Korean J. Math. **18** (2010), No. 2, pp. 119–132

ON STRONG REVERSIBLE RINGS AND THEIR EXTENSIONS

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Abstract. P. M. Cohn called a ring R *reversible* if whenever $ab = 0$, then $ba = 0$ for $a, b \in R$. In this paper, we study an extension of a reversible ring with its endomorphism. An endomorphism α of a ring R is called *strong right* (resp., *left*) *reversible* if whenever $a\alpha(b) = 0$ (resp., $\alpha(a)b = 0$) for $a, b \in R$, $ba = 0$. A ring R is called *strong right* (resp., *left*) *α-reversible* if there exists a strong right (resp., left) reversible endomorphism α of R, and the ring R is called *strong α-reversible* if R is both strong left and right α-reversible. We investigate characterizations of strong α-reversible rings and their related properties including extensions. In particular, we show that every semiprime and strong α-reversible ring is α-rigid and that for an α-skew Armendariz ring R, the ring R is reversible and strong α-reversible if and only if the skew polynomial ring $R[x; \alpha]$ of R is reversible.

1. Introduction

Throughout this paper R denotes an associative ring with identity. Cohn [2] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A reversible ring is a generalization of a reduced ring (i.e., it has no nonzero nilpotent elements). Recently, the concept of the reversibility of elements at zero is extended to one of both an endomorphism and an element in a ring. From [1, Definitoion 2.1], an endomorphism α of a ring R is called *right* (resp., *left*) *reversible* if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called *right* (resp., *left*) *α-reversible* if there exists a right (resp., left) reversible endomorphism α

Received January 12, 2010. Revised April 22, 2010. Accepted May 4, 2010.
2000 Mathematics Subject Classification: 16W20, 16U80, 16S36.
Key words and phrases: reduced ring, endomorphism, (strong)reversible ring, skew Armendariz ring.
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of R. The ring R is *α-reversible* if it is both right and left α-reversible. Consider the reverse condition of a right α-reversible ring R,

$$(*) \quad a\alpha(b) = 0 \ \text{ for } \ a, b \in R \ \text{ implies } \ ba = 0.$$

The following example illuminates that there exists a right α-reversible ring which does not satisfy (*).

Example 1.1. Let $\mathbb{Z}$ be the ring of integers. Consider a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then R is right α-reversible, but not reversible by [1, Example 2.2]. Note that R does not satisfy (*): For $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$ with $A\alpha(B) = O$, we have $BA \neq O$.

Recall that an endomorphism α of a ring R is called *rigid* [8] if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and R is called an *α-rigid* ring [3] if there exists a rigid endomorphism α of R. Rigid endomorphisms of a ring are monomorphisms, and α-rigid rings are reduced rings by [3, Proposition 5].

Proposition 1.2. *Let α be an endomorphism of a ring R.*

(1) *A ring R satisfies (*) if and only if R is a right α-reversible ring and α is a monomorphism.*

(2) *R is an α-rigid ring if and only if R is a semiprime ring which satisfies (*).*

Proof. (1)Suppose that R satisfies (*). For $a, b \in R$ if $ab = 0$ then $\alpha(a)\alpha(b) = \alpha(ab) = 0$, and so $b\alpha(a) = 0$ by (*). Thus R is right α-reversible. Now, if $\alpha(a) = \alpha(b)$ then $\alpha(a - b) = 0$, and so $a = b$ by (*). Thus α is a monomorphism. Conversely, assume that R is right α-reversible with a monomorphism α. If $a\alpha(b) = 0$ for $a, b \in R$ then $\alpha(a)\alpha(b) = 0$ and hence, we get $ab = 0$ by the assumption, concluding that R satisfies (*).

(2)Let R be an α-rigid ring. Then R is a reduced ring (and so a semiprime ring) and α is a monomorphism by [3, p. 218]. Assume that $a\alpha(b) = 0$ for $a, b \in R$. Then $\alpha(ba)\alpha(\alpha(ba)) = \alpha(b)\alpha(a\alpha(b))\alpha(\alpha(a)) = 0$. Since R is α-rigid, $\alpha(ba) = 0$ and thus $ba = 0$. Therefore R satisfies (*). Conversely, assume that $a\alpha(a) = 0$ for $a \in R$. For any $r \in R$,

$0 = a\alpha(a)\alpha(r) = a\alpha(ar)$. Since R satisfies (*), we have $\alpha(ara) = 0$ for any $r \in R$, and so $aRa = 0$ by (1). Thus $a = 0$, since R is a semiprime ring. Therefore R is an α-rigid ring. □

COROLLARY 1.3. [7, Lemma 2.7] *A ring R is reduced if and only if R is a semiprime and reversible ring.*

Proof. It directly follows from Proposition 1.2(2), letting $\alpha = id_R$ where id_R denotes the identity endomorphism of a ring R. □

The following example shows that the conditions "R is a semiprime ring" and "R satisfies (*)" in Proposition 1.2(2) are not superfluous, respectively.

