MIXED CHORD-INTEGRALS OF STAR BODIES

LU FENGHONG

Abstract. The mixed chord-integrals are defined. The Fenchel-Aleksandrov inequality and a general isoperimetric inequality for the mixed chord-integrals are established. Furthermore, the dual general Bieberbach inequality is presented. As an application of the dual form, a Brunn-Minkowski type inequality for mixed intersection bodies is given.

1. Introduction and main results

In [6, 9] Lutwak posed the notion of the mixed width-integrals of convex bodies (compact, convex subsets with non-empty interiors) and obtained a great many properties in common with the dual quermassintegrals (see [7]). It exists closely relations (see [5, 16]) between the mixed width-integrals and mixed projection bodies (see [10, 12]). In [8] Lutwak established a general Bieberbach inequality. Motivated by the ideas of Lutwak, we shall introduce the definitions of the mixed chord-integrals of star bodies and establish a dual general Bieberbach inequality.

The setting for this paper is \(n\)-dimensional Euclidean space \(\mathbb{R}^n\). Let \(K^n\) denote the set of convex bodies (compact, convex subsets with non-empty interiors) and \(K_n^o\) denote the subset of \(K^n\) that contains the origin in their interiors in \(\mathbb{R}^n\). Let \(S^{n-1}\) denote the unit sphere in \(\mathbb{R}^n\). The volume of the unit \(n\)-ball, \(U\), will be denoted by \(\omega_n\). If \(K \in K^n\), then the support function of \(K\), \(h_K = h(K, \cdot) : \mathbb{R}^n \rightarrow (0, \infty)\), is defined by

\[
(1.1) \quad h(K, u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}
\]

where \(u \cdot x\) denotes the standard inner product of \(u\) and \(x\). For \(K \in K^n\) and \(u \in S^{n-1}\), \(b(K, u)\) is half the width of \(K\) in the direction \(u\).

\[\text{Received May 14, 2008; Revised December 15, 2008.}
\]
\[\text{2000 Mathematics Subject Classification. 52A40.}
\]
\[\text{Key words and phrases. mixed intersection bodies, mixed chord-integrals, Fenchel-Aleksandrov inequality, Bieberbach inequality.}
\]
\[\text{Supported by the Sciences Foundation of Shanghai (No.071605123) and by Innovation Program of Shanghai Municipal Education Commission (No.10YZ160).}
\]
\[\text{©2010 The Korean Mathematical Society}
\]
Mixed width-integrals, $A(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n \in \mathcal{K}^n$ was defined by Lutwak (see [9])

$$\tag{1.2} A(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} b(K_1, u) \cdots b(K_n, u) dS(u),$$

where $dS$ is the $(n-1)$-dimensional volume element on $S^{n-1}$. More in general, for a real number $p \neq 0$, the mixed width-integrals of order $p$, $A_p(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n \in \mathcal{K}^n$ was also defined by Lutwak (see [9])

$$\tag{1.3} A_p(K_1, \ldots, K_n) = \omega_n \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} b(K_1, u)^p \cdots b(K_n, u)^p dS(u) \right]^{1/p},$$

Furthermore, Lutwak in [9] showed an isoperimetric inequality involving the mixed width-integrals which generalized an inequality obtained by Chakerian (see [1, 2]).

**Theorem 1**. If $K_1, \ldots, K_n \in \mathcal{K}^n$, then

$$\tag{1.4} V(K_1) \cdots V(K_n) \leq A^n(K_1, \ldots, K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are $n$-balls.

A more general version of inequality (1.4) is obtained by introducing indexed mixed width-integrals.

**Theorem 2**. If $K_1, \ldots, K_n \in \mathcal{K}^n$, $p \neq 0$ and $-1 \leq p \leq \infty$, then

$$\tag{1.5} V(K_1) \cdots V(K_n) \leq A^n_p(K_1, \ldots, K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are $n$-balls.

Let $\mathcal{S}_0^n$ denote the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. In this paper, we give the definitions of the mixed chord-integrals, $B(L_1, \ldots, L_n)$ and for a real number $p \neq 0$, the mixed chord-integrals of order $p$, $B_p(L_1, \ldots, L_n)$, of $L_1, \ldots, L_n \in \mathcal{S}_0^n$ and the $p$-chord, $\tilde{d}_p(L)$, of $L \in \mathcal{S}_0^n$; respectively. We mainly obtain the following results.

