SOME APPLICATIONS OF $q$-DIFFERENTIAL OPERATOR

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Abstract. In this paper, we use $q$-differential operator to recover the finite Heine $_2\Phi_1$ transformations given in [3]. Applying that, we also obtain some terminating $q$-series transformation formulas.

1. Introduction

Recently, G. E. Andrews [3] derived several finite Heine $_2\Phi_1$ transformations from the terminating Sears $_3\Phi_2$ transformation. Then he used them to give two finite Rogers-Ramanujan type identities. In this paper, by using the properties of $q$-differential operators, we also obtain the finite Heine $_2\Phi_1$ transformation and the following finite $q$-series transformations

\begin{equation}
\sum_{j=0}^{M} \frac{1}{\binom{M}{j}} (-1)^j q^{(j)^2+2j} \frac{1}{1-a_1q^j} = \frac{(q;q)_M}{(a_1;q)_{M+1}} \sum_{j=0}^{M} (a_1;q)_j q^j, \tag{1}
\end{equation}

\begin{equation}
\sum_{k=0}^{M} \frac{(-1)^k k^2 (-q^2;q^2)_k}{(q^4;q^4)_k (q^2;q^2)_{M-k}} = \frac{(q^2;q^2)_M}{(q^4;q^4)_M}, \tag{2}
\end{equation}

\begin{equation}
\sum_{k=0}^{M} \frac{q^{k^2-sk} (-q^s;q^2)_k}{(q;q)_{2k} (q^2;q^2)_{M-k}} = \frac{(-q^{1-s};q^2)_M}{(q;q)_{2M}}, \quad s = 0, 1, \tag{3}
\end{equation}

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\[
\sum_{k=0}^{M} q^{3k^2 - sk} = \frac{1}{(-q^2-s;q^2)_M} \sum_{k=0}^{M} q^{k^2 + (1-s)k} (q^2;q^2)_M^{-k}, \quad s = 0, 1,
\]

where
\[
(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \ldots
\]

and
\[
(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).
\]

As \( M \to \infty \), the second identity reduces to the identity appearing in [14, p. 152, Eq. (4)] (or [13, p. 99, Eq. (A.4)]). If \( s = 0, M \to \infty \), the third tends to the identity appearing in Slater’s paper [14, p. 156, Eq. (47)] (or [4, p. 252, Eq. (11.2.1)], [13, p. 104, Eq. (A.47)]).

Throughout the paper, we take \( 0 < |q| < 1 \). And we also use the following notations
\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,
\]
\[
(r \Phi_s (a_1, \ldots, a_r; b_1, \ldots, b_s; q, x)) = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r; q]}{[b_1, b_2, \ldots, b_s; q]} \frac{(-1)^n q^{n(n-1)/2}}{x^n},
\]
and
\[
\binom{n}{k} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.
\]

2. Some lemmas

Recall that the \( q \)-differential operator \( D_q \) and \( q \)-shifted operator \( \eta \) (cf. [6, 7, 10-12]), acting on the variable \( x \), are defined by:
\[
D_q \{ f(x) \} = \frac{f(x) - f(xq)}{x} \quad \text{and} \quad \eta \{ f(x) \} = f(xq).
\]

We can prove, by means of induction, the explicit formulae (cf. [10, 11])
\[
D_q^n \left\{ \frac{(x\omega; q)_\infty}{(xs; q)_\infty} \right\} = s^n \frac{(\omega/ s; q)_n (x\omega q^n; q)_\infty}{(xs; q)_\infty},
\]
Some Applications of $q$-Differential Operator

(6) \[ D^n_q \{ f(x) \} = x^{-n} \sum_{k=0}^{n} q^k \frac{(q^{-n}; q)_k}{(q; q)_k} f(q^k x) \]

and the $q$-Leibniz rule for the product of two functions

(7) \[ D^n_q \{ f(x)g(x) \} = \sum_{k=0}^{n} \binom{n}{k} q^{(k-n)} D^k_q \{ f(x) \} D^{n-k}_q \{ g(x^k) \}. \]