EXAMPLE 1.4. (1) Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}$. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. Then it can be easily checked that the ring R is neither semiprime nor α-rigid; while R is right α-reversible by [1, Example 2.7(i)]. Since α is an automorphism, the ring R satisfies (*), by Proposition 1.2(1).

(2) Let $R = F[x]$ be the polynomial ring over a field F. Define $\alpha : R \to R$ by $\alpha(f(x)) = f(0)$ where $f(x) \in R$. Then R is a commutative domain and so it is semiprime. Since α is not a monomorphism, R does not satisfy (*), by Proposition 1.2(1). Note that R is not α-rigid by [4, Example 5(2)].

2. Properties of strong α-reversible rings

Based on Proposition 1.2, we define the following.

DEFINITION 2.1. For an endomorphism α of a ring R, α is called *strong right* (resp., *left*) *reversible* if whenever $a\alpha(b) = 0$ (resp., $\alpha(a)b = 0$), we get $ba = 0$ for $a, b \in R$, and R is called a *strong right* (resp., *left*) *α-reversible* ring if there exists a strong right (resp., left) reversible endomorphism α. A ring R is called *strong α-reversible* if it is both strong left and right α-reversible.

The following results are a direct consequence of routine computations.

REMARK 2.2. (1) Let α be an endomorphism of a ring R. (i) Every subring S of a strong right (resp., left) α-reversible ring R with $\alpha(S) \subseteq S$ is strong right (resp., left) α-reversible. (ii) Every α-rigid ring is strong α-reversible and every strong right (resp., left) α-reversible is right (resp., left) α-reversible. (iii) Any strong α-reversible ring R is left-right symmetric whenever either $\alpha^2 = id_R$ or R is a reversible ring, where id_R denotes the identity endomorphism of R. (iv) R is a strong right (resp., left) α-reversible ring if and only if $ab = 0 \Leftrightarrow b\alpha(a) = 0$ (resp., $ab = 0 \Leftrightarrow \alpha(b)a = 0$) for $a, b \in R$.

(2) Let R_γ be a ring and α_γ an endomorphism of R_γ for each $\gamma \in \Gamma$. Then, for the product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ and the endomorphism $\bar{\alpha} : \prod_{\gamma\in\Gamma} R_\gamma \to \prod_{\gamma\in\Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma\in\Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma\in\Gamma}$, $\prod_{\gamma\in\Gamma} R_\gamma$ is strong right (resp., left) $\bar{\alpha}$-reversible if and only if each R_γ is strong right (resp., left) α_γ-reversible.

We have the basic equivalences for strong right (resp., left) α-reversible rings as follows. For a nonempty subset S of a ring R, $r_R(S)$ and $\ell_R(S)$ denote the *right* and *left annihilator* of S in R, respectively.

PROPOSITION 2.3. *For a ring R with an endomorphism α, the following statements are equivalent:*

(1) *R is strong right (resp., left) α-reversible.*

(2) *$\ell_R(\alpha(S)) = r_R(S)$ (resp., $r_R(\alpha(S)) = \ell_R(S)$) for each subset S of R.*

(3) *For each $a \in R$, $\ell_R(\alpha(a)) = r_R(a)$ (resp., $r_R(\alpha(a)) = \ell_R(a)$).*

(4) *For any two nonempty subsets A and B of R, $A\alpha(B) = 0$ (resp., $\alpha(A)B = 0$) if and only if $BA = 0$.*

Proof. (1)⇒(2) For $a \in R$ and $S \subseteq R$, $a \in \ell_R(\alpha(S))$ if and only if $a\alpha(S) = 0$ if and only if $Sa = 0$ by (1), if and only if $a \in r_R(S)$. (2)⇒(3) and (4)⇒(1) are straightforward. (3)⇒(4) Let A and B be nonempty subsets. Then $A\alpha(B) = 0$ if and only if $a\alpha(b) = 0$, for any $a \in A$ and $b \in B$ if and only if $a \in \ell_R(\alpha(b)) = r_R(b)$ if and only if $ba = 0$ by (3), if and only if $BA = \sum_{a\in A, b\in B} ba = 0$. The case of a strong left α-reversible ring can be proved similarly. □ □

A ring R is reversible if R is strong one-sided id_R-reversible. Any domain is strong α-reversible for a monomorphism α of R, but the converse does not hold by Example 1.4(1); while there is a commutative reduced ring which is not a strong right α-reversible ring by the next example.