The analog of Theorem 1 (i.e., the general isoperimetric inequality involving the mixed width-integrals) for the mixed chord-integrals is obtained:

**Theorem 1**. If $L_1, \ldots, L_n \in \mathcal{S}_0^n$, then

$$\tag{1.6} B^n(L_1, \ldots, L_n) \leq V(L_1) \cdots V(L_n),$$

with equality if and only if $L_1, \ldots, L_n$ are dilates and centered.

A more general version of inequality (1.6) is obtained:

**Theorem 2**. If $L_1, \ldots, L_n \in \mathcal{S}_0^n$ and $-\infty \leq p \leq 1$, then

$$\tag{1.7} B^n_p(L_1, \ldots, L_n) \leq V(L_1) \cdots V(L_n),$$

with equality if and only if $L_1, \ldots, L_n$ are $n$-balls.
Theorem 3. If $L_1, \ldots, L_n \in S_n^o$ and $1 < m \leq n$, then
\begin{equation}
B^m(L_1, \ldots, L_n) \leq \prod_{i=0}^{m-1} B(L_1, \ldots, L_{n-i}, \ldots, L_{n-i}),
\end{equation}
with equality if and only if $L_{n-m+1}, L_{n-m+2}, \ldots, L_n$ are all of similar chord.

A more general version of inequality (1.8) is obtained:

Theorem 4. If $L_1, \ldots, L_n \in S_n^o$ and $1 < m \leq n$, then for $p > 0$
\begin{equation}
B^m_p(L_1, \ldots, L_n) \leq \prod_{i=0}^{m-1} B_p(L_1, \ldots, L_{n-i}, \ldots, L_{n-i}),
\end{equation}
with equality if and only if $L_{n-m+1}, L_{n-m+2}, \ldots, L_n$ are all of similar chord. For $p < 0$, inequality (1.9) is reversed.

Theorem 3 and Theorem 4 are just analogs of the Fenchel-Aleksandrov inequality for the dual mixed volumes (see [7]).

Moreover, we obtain the dual general Bieberbach inequality as follows.

Theorem 5. If $L \in S_n^o$ and $-\infty \leq p < n$, then
\begin{equation}
V(L) \geq \omega_n \tilde{\alpha}_p(L)^n,
\end{equation}
with equality if and only if $L$ is an $n$-ball.

Theorem 5 is just a dual form of the following general Bieberbach inequality which was shown by Lutwak [8].

Theorem 5*. If $K \in K^n$ and $-n < p \leq \infty$, then
\begin{equation}
V(K) \leq \omega_n \tilde{\alpha}_p(K)^n,
\end{equation}
with equality if and only if $K$ is an $n$-ball.

As an application of Theorem 5, we establish a Brunn-Minkowski type inequality for mixed intersection bodies which defined in [11, 15].

Theorem 6. If $L_1, L_2 \in S_n^o$ and $i \leq n-1$, then
\[
\tilde{W}_i(I(L_1 \# L_2))^{1/(n-i)} \leq \tilde{W}_i(I L_1)^{1/(n-i)} + \tilde{W}_i(I L_2)^{1/(n-i)},
\]
with equality if and only if $L_1$ and $L_2$ are dilates. If $n-1 < i < n$ or $i > n$, then this inequality is reversed.

In particular, for $i = 0$, we obtain:

Corollary 1. If $L_1, L_2 \in S_n^o$, then
\[
V(I(L_1 \# L_2))^{1/n} \leq V(I L_1)^{1/n} + V(I L_2)^{1/n},
\]
with equality if and only if $L_1$ and $L_2$ are dilates.
2. Mixed chord-integrals

For a compact subset \( L \) of \( \mathbb{R}^n \), which is star-shaped with respect to the origin, we shall use \( \rho(L, \cdot) \) to denote its radial function; i.e., for \( u \in S^{n-1} \),
\[
\rho(L, u) = \max\{\lambda > 0 : \lambda u \in L\}.
\]
If \( \rho(L, \cdot) \) is continuous and positive, \( L \) will be called a star body. Two star bodies \( K \) and \( L \) are said to be dilates if \( \rho_K(u)/\rho_L(u) \) is independent of \( u \in S^{n-1} \).