In [6, 7], we have constructed the following $q$-exponential operator

(8) \[ _1 \Phi_0 \left( \frac{b}{c}; q, cD_q \right) = \sum_{n=0}^{\infty} \frac{(b; q)_n (cD_q)^n}{(q; q)_n}, \]

and gave some applications of it. In this paper, we will use the case of $b = q^{-M}$,

(9) \[ _1 \Phi_0 \left( \frac{q^{-M}}{c}; q, cD_q \right) = \sum_{n=0}^{M} \frac{(q^{-M}; q)_n (cD_q)^n}{(q; q)_n} \]

and the following more general finite $q$-exponential operator

(10) \[ _2 \Phi_1 \left( q^{-M}; a_1/ b_1, q; q, cD_q \right) = \sum_{n=0}^{M} \frac{(q^{-M}; q)_n (cD_q)^n}{(q; q)_n} \]

where $M$ is a non-negative integer.

Letting \[ F(x) = \left[ xc_1, \; xc_2, \ldots, \; xc_r ; q \right]_\infty, \]
from (6), we have:

**Lemma 2.1.** For complex numbers $x, a_i, b_i, i = 1, 2, \ldots, r,$

(11) \[ D^n_q \{ F(x) \} = x^{-n} F(x) \sum_{k=0}^{n} \frac{q^{-n}, \; xd_1, \; xd_2, \ldots, \; xd_r ; q}{q, \; xc_1, \; xc_2, \ldots, \; xc_r ; q} \]

From (7) and (9), we obtain the next lemma.

**Lemma 2.2.** We have

(12) \[ _1 \Phi_0 \left( \frac{q^{-M}}{c} ; q, cD_q \right) \left\{ \frac{(c_1 x_1 ; q)_\infty}{(d_1 x_1, d_2 x_2)_\infty} \right\} = (cd_2 q^{-M} ; q)_M \left( c_1 x_1 ; q \right)_\infty _3 \Phi_2 \left( q^{-M}, \; c_1 / d_1, \; x_2 ; q, cd_1 \right). \]

The identity above is a special case of an identity in [6, p. 21, Eq. (7)].
Lemma 2.3. If \( c = q/d_2 \), then
\[
(13)
\]
\[
2\Phi_1 \left( q^{-M}, \frac{a_1}{b_1}; q, cD_q \right) \left\{ (c_1x; q)_\infty \right\}
\]
\[
= a_M \left( \frac{b_1/a_1;q)_M}{(b_1;q)_M} \frac{(c_1x; q)_\infty}{(d_1x, d_2x)_\infty} \Phi_2 \left( q^{-M}, a_1, d_2x, \frac{c_1/d_1}{c_1x}, q^{-1}a_1/b_1; q, qd_1/b_1d_2 \right). \]
\]
Proof. From (7), we have
\[
(14)
\]
\[
2\Phi_1 \left( q^{-M}, \frac{a_1}{b_1}; q, cD_q \right) \left\{ (c_1x; q)_\infty \right\}
\]
\[
= \sum_{k=0}^{M} \left[ q^{-M}, \frac{a_1}{b_1}; q, cD_q, k \right] \left\{ (c_1x; q)_\infty \right\}
\]
\[
\times \sum_{j=0}^{M-k} \left[ q^{-M-k}, \frac{a_1q^k}{b_1q^k}; q, c, j \right] \left\{ (c_1x; q)_\infty \right\}
\]
\[
= a_M \left( \frac{b_1/a_1;q)_M}{(b_1;q)_M} \frac{(c_1x; q)_\infty}{(d_1x, d_2x)_\infty} \Phi_2 \left( q^{-M}, a_1, d_2x, \frac{c_1/d_1}{c_1x}, q^{-1}a_1/b_1; q, qd_1/b_1d_2 \right). \]
\]
This completes the proof. \( \square \)

3. Main results and special cases

Theorem 3.1 (cf. [8], p. 11). The \textit{q-Chu-Vandermonde summation}
\[
(15)
\]
\[
2\Phi_1 \left( q^{-n}, \frac{a_1}{d_1}; q, q \right) = a^n / (d_1q)_n\]
\]
Proof. Setting \( F(x) = (xc_1; q)_\infty / (xd_1; q)_\infty \) in (11), and then putting \( xc_1 = d, xd_1 = a \), we complete the proof. \( \square \)

