/2\pi R)^{1/2} H(\theta)\exp[Kz - iKR + i\pi/4]\end{aligned} \tag{5}$$

where function $H(\theta)$ is a Kochin function given as

$$H(\theta) = -\iint_s \left(\frac{\partial\phi_B}{\partial n} - \phi_B\frac{\partial}{\partial n}\right)\exp(Kz + iKx\cos\theta + iKy\sin\theta)ds \tag{6}$$

where subscript s denotes the body surface and unit normal n is drawn to the fluid domain from the body. A circular cylinder, S', of infinite radius with vertical z-axis at the origin of the coordinates is taken as a momentum control surface. The change in the linear momentum can be written in a vector form as

$$\frac{d\overrightarrow{M}}{dt} = \iint_{S'} p\vec{n}ds + \iint_{S'} \rho\vec{q}(\vec{n}\bullet\vec{q})ds - \vec{F} \tag{7}$$

where $\vec{n}$ is a unit normal drawn inward to S', p is the pressure, ρ is the water density, $\vec{q} = -\nabla\Phi$ is the fluid velocity and $\vec{F}$ is the force acting on the body. The pressure on the control surface at the infinite radius is written as

$$\frac{p}{\rho} = \frac{\partial\Phi}{\partial t} - gz - \frac{1}{2}\left\{\left(\frac{\partial\Phi}{\partial z}\right)^2 + \left(\frac{\partial\Phi}{\partial R}\right)^2 + \left(\frac{\partial\Phi}{\partial\theta}\right)^2/R^2\right\} \tag{8}$$

in terms of the partial derivatives. We also have

$$\vec{n}\bullet\vec{q} = \frac{\partial\Phi}{\partial R} \tag{9}$$

Basically the vector component forces can be deduced from Eq. (7). Taking only the x-component for simplicity, the force in the x-direction is obtained from Eq. (7) as

$$\begin{aligned}F_x = & -\int_{-\infty}^{\zeta} dz\int_0^{2\pi} p\cos\theta\,Rd\theta \\ & -\rho\int_{-\infty}^{\zeta} dz\int_0^{2\pi}\left(\frac{\partial\Phi}{\partial R}cos\theta - \frac{\partial\Phi}{\partial\theta}sin\theta/R\right)\frac{\partial\Phi}{\partial R}Rd\theta \\ & -\frac{dM_x}{dt}\end{aligned} \tag{10}$$

where the wave elevation is given by

$$\zeta = \frac{1}{g}\frac{\partial\Phi}{\partial t}\Big|_{z=0} \tag{11}$$

and M_x is the x-directional momentum. The force acting on the circular cylinder, S', is expressed by Eq. (10). The physical interpretation of the force in (10) is that Eq. (10) is the force acting on the circular cylinder, S', caused by the incident and disturbance wave fields by the body. This force has a close relationship with the forces acting on the body, but is not necessarily equal to the force on the body.

Eq. (10) is composed of forces with different characteristics including first order force, mean drift force, and second order slowly varying force. The slowly varying force in bichromatic waves can be identified from Eq. (10).

2.2 Velocity potentials caused by the bichromatic waves

The linear incident and disturbed velocity potentials of the bichromatic unidirectional waves with circular frequencies σ_i and σ_j are written in a simple sum as,

$$\Phi = Re\left[(\phi_{iI} + \phi_{iB})e^{i\sigma_i t} + (\phi_{jI} + \phi_{jB})\,e^{i\sigma_j t}\right] \tag{12}$$

Here, the subscripts i and j denote the frequency related quantities, and the subscripts I and B represent the incident and body disturbed quantities, respectively.