If \( K \in \mathcal{K}_n^o \), the polar body of \( K \), \( K^* \), is defined by
\[
K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, x \in K\}.
\]
Obviously, we have \((K^*)^* = K\). From the definition (2.2), we also know that: If \( K \in \mathcal{K}_n^o \), then the support and radial function of \( K^* \), the polar body of \( K \), are respectively defined by
\[
h_{K^*} = \frac{1}{\rho_K} \quad \text{and} \quad \rho_{K^*} = \frac{1}{h_K}.
\]
The polar coordinate formula for volume of body \( L \) in \( \mathbb{R}^n \) is
\[
V(L) = \frac{1}{n} \int_{S^{n-1}} \rho(L, u)^n dS(u).
\]
If \( L \in \mathcal{S}_n^o \) and \( u \in S^{n-1} \), we let
\[
d(L, u) = \frac{1}{2} \rho(L, u) + \frac{1}{2} \rho(L, -u),
\]
i.e., \( d(L, u) \) denotes half the chord of \( L \) in the direction \( u \). Star bodies \( L_1, \ldots, L_n \) are said to have similar chord if there exist constants \( \lambda_1, \ldots, \lambda_n > 0 \) such that \( \lambda_1 d(L_1, u) = \cdots = \lambda_n d(L_n, u) \) for all \( u \in S^{n-1} \); they are said to have constant chord jointly if the product \( d(L_1, u) \cdots d(L_n, u) \) is constant for all \( u \in S^{n-1} \). For reference see Gardner ([3]) and Schneider ([14]).

Following Lutwak, we define the mixed chord integral of star bodies: For \( L_1, \ldots, L_n \in \mathcal{S}_n^o \), the mixed chord-integral, \( B(L_1, \ldots, L_n) \), of \( L_1, \ldots, L_n \) is defined by
\[
B(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} d(L_1, u) \cdots d(L_n, u) dS(u).
\]
By this definition, \( B \) is a map
\[
B : \underbrace{\mathcal{S}_n^o \times \cdots \times \mathcal{S}_n^o}_n \rightarrow \mathbb{R}.
\]
We list some of its elementary properties.

(i) (Continuity) The mixed chord-integral \( B(L_1, \ldots, L_n) \) is a continuous function of \( L_1, \ldots, L_n \in \mathcal{S}_n^o \).

(ii) (Positivity) For \( L_1, \ldots, L_n \in \mathcal{S}_n^o \), \( B(L_1, \ldots, L_n) > 0 \).

(iii) (Positively homogeneous) If \( L_1, \ldots, L_n \in \mathcal{S}_n^o \) and \( \lambda_1, \ldots, \lambda_n > 0 \), then
\[
B(\lambda_1 L_1, \ldots, \lambda_n L_n) = \lambda_1 \cdots \lambda_n B(L_1, \ldots, L_n).
\]
(iv) (Monotonicity for set inclusion) If $K_i, L_i \in S^n_o$, $K_i \subset L_i$ and $1 \leq i \leq n$, then
\[ B(K_1, \ldots, K_n) \leq B(L_1, \ldots, L_n), \]
with equality if and only if $K_i = L_i$ for $1 \leq i \leq n$.

(v) (Change under linear transformations) If $L_1, \ldots, L_n \in S^n_o$ and $\phi \in GL(n)$, then
\[ B(\phi L_1, \ldots, \phi L_n) = |\det \phi| B(L_1, \ldots, L_n). \]

Just as the width-integrals $B_i(K)$ (see [6]) of $K \in K^n$, are defined to be the special mixed width-integrals
\[ A(K, \ldots, K_{i-1}, U, \ldots, U_{n-1}), \]
the chord-integrals $D_i(L)$ of $L \in S^n_o$, can be defined as the special mixed chord-integral
\[ B(L, \ldots, L_{i-1}, U, \ldots, U_{n-1}). \]