Theorem 3.2 (cf. [8, p. 16, Eq. (1.9.11)]). Suppose \( n > m_1 + \cdots + m_r \). Then we have
\[
(16)
\]
\[
\Phi_r \left( q^{-n}, xc_1q^{m_1}, \ldots, xc_rq^{m_r}; q, q \right) = 0.\]
\]
Proof. Setting \( d_i = c_iq^{m_i}, m_i = 0, 1, \ldots, \infty, i = 1, 2, \ldots, r \), in (11), we complete the proof. \( \square \)

Theorem 3.3. We have
\[
(17)
\]
\[
\sum_{m=0}^{n} \sum_{j=0}^{M} \left( q^{-n}, a_1q^m; (q, d_1; q)_m \right) \left( q^{-M}, a_1q^m; q, b_1q^j \right) q^{m+j} = a_M \left( \frac{b_1/a_1;q)_M}{(b_1; q)_M} \frac{a^n(d/a; q)_n}{(d; q)_n} \Phi_2 \left( q^{-M}, a_1, a_1, q^{-1}a_1/b_1; q, q^2/db_1 \right). \]
Proof. We rewrite (14) as follows
\begin{equation}
\sum_{m=0}^{n} \frac{(q^{-n}; q)_m}{(q, d; q)_m} q^m \frac{1}{(aq^m; q)_\infty} = \frac{(-1)^n q^{-\frac{1}{2}}}{(d, q)_n} \frac{(a q^{1-n}/d; q)_\infty}{(aq/d, a; q)_\infty}.
\end{equation}

Applying the operator $2\Phi_1 \left( \genfrac{}{}{0pt}{}{q^{-M}, a_1}{b_1}; q, b \right)$ to both sides of (17) with respect to the variable $a$, from (13), we complete the proof.

Corollary 3.1. We have
\begin{equation}
\sum_{m=0}^{n} \sum_{j=0}^{M} \frac{(q^{-n}; q)_m (q^{-M}; q)_j}{(q, q)_m (q; q)_j} q^{m+j} (1 - a q^m) (1 - a q^j)
= \frac{(q; q)_M}{(a; q)_M + 1} \frac{(q; q)_n}{(q, q)_n} \sum_{k=0}^{\min(M, n)} \left( \frac{a, a_1; q}{q, q} \right)_k (-1)^k q^{-\frac{k}{2}} a_1^{M-k} a^n - k.
\end{equation}

Proof. If set $b_1 = qa_1$ and $d = qa$ in (16), we complete the proof.

Theorem 3.4. We have
\begin{equation}
\sum_{m=0}^{n} \sum_{j=0}^{M} \frac{(q^{-n}; q)_m (q^{-M}; q)_j}{(q, d; q)_m (q, b_1; q)_j} d^j q^{m+j} = \frac{a_1^M b_1 (b_1/a_1; q)_M}{(b_1; q)_M} a^n \frac{(d/a_1; q)_n}{(d; q)_n} q^{-1-n} \Phi_2 \left( \genfrac{}{}{0pt}{}{q^{-1-n}/d}{d/a_1}; q, d/b_1 \right).
\end{equation}

Proof. We rewrite (17) as follows
\begin{equation}
\sum_{m=0}^{n} \frac{(q^{-n}; q)_m}{(q, d; q)_m} q^m \frac{1}{(aq^m; q)_\infty} = \frac{(-1)^n q^{-\frac{1}{2}}}{(d, q)_n} \frac{(a q^{-1-n}/d; q)_\infty}{(a q/d, a; q)_\infty}.
\end{equation}

Applying the operator $2\Phi_1 \left( \genfrac{}{}{0pt}{}{q^{-M}, a_1}{b_1}; q, d \right)$ to both sides of (20) with respect to the variable $a$, using (13), we complete the proof.