Example 2.4. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider a ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a commutative reduced ring. Let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R. Note that R is not strong right α-reversible. For $a = (1, 0) = b \in R$, $a\alpha(b) = 0$ but $ba = (0, 1) \neq 0$.

Proposition 2.5. *For a reversible ring R with an endomorphism α, the following statements are equivalent:*

(1) *R is strong α-reversible.*

(2) *R is strong right α-reversible.*

(3) *If either $a\alpha^n(b) = 0$ or $\alpha^n(a)b = 0$ for a positive integer n and $a, b \in R$, then $ab = 0$. Conversely, $ab = 0$ for $a, b \in R$ implies $a\alpha^m(b) = 0$ and $\alpha^m(a)b = 0$ for any positive integer m.*

(4) *For each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$.*

Proof. (1)⇒(2), (3)⇒(1) and (3)⇒(4) are obvious. (2)⇒(3) Suppose that $a\alpha^n(b) = 0$ for a positive integer n and $a, b \in R$. Since R is reversible and strong right α-reversible, $a\alpha^n(b) = 0$ implies $0 = \alpha^{n-1}(b)a = a\alpha^{n-1}(b) = \ldots = ba = ab$. Similarly, $\alpha^n(a)b = 0$ yields $ab = 0$. The remainder is clear by hypothesis. (4)⇒(1) follows from definition. □

A ring R is called *Armendariz* [10] if whenever the product of any two polynomials in $R[x]$ over R is zero, then so is the product of any pair of coefficients from the two polynomials, where $R[x]$ denotes the polynomial ring with an indeterminate x over R. Every reduced ring is Armendariz. In [4], the Armendariz property of a ring was extended to one of the skew polynomial ring, which is a generalization of an α-rigid ring. For an endomorphism α of a ring R, the *skew polynomial* ring $R[x; \alpha]$ of R consists of the polynomial in x with coefficients in R written on the left, subject to the relation $xr = \alpha(r)x$ for all $r \in R$. A ring R is called *α-skew Armendariz* [4, Definition] if for $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x; \alpha]$, $p(x)q(x) = 0$ implies $a_i\alpha^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Any α-rigid ring is α-skew Armendariz.

According to [1, Theorem 2.9(i)], every reduced and right α-reversible ring is α-skew Armendariz, and thus reduced and strong right α-reversible rings are α-skew Armendariz. Moreover, every semiprime and strong right α-reversible ring is α-rigid by Proposition 1.2(2). Hence, we may ask whether R is an α-skew Armendariz ring when R is reversible and strong right α-reversible. However, the possibility is eliminated by Example 1.4(1): In fact, the ring R with the endomorphism α in Example

1.4(1) is a reversible and strong α-reversible ring (which is not a reduced ring). For $p(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} x \in R[x;\alpha]$, we have $p^2(x) = 0$, but $\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \neq O$, and thus R is not α-skew Armendariz.

Theorem 2.6. *Let α be an endomorphism of a ring R. Assume that R is an α-skew Armendariz ring. Then R is reversible and strong α-reversible if and only if the skew polynomial ring $R[x;\alpha]$ of R is reversible.*

Proof. Assume that R is a reversible and strong α-reversible ring. Let $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$. Since R is α-skew Armendariz, we get $a_i\alpha^i(b_j) = 0$ for any i and j. Then $b_j a_i = 0$ and so $b_j\alpha^j(a_i) = 0$ for any i and j, by Proposition 2.5. Hence, $R[x;\alpha]$ is reversible. Conversely, suppose that $R[x;\alpha]$ is a reversible ring. Then R is reversible as a subring of $R[x;\alpha]$. Suppose that $a\alpha(b) = 0$ for $a, b \in R$. Let $p(x) = ax$ and $q(x) = b$ in $R[x;\alpha]$. Then $p(x)q(x) = a\alpha(b)x = 0 \in R[x;\alpha]$. Since $R[x;\alpha]$ is reversible, we get $0 = q(x)p(x) = bax$, and so $ba = 0$. Thus R is strong right α-reversible, without the assumption that R is α-skew Armendariz. □

The next result is a direct consequence of Theorem 2.6.