Now we generalize the notion of the mixed chord-integral of star bodies: For $L_1, \ldots, L_n \in S^n_o$ and a real number $p \neq 0$, the mixed chord-integral of order $p$, $B_p(L_1, \ldots, L_n)$, of $L_1, \ldots, L_n$ is defined by
\[ B_p(L_1, \ldots, L_n) = \omega_n \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_n, u)^p dS(u) \right]^{1/p}. \]
Specially $p = 1$, then definition (2.7) is just definition (2.6). For $p$ equal to $-\infty, 0 \text{ or } \infty$ we respectively define the mixed chord-integral of order $p$ by
\[ B_p(L_1, \ldots, L_n) = \lim_{s \to p} B_s(L_1, \ldots, L_n). \]

As a direct consequence of Jensen’s inequality [4] we have:

**Proposition 1.** If $L_1, \ldots, L_n \in S^n_o$ and $-\infty \leq p < q \leq \infty$, then
\[ B_p(L_1, \ldots, L_n) \leq B_q(L_1, \ldots, L_n), \]
with equality if and only if $L_1, \ldots, L_n$ have constant chord jointly.

The well-known Blaschke-Santaló inequality (see [13]) can be stated: For $K \in K^n_o$, then
\[ V(K) V(K^*) \leq \omega_n^2, \]
with equality if and only if $K$ is an $n$-dimensional ellipsoid.

By combining the well-known Blaschke-Santaló inequality with Theorem 2, we obtain the Blaschke-Santaló type inequality for the mixed chord-integral of order $p$ (the mixed chord-integral when $p = 1$).

**Corollary 2.** If $K_1, \ldots, K_n \in K^n_o$, then
\[ B_p(K_1, \ldots, K_n) B_p(K_1^*, \ldots, K_n^*) \leq \omega_n^2, \]
with equality if and only if $K_1, \ldots, K_n$ are $n$-balls.
In particular, if $L_1 = \cdots = L_{n-i} = L$ and $L_{n-i+1} = \cdots = L_n = U$, then Theorem 1 becomes:

**Corollary 3.** If $L \in S^n_o$ and $0 \leq i \leq n$, then

$$D_i^n(L) \leq \omega_n V(L)^{n-i},$$

with equality if and only if $L$ is an $n$-ball.

Combing Theorems 1 and 2 with Theorems 1 and 2, we obtain the following relations between $B(L_1, \ldots, L_n)$, $B_p(L_1, \ldots, L_n)$ and $A(L_1, \ldots, L_n)$, $A_p(L_1, \ldots, L_n)$, respectively.

**Corollary 4.** If $K_1, \ldots, K_n \in K^n$, then

$$B(K_1, \ldots, K_n) \leq A(K_1, \ldots, K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are $n$-balls.

**Corollary 5.** If $K_1, \ldots, K_n \in K^n$ and $-1 \leq p \leq 1$, then

$$B_p(K_1, \ldots, K_n) \leq A_p(K_1, \ldots, K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are $n$-balls.

**Proof of Theorem 1.** For $L_1, \ldots, L_n \in S^n_o$. From definition (2.6), H"older integral inequality [4], definition (2.5), Minkowski integral inequality [4] and formula (2.4), we have

$$B(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-i}} d(L_1, u) \cdots d(L_n, u) dS(u)$$

$$= n^{-1/n} \| d(L_1, u) \|_n \cdots n^{-1/n} \| d(L_n, u) \|_n$$

$$= n^{-1/n} \left( \frac{1}{2} \rho(L_1, u) + \frac{1}{2} \rho(L_1, -u) \right) \cdots n^{-1/n} \left( \frac{1}{2} \rho(L_n, u) + \frac{1}{2} \rho(L_n, -u) \right)$$

$$\leq V^{\frac{1}{n}}(L_1) \cdots V^{\frac{1}{n}}(L_n).$$

In view of the equality conditions of H"older integral inequality and Minkowski integral inequality, equality of inequality (1.6) holds if and only if $L_1, \ldots, L_n$ are dilates and centered. Thus we obtain the conclusion.