From Eqs. (8) and (11), the leading order of Eq. (10) with respect to wave slope can be rewritten as

$$\begin{aligned}F_x = & -\rho\int_{-\infty}^{0} dz\int_0^{2\pi}\frac{\partial\Phi}{\partial t}\cos\theta\,Rd\theta - \frac{\rho g}{2}\int_0^{2\pi}\zeta^2|_{z=0}\cos\theta\,Rd\theta \\ & + \frac{1}{2}\rho\int_{-\infty}^{0} dz\int_0^{2\pi}\left\{\left(\frac{\partial\Phi}{\partial z}\right)^2 - \left(\frac{\partial\Phi}{\partial R}\right)^2\right\}\cos\theta\,Rd\theta - \frac{dM_x}{dt}\end{aligned} \tag{13}$$

The derivatives in Eq. (13) are obtained from Eqs. (5), (11), and (12). The first term in Eq. (13) is the first order force and the others are second order or higher with respect to the wave slope. Because we are concerned with the second order forces, the first order term is not treated in the present analysis.

The remaining terms in Eq. (13) are composed of second order forces such as the mean drift, slowly varying, and fast varying forces. The mean drift force was well explained by Maruo (1960) for a body in a monochromatic wave. In the present research, only the terms that contribute to the slowly varying force are collected and considered (Lee, 2008).

The second order forces that contribute the slowly varying force are written as

$$F_{xij}=-\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi} Re(E_{iI}+E_{iB})(E_{jI}+E_{jB})^* e^{i(\sigma_i-\sigma_j)t}\cos\theta\, Rd\theta \quad (14)$$
$$+\frac{\rho K_iK_j}{2(K_i+K_j)}\int_0^{2\pi} Re[\{(E_{iI}+E_{iB})(E_{jI}+E_{jB})^*$$
$$-\{(-i\cos(\theta-\alpha)E_{iI}+(-i)E_{iB})$$
$$((-i\cos(\theta-\alpha)E_{jI}+(-i)E_{jB})^*\}\}e^{i(\sigma_i-\sigma_j)t}]\cos\theta\, Rd\theta$$
$$-\frac{dM_{xij}}{dt}$$

where the superscript * denotes the complex conjugate, and

$$E_{iI}=c_ih_i\exp(-iK_iR\cos(\theta-\alpha)) \quad (15a)$$
$$E_{iB}=-i(K_i/2\pi R)^{1/2}H_i(\theta)\exp(-iK_iR+i\pi/4) \quad (15b)$$
$$E_{jI}=c_jh_j\exp(-iK_jR\cos(\theta-\alpha)) \quad (15c)$$
$$E_{jB}=-i(K_j/2\pi R)^{1/2}H_j(\theta)\exp(-iK_jR+i\pi/4) \quad (15d)$$

The integrand of Eq. (14) combined with Eq. (15a) and Eq. (15c) is highly oscillating for large R. In that case, the integration can be performed using the stationary phase method as shown in Appendix A.

Further observation shows that the integrand in Eq. (14) always has stationary phases at $\theta=\alpha$ or $\theta=\alpha+\pi$. Using this fact, Eq. (14) is then rearranged as

$$F_{xij}=-\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi}\{Re(E_{iI}E_{jI}^*+E_{iB}E_{jI}^* \quad (16)$$
$$+E_{iI}E_{jB}^*+E_{iB}E_{jB}^*)e^{i(\sigma_i-\sigma_j)t}\cos\theta\, R\}\, d\theta$$
$$+\frac{\rho K_iK_j}{2(K_i+K_j)}\int_0^{2\pi}Re\{(1-\cos(\theta-\alpha))$$
$$(E_{iB}E_{jI}^*+E_{iI}E_{jB}^*)\}e^{i(\sigma_i-\sigma_j)t}\cos\theta\, Rd\theta-\frac{dM_{xij}}{dt}$$

Each combination of the integrand in Eq. (14) can be treated independently. Next, the integration will be done term by term and discussed.

The first integral of $E_{iI}E_{jI}^*$ comes from the incident wave interactions, the so called second order Froude-Krylov force caused by the first order potential. Actually, the first integral is $O(R^{1/2})$ at an infinite radius, as shown in Appendix C, while the other integrals are $O(R^0)$. The meaning of this integration is that it is the second order wave interaction force on a fictitious large cylinder, not on the structure. From this fact, we can say that the incident wave-wave interaction effect is not properly modeled with this far field analysis.