Corollary 3.2 (cf. [8, p. 23]). Jackson’s transformation formula
\begin{equation}
\sum_{j=0}^{M} \frac{(q^{-M}, a_1; q)_j}{(q, b_1; q)_j} d^j = \frac{a_1^M b_1 (b_1/a_1; q)_M}{(b_1; q)_M} q^{-M} \frac{q/d}{q^{-M} a_1/b_1} \Phi_1 \left( \genfrac{}{}{0pt}{}{q^{-M}, a_1}{b_1}; q, d/b_1 \right).
\end{equation}

Proof. If set $a \rightarrow 1$ in (19), we complete the proof.

Corollary 3.3. We have
\begin{equation}
\sum_{j=0}^{M} \left[ \frac{M}{j} \right] \frac{(-1)^j q^{-\frac{j+1}{2}}}{(a_1; q)_j} \frac{(b_1/a_1; q)_M}{(b_1; q)_M} d^j \frac{b_1}{a_1} = \frac{b_1 (b_1/a_1; q)_M}{(b_1; q)_M} \sum_{j=0}^{M} \frac{(q, a_1; q^{-1-M}/b_1; q)_j}{(q, a_1; q^{-1-M}/b_1; q)_j}.
\end{equation}
Proof. If set \( q \to 1/q \) in (21), then replacing \( a_1 \) by \( 1/a_1 \), \( b_1 \) by \( 1/b_1 \), we complete the proof. \( \square \)

Setting \( d = 1, b_1 = qa_1 \), (22) tends to:

**Corollary 3.4.** We have

\[
\sum_{j=0}^{M} \left[ \frac{M}{j} \right] (-1)^j q^{(j+2)/2} \frac{1}{1-a_1q^j} = \frac{(q;q)_M}{(a_1;q)_{M+1}} \sum_{j=0}^{M} (a_1;q) j q^j.
\]

**Theorem 3.5.** We have

\[
3 \Phi_2 \left( q^{-M}, \frac{c_1/d_2}{cd_1q^{-M}}, \frac{xd_1}{xc_1}; q, cd_2 \right)
\]

\[
= \frac{(q/cd_2;q)_M}{(q/cd_1;q)_M} \frac{d_2}{d_1}^{M} 3 \Phi_2 \left( q^{-M}, \frac{c_1/d_1}{d_2q^{-M}}, \frac{xd_2}{xc_1}; q, cd_1 \right).
\]

**Proof.** For

\[
1 \Phi_0 \left( q^{-M} : q, cD_q \right) \left\{ (c_1x;q)_\infty \right\} = 1 \Phi_0 \left( q^{-M} : q, cD_q \right) \left\{ (c_1x;q)_\infty \right\},
\]

and applying (12), we complete the proof. \( \square \)

In the identity (24), taking \( q \to 1/q \), then replacing \((x, c, c_1, d_i)\) by \((1/x, c/q, 1/c_1, 1/d_i)\) respectively, where \( i = 1, 2 \), we obtain the following identity:

**Theorem 3.6.** We have

\[
3 \Phi_2 \left( q^{-M}, \frac{c_1/d_2}{cd_1q^{-M}}, \frac{xd_1}{xc_1}; q, q \right)
\]

\[
= \frac{(cd/ab;q)_M}{(cd/ab;q)_M} \frac{d}{d_1}^{M} 3 \Phi_2 \left( q^{-M}, \frac{c_1/d_1}{d_2q^{-M}}, \frac{xd_2}{xc_1}; q, d_2q \right).
\]

**Remark.** An equivalent identity can be found in Andrews’ paper [3, Corollary 4].

**Theorem 3.7.** We have

\[
3 \Phi_2 \left( q^{-M}, \frac{a}{c}, \frac{b}{d}, \frac{cdq^M/ab}{c}, \frac{x}{z}; q, z \right)
\]

\[
= \frac{(cd/ab;q)_M}{(d_1/q)_M} \frac{d}{d_1}^{M} 3 \Phi_2 \left( q^{-M}, \frac{a}{c}, \frac{b}{d}, \frac{cd/ab}{c}, \frac{dx}{z}; q, dq^{ab} \right).
\]