**Proof of Theorem 2.** For $L_1, \ldots, L_n \in S^n_o$ and $-\infty \leq p \leq 1$. By combining Theorem 1 with Proposition 1 we obtain

$$B_p^n(L_1, \ldots, L_n) \leq B_1^n(L_1, \ldots, L_n) \leq V(L_1) \cdots V(L_n).$$

In view of the equality conditions of Theorem 1 and Proposition 1, equality holds if and only if $L_1, \ldots, L_n$ are $n$-balls. Thus we obtain the conclusion.
In order to prove Theorem 3 and Theorem 4 in the introduction, we require the following simple extension of Hölder’s inequality.

**Lemma 1** (see [7]). If $f_0, f_1, \ldots, f_m$ are (strictly) positive continuous functions defined on $S^{n-1}$ and $\lambda_1, \ldots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then

$$\int_{S^{n-1}} f_0(u) f_1(u) \cdots f_m(u) dS(u) \leq \prod_{i=1}^{m} \left( \int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u) dS(u) \right)^{1/\lambda_i},$$

with equality if and only if there exist positive constants $\alpha_1, \ldots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

**Proof of Theorem 3.** For $L_1, \ldots, L_n \in S^n$. Let

$$\lambda_i = m \ (1 \leq i \leq m),$$

$$f_0 = d(L_1, u) \cdots d(L_{n-m}, u) \ (f_0 = 1 \text{ if } m = n),$$

$$f_i = d(L_{n-i+1}, u) \ (1 \leq i \leq m).$$

Using Lemma 1, we have

$$\int_{S^{n-1}} d(L_1, u) \cdots d(L_n, u) dS(u) \leq \prod_{i=1}^{m} \left( \int_{S^{n-1}} d(L_1, u) \cdots d(L_{n-m}, u) d(L_{n-i+1}, u)^m dS(u) \right)^{1/\lambda_i},$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \ldots, L_n$ are all of similar chord. Thus we obtain the conclusion. \qed

**Proof of Theorem 4.** For $L_1, \ldots, L_n \in S^n$. Let

$$\lambda_i = m \ (1 \leq i \leq m),$$

$$f_0 = d(L_1, u)^p \cdots d(L_{n-m}, u)^p \ (f_0 = 1 \text{ if } m = n),$$

$$f_i = d(L_{n-i+1}, u)^p \ (1 \leq i \leq m).$$

Using Lemma 1, we have

$$\int_{S^{n-1}} d(L_1, u)^p \cdots d(L_n, u)^p dS(u) \leq \prod_{i=1}^{m} \left( \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_{n-m}, u)^p d(L_{n-i+1}, u)^{pm} dS(u) \right)^{1/\lambda_i},$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \ldots, L_n$ are all of similar chord. For $p > 0$, we get

$$\omega_n \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_n, u)^p dS(u) \right]^{1/p} \leq \omega_n \prod_{i=1}^{m} \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_{n-m}, u)^p d(L_{n-i+1}, u)^{pm} dS(u) \right]^{1/p \lambda_i},$$
with equality if and only if \( L_{m+1}, L_{m+2}, \ldots, L_n \) are all of similar chord. For \( p < 0 \), inequality above is reversed. Thus we obtain the conclusion. \( \square \)

### 3. Dual general Bieberbach inequality

In [8] Lutwak established a general Bieberbach inequality which has the Bieberbach, Urysohn and harmonic Urysohn inequalities as special cases. Following Lutwak, we give a dual general Bieberbach inequality.

If \( L \in S^n_o \) and \( i \in \mathbb{R} \), then the \( i \)-th dual quermassintegrals is defined by Lutwak (see [7])

\[
\tilde{W}_i(L) = \frac{1}{n} \int_{S^{n-1}} \rho(L, u)^{n-i} dS(u).
\]

Specifically, \( \tilde{W}_0(K) = V(K) \), and \( \tilde{W}_n(K) = \omega_n \).

For \( L \in S^n_o \), the intersection body of \( L \), \( IL \) is the centrally symmetric body whose radial function on \( S^{n-1} \) is given by (see [11]),

\[
\rho(IL, u) = \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(L, v)^{n-1} d\lambda_{n-2}(v),
\]

where \( \lambda_{n-2} \) denotes \((n-2)\)-dimensional Lebesgue measure. For \( u \in S^{n-1} \), \( L \cap u^\perp \) denotes the intersection of \( L \) with the subspace \( u^\perp \) that passes through the origin and is orthogonal to \( u \).