However, we can expect their effect to be limited and proportional to the frequency difference because its limiting behavior is $O(K_i-K_j)R^2)$ as $R\to 0$. As the outer radius or the frequency difference goes to zero, it approaches zero. This means that the far field analysis is applicable to the narrow band wave spectrum without correcting the Froude-Krylov force. If the wave spectrum is wide band, the Froude-Krylov force should be included. This can be done by using direct integration on the body surface or taking an approximation as shown in Appendix B. The effect of the incident wave interaction, meaning the second order Froude-Krylov force, is excluded from the present analysis.

The other integrations in Eq. (16) are $O(R^0)$ and are independent on the fictitious radius, R. Therefore, these terms can be evaluated at the far field and used as the quantities related to the body itself.

Applying the stationary phase method, the second term in Eq. (16) is written as

$$F_{xij2}=-\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi}Re(E_{iI})(E_{jB})^*e^{i(\sigma_i-\sigma_j)t}\cos\theta\, Rd\theta \quad (17)$$
$$=-\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi}Re\{ic_ih_i(K_j/2\pi R)^{1/2}H_j^*(\theta)$$
$$\exp(-iK_iR\cos(\theta-\alpha)+i(K_jR-\pi/4))e^{i(\sigma_i-\sigma_j)t}\}\cos\theta\, Rd\theta$$
$$=-\frac{\rho\sigma_jh_i}{2}(\frac{K_j}{K_i})^{1/2}\cos\alpha\, Re\{(iH_j^*(\alpha)\exp(-i(K_i-K_j)R)$$
$$+H_j^*(\alpha+\pi)\exp(iK_iR+iK_jR))e^{i(\sigma_i-\sigma_j)t}\}$$

where the phase is stationary at $\theta=\alpha$ or $\theta=\alpha+\pi$. The integration Eq. (17) is the sum of the results at both stationary phases. The third term is evaluated also by the method of stationary phase as

$$F_{xij3}=-\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi}Re(E_{iB})(E_{jI})^*e^{i(\sigma_i-\sigma_j)t}\cos\theta\, Rd\theta \quad (18)$$
$$=\frac{\rho\sigma_ih_j}{2}(K_i/K_j)^{1/2}\cos\alpha\, Re[\{iH_i(\alpha)\exp(-iK_iR+iK_jR)$$
$$-H_i(\alpha+\pi)\exp(-iK_iR-iK_jR)\}e^{i(\sigma_i-\sigma_j)t}]$$

The fourth term is directly rewritten as

$$F_{xij4}=-\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi}Re(E_{iB})(E_{jB})^*e^{i(\sigma_i-\sigma_j)t}\cos\theta\, Rd\theta \quad (19)$$
$$=-\frac{\rho g}{4}Re[\int_0^{2\pi}H_i(\theta)H_j^*(\theta)\cos\theta d\theta\, e^{(-i(K_i-K_j)R)}e^{i(\sigma_i-\sigma_j)t}]$$

The second integration in Eq. (16) can also be evaluated as

$$F_{xij5}=\frac{\rho K_iK_j}{(K_i+K_j)}(c_ih_i)(K_j/K_i)^{1/2}\cos\alpha \quad (20)$$
$$Re[H_j^*(\alpha+\pi)\exp(iK_iR+iK_jR)e^{i(\sigma_i-\sigma_j)t}]$$
$$+\frac{\rho K_iK_j}{(K_i+K_j)}(c_jh_j)(K_i/K_j)^{1/2}\cos\alpha$$
$$Re[H_i(\alpha+\pi)\exp(-iK_jR-iK_iR))e^{i(\sigma_i-\sigma_j)t}]$$