**Proof.** In (24), letting \( c \to cq^M \), then replacing \( xc_1 \) by \( c \), \( cd_1 \) by \( d \), \( c_1/d_2 \) by \( a \), last step setting \( cd_1/ad_2 = b \), we complete the proof. \( \square \)

**Remark.** (26) follows from setting \( a = q^{-M} \) in the Sears \( \Phi_2 \) transformation [8, p. 62, Eq. (3.2.7)].

**Corollary 3.5** (cf. [8, p. 10, Eq. (1.4.6)]). Heine’s \( 2 \Phi_1 \) transformation formula

\[
2 \Phi_1 \left( \frac{a}{c}, \frac{b}{d}; q, z \right) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} 2 \Phi_1 \left( \frac{a}{c}, \frac{b}{d}; q, abz/c \right).
\]
Proof. In (25), letting $M \to \infty$, then replacing $c/d_1$ by $z$, $xc_1$ by $c$, $c_1/d_1$ by $a$, last step setting $a_1c/ad_2 = b$, we complete the proof.

Corollary 3.6. We have

$$2\Phi_2\left(a, \frac{b}{c}, \frac{c}{d}; q, cd/ab\right) = \frac{(cd/ab; q)_\infty}{(d; q)_\infty} 2\Phi_2\left(c/a, \frac{c/b}{cd/ab}; q, d\right).$$

Proof. In (25), putting $M \to \infty$, we complete the proof.

4. Some other special cases

Theorem 4.1. We have

$$\sum_{k=0}^{M} \frac{(c_1/d_2; q)_k q^k}{(q, xc_1; q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{(k+1)}(xd_2; q)_k}{(q, xc_1; q)_k (q; q)_M-k} \left(\frac{c_1}{d_2}\right)^k.$$

Proof. In (25), putting $c = d_1q$, then letting $d_1 \to 0$, we complete the proof.

Theorem 4.2. We have

$$\sum_{k=0}^{M} \frac{(c_1/d_2; q)_k (-xd_2)_k q^{(k+1)}q}{(q, xc_1; q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{(k+1)}(xd_2; q)_M-k}{(q; q)_k (q, xc_1; q)_M-k}.$$

Proof. In (28), taking $q \to 1/q$, then replacing $(x, c_1, d_2)$ by $(1/x, 1/c_1, 1/d_2)$ respectively, we complete the proof.

Corollary 4.1. We have

$$\sum_{k=0}^{M} \frac{q^k}{(xd_2; q)_k+1} = \sum_{k=0}^{M} \frac{(-1)^k q^k q^{2k}}{(1-xd_2q^k)(q; q)_k (q; q)_M-k}.$$

Proof. In (28), putting $c_1 = d_2q$, we complete the proof.

Corollary 4.2. We have

$$\sum_{k=0}^{M} \frac{q^k}{(q, xc_1; q)_k} = \sum_{k=0}^{M} \frac{q^{k^2} (xc_1)_k}{(q, xc_1; q)_k (q; q)_M-k}.$$

Proof. In (28), letting $d_2 \to \infty$, we complete the proof.

Corollary 4.3. We have

$$\sum_{k=0}^{M} \frac{q^k}{(q; q)_k^2} = \sum_{k=0}^{M} \frac{q^{k^2+k}}{(q; q)_k (q; q)_M-k}.$$

Proof. In (31), putting $xc_1 = q$, we complete the proof.

Corollary 4.4. We have

$$\sum_{k=0}^{M} \frac{q^{k^2-k} (xc_1)_k}{(q, xc_1; q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{(k+1)}q}{(q; q)_k (q, xc_1; q)_M-k}.$$
Proof. In (29), setting \( d_2 = 0 \), we complete the proof. \( \square \)

**Corollary 4.5.** We have

\[
\sum_{k=0}^{M} \frac{q^{(k+1)/2}(-1)^{k/2}}{(q; q^2)_k} = \sum_{k=0}^{M} \frac{(-1)^{k/2}q^{(k+1)/2}(-q; q)_{M-k}}{(q; q)_{k}(q; q^2)_{M-k}}.
\]