For \( L_1, L_2 \in S^n_o \), the radial addition \( L_1 +_L L_2 \) is defined as the star body whose radial function is given by (see [11]),

\[
\rho(L_1 +_L L_2, \cdot) = \rho(L_1, \cdot) + \rho(L_2, \cdot).
\]

The radial Blaschke linear addition, \( \lambda \cdot K +_\mu L \), is defined by Lutwak (see [11]), whose radial function satisfies for \( u \in S^{n-1} \) (see [11])

\[
\rho(\lambda \cdot K +_\mu L, u)^{n-1} = \lambda \rho(K, u)^{n-1} + \mu \rho(L, u)^{n-1}.
\]

The following properties will be used later: If \( L_1, L_2 \in S^n_o \) and \( \lambda, \mu > 0 \), then

\[
d(L_1 +_L L_2, \cdot) = d(L_1, \cdot) + d(L_2, \cdot),
\]

\[
I(\lambda \cdot K +_\mu L) = \lambda IK +_\mu IL.
\]

For \( K \in K^n \) and a real number \( p \neq 0 \), the \( p \)-width, \( \bar{b}_p(K) \) of \( K \) was defined by Lutwak in ([8])

\[
\bar{b}_p(K) = \left[ \frac{1}{n \omega_n} \int_{S^{n-1}} \bar{b}(K, u) dS(u) \right]^{1/p},
\]

where the definition (3.6) differs slightly from that of Lutwak (see [8]) in that we multiply a constant factor. For \( p \) equal to \(-\infty, 0 \) or \( \infty \) the \( p \)-width of \( K \) was defined by

\[
\bar{b}_p(K) = \lim_{s \to p} \bar{b}_s(K).
\]
In order to establish the dual general Bieberbach inequality, we shall also introduce the dual concept of the $p$-width of convex body: For $L \in S^n_0$ and a real number $p \neq 0$, the $p$-chord, $\tilde{d}_p(L)$ of $L$ was defined by

\[(3.7)\quad \tilde{d}_p(L) = \left[ \frac{1}{n\omega_n} \int_{S^n_{n-1}} d_p(L, u) dS(u) \right]^{1/p}.\]

For $p$ equal to $-\infty$, 0 or $\infty$ the $p$-chord of $L$ was defined by

$$\tilde{d}_p(L) = \lim_{s \to p} \tilde{d}_s(L).$$

For a fixed $p$, the $p$-chord is a map $\tilde{d}_p : S^n_0 \to \mathbb{R}$.

Similar to the mixed chord-integral, it is positive, continuous, bounded, homogeneous of degree one and monotone under set inclusion.

From Theorem 5 and Theorem 5$^*$ we can obtain the relation between the dual general Bieberbach inequality and the general Bieberbach inequality as follows.

**Corollary 6.** If $K \in K^n$ and $-n < p < n$, then

$$\tilde{d}_p(K) \leq \bar{b}_p(K),$$

with equality if and only if $K$ is an $n$-ball.

We shall prove the following relation between the $p$-chord of convex body and the $p$-chord of polar for convex body.

**Theorem 7.** If $K \in K^n_0$ and $-\infty \leq p < n$, then

$$\tilde{d}_p(K)\tilde{d}_p(K^*) \leq 1,$$

with equality if and only if $K$ is an $n$-ball.

Theorem 7 is just the dual of the following relation between the $p$-width of convex body and the $p$-width of polar for convex body which was shown by Lutwak (see [8]).

**Theorem 7$^*$**. If $K \in K^n_0$ and $-n < p \leq \infty$, then

$$\tilde{b}_p(K)\tilde{b}_p(K^*) \geq 1,$$

with equality if and only if $K$ is an $n$-ball.

Furthermore, we establish the following Brunn-Minkowski inequality for the $p$-chord of star bodies.