The slowly varying force in $O(R^0)$ is then obtained by summing Eqs. (17), (18), (19) and (20) as

$$
\begin{aligned}
F_{xij} &= F_{xij2} + F_{xij3} + F_{xij4} + F_{xij5} \\
&= -\frac{\rho\sigma_j h_i}{2}(K_j/K_i)^{1/2}\cos\alpha\, Re[iH_j^*(\alpha)e^{(-i(K_i-K_j)R)}e^{i(\sigma_i-\sigma_j)t}] \\
&+ \frac{\rho\sigma_i h_j}{2}(K_i/K_j)^{1/2}\cos\alpha\, Re[iH_i(\alpha)e^{(-i(K_i-K_j)R)}e^{i(\sigma_i-\sigma_j)t}] \\
&- \frac{\rho K_i K_j}{4\pi} Re[\int_0^{2\pi} H_i(\theta)H_j^*(\theta)\cos\theta d\theta\, e^{(-i(K_i-K_j)R)}e^{i(\sigma_i-\sigma_j)t}] \\
&- \frac{\rho g h_i h_j}{(K_j K_i)^{1/2}} \frac{(\sqrt{K_i}-\sqrt{K_j})^2}{2(K_i+K_j)} cos\alpha \\
& Re\{ \frac{K_j}{c_j h_j} H_j^*(\alpha+\pi)\exp(iK_iR+iK_jR)) \\
&+ \frac{K_i}{c_i h_i} H_i(\alpha+\pi)\exp(-iK_iR-iK_jR)\} e^{i(\sigma_i-\sigma_j)t} \}
\end{aligned}
\tag{21}
$$

A pattern can be found in the components of Eq. (21). The first three terms have the same outgoing wave pattern with constant amplitudes. The last term shows a standing wave pattern with a long wave length and is second order with respect to the frequency difference, $O((\sigma_i - \sigma_j)^2)$. This is maybe due to the reflected waves in the negative wave direction. The terms in Eq. (21) can not be simply added up to give the magnitude. The magnitude of the slowly varying force is given as a bounded value

$$
\begin{aligned}
& F_{xij} / \frac{\rho g h_i h_j}{2\sqrt{K_i K_j}} \\
& \leq \Bigg| i\cos\alpha\Big(\frac{K_i H_i(\alpha)}{c_i h_i} - \frac{K_j H_j^*(\alpha)}{c_j h_j}\Big) \\
& - \frac{1}{2\pi}\int_0^{2\pi} \frac{K_i H_i(\theta)}{c_i h_i}\frac{K_j H_j^*(\theta)}{c_j h_j} cos\theta d\theta \Bigg| \\
& + \cos\alpha\Big(\frac{(\sqrt{K_i}-\sqrt{K_j})^2}{K_i+K_j}\Big)\Big(\Big|\frac{K_i H_i(\alpha+\pi)}{c_i h_i}\Big| + \Big|\frac{K_j H_j^*(\alpha+\pi)}{c_j h_j}\Big|\Big)
\end{aligned}
\tag{22}
$$

in normalized forms. The result shows that the slowly varying force can be evaluated only by the information from the Kochin functions at both component frequencies.

3. Discussion

The validity can be shown by taking the limit to Eq. (22). If σ_j approaches σ_i, the slowly varying force of Eq. (22) becomes the mean drift force that was derived by Maruo (1960). From the energy conservation, the Kochin function has the relation

$$H(\alpha) - H^*(\alpha) = -\frac{iK}{2\pi ch}\int_0^{2\pi} |H(\theta)|^2 d\theta \tag{23}$$

Applying Eq. (23) to Eq. (22), the mean drift force is obtained as

$$F_x^{(2)} = \frac{\rho K_i^2}{4\pi}\int_0^{2\pi} |H_i(\theta)|^2(\cos\alpha - \cos\theta)d\theta \tag{24}$$

This is exactly twice the value derived by Maruo (1960). Because we are dealing with bichromatic waves, the final result is double the drift force of a single wave.