Proof. In (29), taking \( xc_1 = q, c_1 = -d_2 \), we complete the proof. \( \square \)

**Theorem 4.3.** We have

\[
\sum_{k=0}^{M} (-1)^k q^{k^2-k}(a, b; q^2)_k \frac{(cd/ab)}{k} = \frac{(cd/ab)_M}{(d; q^2)_M} \sum_{k=0}^{M} d^k(-1)^k q^{k^2-k}(c/a, c/b; q^2)_k.
\]

Proof. In (26), letting \( q \to q^2 \), we complete the proof. \( \square \)

**Corollary 4.6.** We have

\[
\sum_{k=0}^{M} (-1)^k q^{k^2} (-q; q^2)_k = \frac{(q^2)_M}{(q^2)_M}.
\]

Proof. In (35), letting \( c = b, d = -q^2, a = -q \), we complete the proof. \( \square \)

**Corollary 4.7.** We have

\[
\sum_{k=0}^{M} q^{k^2-sk} (-q; q^2)_k = \frac{(-q^{1-s}; q^2)_M}{(q; q)_{2M}}
\]

where \( s = 0, 1 \).

Proof. In (35), letting \( c = b, d = q, a = -q, -1 \), we complete the proof. \( \square \)

**Corollary 4.8.** We have

\[
\sum_{k=0}^{M} q^{k^2+(2-s)k} (-q^s; q^2)_k = \frac{(-q^{3-s}; q^2)_M}{(q; q)_{2M+1}}
\]

where \( s = 0, 1, 2 \).

Proof. In (35), letting \( c = b, d = q^3, a = -q^2, -q, -1 \), we complete the proof. \( \square \)

**Corollary 4.9.** We have

\[
\sum_{k=0}^{M} \frac{q^{k^2-sk}}{(q^2; q)_{2k}(q^2; q^2)_{M-k}} = \frac{1}{(-q^{2-s}; q^2)_M} \sum_{k=0}^{M} \frac{q^{k^2+(1-s)k}}{(q; q)_{2k}(q^2; q^2)_{M-k}}.
\]
where \( s = 0, 1 \).

Proof. In (35), letting \( c = q, d = -q^2, -q, a, b \to \infty \), we complete the proof. □

Corollary 4.10. We have

\[
\sum_{k=0}^{M} \frac{q^{3k^2 + sk}}{(q; q)_{2k+1}} = \frac{1}{(-q^4; q^2)_M} \sum_{k=0}^{M} \frac{q^{k^2+(s-1)k}}{(q; q)_{2k+1}}.
\]

where \( s = 0, 1, 2, 3 \).

Proof. In (35), letting \( c = q^3, d = -q^3, -q, -1, a, b \to \infty \), we complete the proof. □

Corollary 4.11. We have

\[
\sum_{k=0}^{M} \frac{q^{2k^2} (q; q^2)^k}{(-q^4; q^2)_k(q^4; q^4)_k(q^2; q^2)_M} = \frac{1}{(-q^4; q^2)_M} \sum_{k=0}^{M} \frac{q^{k^2} (-q; q^2)_k}{(q^4; q^4)_k(q^2; q^2)_M},
\]

where \( s = 0, 1, 2, 3 \).

Proof. In (35), letting \( c = q^2, d = -q^2, a = q, b \to \infty \), we complete the proof. □

Letting \( M \to \infty \), if \( x_{c1} = q \), (33) tends to (cf. [1, p. 33, Eq. (1.1)] or [3, p. 1, Eq. (1.2)])

\[
\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k^2} = \frac{1}{(q; q)}.
\]

Equation (34) reduces to

\[
\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k^2} = \frac{(-q; q)_\infty}{(q; q)_\infty}.
\]

Equation (39) turns to

\[
\sum_{k=0}^{\infty} \frac{q^{3k^2 - sk}}{(q^2; q^2)_k(q^4; q^4)_k} = \frac{1}{(-q^2; q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2 + (1-s)k}}{(q; q)_{2k}},
\]

where \( s = 0, 1 \).