**Theorem 8.** For $L_1, L_2 \in S^n_0$, $p \geq 1$ and $\alpha \in [0, 1]$, then

\[(3.8)\quad \tilde{d}_p(L_1 + L_2) \leq \tilde{d}_p(\alpha L_1 + (1-\alpha)L_2) + \tilde{d}_p((1-\alpha)L_1 + \alpha L_2) \leq \tilde{d}_p(L_1) + \tilde{d}_p(L_2),\]

in each inequalities, with equality if and only if $L_1$ and $L_2$ have similar chord. If $p < 1$ and $p \neq 0$, then inequality (3.8) is reversed.
Proof of Theorem 5. For \( L \in S^n \), taking \( p = n \), from definition (3.7), definition (2.5), Minkowski integral inequality (see [4]) and formula (2.4), we get

\[
\tilde{d}_n(L) = \left[ \frac{1}{n\omega_n} \int_{S^{n-1}} d^n(L, u) dS(u) \right]^{1/n} \\
\leq \omega_n^{-1/n} \left[ \frac{1}{n} \int_{S^{n-1}} \rho^n(L, u) dS(u) \right]^{1/n} \\
= \omega_n^{-1/n} V(L)^{1/n},
\]

with equality if and only if \( L \) is centered. From Jensen’s inequality (see [4]), it follows that for \(-\infty \leq p < n\)

\[
(3.10) \quad \tilde{d}_n(L) \geq \tilde{d}_p(L),
\]

with equality if and only if \( L \) is of constant chord.

Combing these inequalities above, we have

\[
V(L) \geq \omega_n \tilde{d}_p(L)^n,
\]

with equality if and only if \( L \) is an \( n \)-ball. \( \square \)

Proof of Theorem 7. For \( K \in K^n \), combing inequality (3.9) with Blaschke-Santaló inequality, we obtain

\[
(3.11) \quad \omega_n \tilde{d}_n(K)^{-n} \geq V(K^*),
\]

with equality if and only if \( L \) is an \( n \)-ball. By combing inequality (3.10) and inequality (3.11), we have for \(-\infty \leq p < n\),

\[
(3.12) \quad \omega_n \tilde{d}_p(K)^{-n} \geq V(K^*),
\]

with equality if and only if \( L \) is an \( n \)-ball.

According to the inequality (3.11), inequality (3.12) and the dual general Bieberbach inequality, we obtain the desired result. \( \square \)

Proof of Theorem 8. For \( L_1, L_2 \in S^n \), \( p \geq 1 \) and \( \alpha \in [0, 1] \). From definition (3.7) and formula (3.4), Minkowski integral inequality, definition (3.7) and (3.4) again, Minkowski integral inequality again, definition (3.7) again, it follows that

\[
\tilde{d}_p(L_1 + L_2) = (n\omega_n)^{-1/p} \|d(L_1 + L_2, u)\|_p \\
= (n\omega_n)^{-1/p} \|d(L_1, u) + d(L_2, u)\|_p
\]
in each inequalities, with equality if and only if $L_1$ and $L_2$ have similar chord. In view of the inverse Minkowski integral inequality, similar above the proof, the cases of $p < 1$ and $p \neq 0$ can also be proved. Here we omit the details, i.e., if $p < 1$ and $p \neq 0$, then this inequality is reversed. □

Proof of Theorem 6. For $L_1, L_2 \in S^n_0$ and $i \leq n-1$. Since $I(L_1 \circ L_2)$, $IL_1$ and $IL_2$ are centered, then

$$d(I(L_1 \circ L_2), u) = \rho(I(L_1 \circ L_2), u), d(IL_1, u) = \rho(IL_1, u), d(IL_2, u) = \rho(IL_2, u).$$

Combining Theorem 8 with definition (3.1) and equalities above, we have

$$\bar{W}_i(I(L_1 \circ L_2))^{1/(n-i)} \leq \bar{W}_i(IL_1)^{1/(n-i)} + \bar{W}_i(IL_2)^{1/(n-i)},$$

with equality if and only if $L_1$ and $L_2$ are dilates. If $n-1 < i < n$ or $i > n$, then this inequality is reversed. □

Acknowledgements. The author wish to thank the referee(s) for their many excellent suggestions for improving the original manuscript.

References


Department of Mathematics
Shanghai University of Electric Power
Shanghai, 200090, P. R. China
E-mail address: lufengred@163.com