Table 1 Magnitude comparison of the slowly varying horizontal force on the vertical circular cylinder, $F/(\rho g h_i h_j/\sqrt{K_i K_j}$

Ka	1.0	1.2	1.4	1.6	1.8	2.0
	0.665	0.652	0.634	0.620	0.601	0.571
Ka		0.638	0.619	0.608	0.593	0.568
1.0	0.666		0.605	0.600	0.594	0.577
1.2	0.647	0.636		0.603	0.606	0.598
1.4	0.612	0.612	0.603		0.618	0.619
1.6	0.578	0.588	0.594	0.600		0.627
1.8	0.552	0.567	0.583	0.602	0.615	
2.0	0.534	0.547	0.566	0.593	0.615	0.624

A brief numerical test is next performed and compared with the existing result. The Kochin function for a vertical infinite depth circular cylinder can be obtained by applying Eq. (6) as

$$H(\theta) = -\frac{2ich}{K}\sum_{m=0}^{\infty}\frac{\epsilon_m J_m{}'(Ka)}{H_m^{(2)}{}'(Ka)} cosm\theta \tag{25}$$

where a is the radius of the cylinder, $H^{(2)}(Ka)$ is the second kind Hankel function, and $\varepsilon_0 = 1$, and $\varepsilon_m = 2$ for $m = 1, 2, 3, ...$ Applying the Kochin function in Eq. (25) to Eq. (22) yields the slowly varying forces as shown in Table 1. The upper half of the table shows the present results, while the lower half shows those from Kim and Yue (1990) for a finite depth cylinder $D/a = 4$ obtained by the near field method. It should be noted that only the effect of the first order potential is compared in the table.

The diagonal terms in Table 1 are the second order mean drift forces, and the off diagonal terms are the slowly varying drift forces caused by the first order potentials. Though there is a difference in the depths of the vertical column, the depth effect is negligible for the given wave numbers because the difference in the diagonal terms that show the depth effect is insignificant. The magnitude of the force difference is within 10% and the values show similar trends. The difference, however, may be caused by the incident wave interactions, which are neglected in the far-field method. It can be concluded that the present analysis can evaluate the second order slowly varying force caused by the first order potential with meaningful accuracy.

4. Conclusions

A far field solution for the slowly varying force acting on an arbitrarily shaped surface body in bichromatic waves was presented. The closed form solution showed that once the Kochin function was evaluated at each frequency, the slowly varying force due to the first order potential could be predicted with the combination of Kochin functions.

The far field formulation for the time dependent force showed that the second order forces caused by the body disturbance are independent of the evaluation radius, $O(R^0)$ and can be evaluated at the far field. In contrast the incident wave interaction force is $O(R^{1/2})$ and can not be evaluated at the far field.

The derived force formula is so simple that only the Kochin functions are needed to predict the slowly varying force. This makes it possible to avoid calculating wave elevations at the water line and particle velocities on the wetted surface as are usually done in the near field method. Moreover, a comparison shows that the present results matched those of the near field method fairly well.

Although the other effects, such as those of the second order incident and diffraction potentials, are needed to give more complete solutions, it is expected that easiness in the analysis and applicability to arbitrary body shapes of the structure will give much flexibility in the theoretical development and practical engineering with the presented far field method.

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Appendix

Appendix A The stationary phase method

Let a function be defined as the integration on the finite contour like

$$J(x,t) = \int_c e^{i\phi(\omega)} F(\omega) d\omega \tag{A.1}$$

It is assumed that $F(\omega)$ is a moderately varying function, while $\phi(\omega)$ is fast varying and has a vanishing derivative at $\omega = \omega_s$. Using the relation

$$\phi(\omega) = \phi_s + \frac{1}{2}\phi_s''(\omega-\omega_s)^2 + \ldots \tag{A.2}$$

the integral over a contour from zero to positive infinity is written as

$$J_s \simeq F(\omega_s)e^{i\phi_s}\int_0^\infty e^{\frac{i}{2}\phi_s''(\omega-\omega_s)^2} d\omega \tag{A.3}$$