Equation (40) tends to

\[
\sum_{k=0}^{\infty} \frac{q^{3k^2 + sk}}{(q; q)_{2k+1}(-q^4; q^2)_k} = \frac{1}{(-q^4; q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2 + (s-1)k}}{(q; q)_{2k+1}},
\]

where \( s = 0, 1, 2, 3 \).

Applying these relations above, then using the identities

\[
\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_{2k}} = \frac{(q^2; q^2; q^{10}; q^{10})_\infty}{(q; q)_\infty} \cdot \frac{(q^4; q^{14}; q^{20})_\infty}{(q; q)_\infty},
\]
We have improvement of an earlier version of this paper. In [2, 3, 4, 9, 15], the authors used Eq. (19), [14, p. 156, Eq. (46)], [4, p. 252, Eq. (11.2.7)] and [14, p. 156, Eq. (44)], respectively. In Slater’s paper [14, p. 162, Eq. (98), (94), (99), (96), respectively] (or cf. [4, p. 252, Eq. (11.2.1)–Eq. (11.2.4)]), we have shown in Slater’s paper [14, p. 154, (98), (94), (99), (96), respectively] (or cf. [4, p. 252, Eq. (11.2.1)–Eq. (11.2.4)]), we have

\[\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_{2k+1}} = \frac{(q^3, q^7, q^{10}; q^{10})_\infty (q^4, q^{16}; q^{20})_\infty}{(q; q)^\infty},\]

\[\sum_{k=0}^{\infty} q^{k^2+k} \frac{q}{(q; q)_{2k}} = \frac{(q, q^9, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q; q)^\infty},\]

\[\sum_{k=0}^{\infty} q^{k^2+2k} \frac{q}{(q; q)_{2k+1}} = \frac{(q^4, q^6, q^{10}; q^{10})_\infty (q^7, q^{18}; q^{20})_\infty}{(q; q)^\infty},\]

shown in Slater’s paper [14, p. 162, Eq. (98), (94), (99), (96), respectively] (or cf. [4, p. 252, Eq. (11.2.1)–Eq. (11.2.4)]), we have

\[\sum_{k=0}^{\infty} q^{3k^2} \frac{q}{(q; q^2)_k(q^4; q^4)_k} = \frac{(q, q^9, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q; q)^\infty (-q^2; q^2)^\infty},\]

\[\sum_{k=0}^{\infty} q^{3k^2-k} \frac{q}{(q^2; q^2)_k(q^2; q^4)_k} = \frac{(q^2, q^8, q^{10}; q^{10})_\infty (q^6, q^{14}; q^{20})_\infty}{(q; q)^\infty (-q^2; q^2)^\infty},\]

\[\sum_{k=0}^{\infty} \frac{q^{3k^2+2k}}{(q^2; q^2)_{k+1}(q^4; q^4)_k} = \frac{(q^3, q^7, q^{10}; q^{10})_\infty (q^4, q^{16}; q^{20})_\infty}{(q; q)^\infty (-q^2; q^2)^\infty},\]

\[\sum_{k=0}^{\infty} q^{3k^2+2k} \frac{q}{(q; q)_{2k+1}(-q; q^2)_{k+1}} = \frac{(q^4, q^6, q^{10}; q^{10})_\infty (q^7, q^{18}; q^{20})_\infty}{(q; q)^\infty (-q^2; q^2)^\infty}.\]

Equations (49), (50), (51) and (52) are equivalent to the identities [14, p. 154, Eq. (19)], [14, p. 156, Eq. (46)], [4, p. 252, Eq. (11.2.7)] and [14, p. 156, Eq. (44)] respectively. In [2, 3, 4, 9, 15], the authors used q-series transformations to obtain many Rogers-Ramanujan type identities. Here, we will present a new identity by using this method. From the identity in Slater’s list [14, p. 154, Eq. (44)], combined with (41), we get the new identity.

**Corollary 4.12.** We have

\[\sum_{k=0}^{\infty} q^{2k^2} \frac{(q; q^2)_k}{(-q; q^2)_k(q^4; q^4)_k} = \frac{(q^3, q^5, q^7; q^6)_\infty}{(q^2; q^2)^\infty}.\]

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References


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