When $\phi_s'' > 0$, the integral is transformed into

$$J_s \simeq F(\omega_s)e^{i\phi_s}\sqrt{\frac{4\pi}{\phi_s''}}\int_0^\infty [\cos(\frac{\pi t^2}{2}) + i\sin(\frac{\pi t^2}{2})]dt \tag{A.4}$$

where $\frac{\pi t^2}{2} = \frac{1}{2}\phi_s''(\omega-\omega_s)^2$, $\sqrt{\pi}\, t = \sqrt{\phi_s''}(\omega-\omega_s)$, $d\omega = \sqrt{\frac{\pi}{\phi_s''}}\, dt$ are used. The integral is a Fresnel integral and can be shown as

$$\int_0^\infty \cos(\pi t^2/2)dt = \int_0^\infty \sin(\pi t^2/2)dt = \frac{1}{2} \tag{A.5}$$

It follows that

$$J_s \simeq F(\omega_s)e^{i\phi_s}\sqrt{\frac{2\pi}{\phi_s''}}\, e^{i\pi/4} \tag{A.6}$$

And if $\phi_s'' < 0$, taking conjugate

$$\begin{aligned} J_s &\simeq F(\omega_s)e^{i\phi_s}\int_0^\infty e^{\frac{i}{2}\phi_s''(\omega-\omega_s)^2} d\omega \\ &= F(\omega_s)e^{i\phi_s} conj(\int_0^\infty e^{-\frac{i}{2}\phi_s''(\omega-\omega_s)^2} d\omega) \\ &= F(\omega_s)e^{i\phi_s}\sqrt{\frac{2\pi}{-\phi_s''}}\, e^{-i\pi/4} \end{aligned} \tag{A.7}$$

Appendix B The effect of the incident wave interaction

The incident wave-wave second order interaction is written from Eq. (14) as

$$F_{xij1} = -\frac{\rho\sigma_i\sigma_j}{2g}\int_0^{2\pi} Re(E_{iI})(E_{jI}^*)e^{i(\sigma_i-\sigma_j)t}\cos\theta \circ Rd\theta$$

$$= \frac{\rho c_i h_i c_j h_j \sigma_i \sigma_j}{2g} Re[-\frac{2i}{\pi} R\cos\alpha J_1(-(K_i - K_j)R)e^{i(\sigma_i - \sigma_j)t}] \quad \text{(A.8)}$$
$$= \frac{\rho g h_i h_j}{\pi} Re[iR\cos\alpha J_1((K_i - K_j)R)e^{i(\sigma_i - \sigma_j)t}]$$

where J_1 is the Bessel function of the first kind. The limiting values of this quantity are obtained by taking the limits as

$$\lim_{R\to 0} F_{x\,ij1} = \lim_{R\to 0} \frac{\rho g h_i h_j}{\pi} Re[iR\cos\alpha J_1((K_i - K_j)R)e^{i(\sigma_i - \sigma_j)t}] \quad \text{(A.9)}$$
$$= -\frac{\rho g h_i h_j}{\pi}(K_i - K_j)R^2 \cos\alpha \sin(\sigma_i - \sigma_j)t$$
$$\approx O((K_i - K_j)R^2)$$

When the radius goes to infinite, the asymptotic value is given as

$$\lim_{R\to\infty} F_{x\,ij1} = \lim_{R\to\infty} \frac{\rho g h_i h_j}{\pi} Re[iR\cos\alpha J_1((K_i - K_j)R)e^{i(\sigma_i - \sigma_j)t}] \quad \text{(A.10)}$$
$$\approx -\frac{\rho \sigma_i \sigma_j}{g} \cos\alpha \sqrt{\frac{2R}{(K_i - K_j)}} \cos((K_i - K_j)R)\sin(\sigma_i - \sigma_j)t$$
$$\approx O(R^{1/2})$$

2010년 10월 8일 원고 접수
2010년 10월 27일 심사 완료
2010년 12월 17일 게재 확정