Corollary 2.7. *If R is an Armendariz ring, then R is reversible if and only if $R[x]$ is reversible.*

3. Extensions of strong α-reversible rings

Lemma 3.1. *For a ring R with an endomorphism α, if R is strong right (resp., left) α-reversible, then $\alpha(1) = 1$ where 1 is the identity of R. In this case, $\alpha(e) = e$ for any $e^2 = e \in R$.*

Proof. $(1-\alpha(1))\alpha(1) = 0$ implies $\alpha(1) = 1$ since R is strong right α-reversible. Now, let $e^2 = e \in R$. Then $(1-e)e = 0$ and $e(1-e) = 0$. Since R is strong right α-reversible, $\alpha(1-e)\alpha(e) = 0$ and $\alpha(e)\alpha(1-e) = 0$ imply $e\alpha(1-e) = 0$ and $(1-e)\alpha(e) = 0$. Thus $\alpha(e) = e\alpha(e)$ and $e = e\alpha(e)$ since $\alpha(1) = 1$. Hence, $\alpha(e) = e$. □

We consider the Jordan extension and Dorroh extension of strong α-reversible rings. Recall that for a monomorphism α of a ring R an over-ring A of R is a *Jordan extension* of R if α can be extended to an automorphism of A and $A = \cup_{k=0}^{\infty}\alpha^{-k}(R)$. Jordan [6] showed, with the use of left localization of the Ore extension $R[x;\alpha]$ with respect to the set of powers of x, that for any pair (R,α), such an extension A always exists. On the other hand, for an algebra R over a commutative ring S, the *Dorroh extension* of R by S is the ring $D = R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

Proposition 3.2. *Let α be an endomorphism of a ring R.*

(1) *Assume that S is a ring and $\sigma : R \to S$ is a ring isomorphism. Then R is a strong right (resp., left) α-reversible ring if and only if S is a strong right (resp., left) $\sigma\alpha\sigma^{-1}$-reversible ring.*

(2) *Assume that e is a central idempotent of a ring R. Then eR and $(1 - e)R$ are strong right (resp., left) α-reversible if and only if R is strong right (resp., left) α-reversible.*

(3) *Assume that A is the corresponding Jordan extension of R. Then R is a strong right (resp., left) α-reversible ring if and only if A is a strong right (resp., left) α-reversible ring.*

(4) *Assume that S is a domain. Then R is a strong right (resp., left) α-reversible ring if and only if the Dorroh extension D of R by S is strong right (resp., left) $\bar{\alpha}$-reversible.*

Proof. (1) Let R be a strong right α-reversible ring and $a'\sigma\alpha\sigma^{-1}(b') = 0$ for $a', b' \in S$. Since σ is an isomorphism, $a' = \sigma(a)$ and $b' = \sigma(b)$ for some $a, b \in R$. Then $0 = \sigma(a)\sigma\alpha\sigma^{-1}(\sigma(b)) = \sigma(a\alpha(b))$, and so $a\alpha(b) = 0$ and $ba = 0$ since R is strong right α-reversible and σ is an isomorphism. Hence, $0 = \sigma(ba) = b'a'$ and therefore S is strong right $\sigma\alpha\sigma^{-1}$-reversible. Conversely, let $a\alpha(b) = 0$ for $a, b \in R$. Then $0 = \sigma(a\alpha(b)) = \sigma(a)\sigma\alpha\sigma^{-1}(\sigma(b))$, and thus $\sigma(b)\sigma(a) = 0$ since S is strong right $\sigma\alpha\sigma^{-1}$-reversible. Hence, $ba = 0$, entailing R is strong right α-reversible.

If R is a strong right α-reversible ring, so is any subring S with $\alpha(S) \subseteq S$ and $\alpha(e) = e$ for any $e^2 = e \in R$ by Lemma 3.1. Hence, it is enough for (2), (3) and (4) to show the necessity.

(2) Suppose that eR and $(1-e)R$ are strong right α-reversible. Let $a\alpha(b)=0$ for $a, b \in R$. Then $ea\alpha(eb)=0$ and $(1-e)a\alpha((1-e)b)=0$. By hypothesis, we get $0=ebea=eba$ and $0=(1-e)b(1-e)a=(1-e)ba$. Thus $ba=eba+(1-e)ba=0$, and therefore R is strong right α-reversible.

(3) Suppose that R is strong right α-reversible and $a\alpha(b)=0$ for $a, b \in A$. Then α is a monomorphism by Proposition 1.2(1). By the definition of A, there exists $k \geq 0$ such that $\alpha^k(a)$, $\alpha^k(b) \in R$. Then $\alpha^k(a)\alpha(\alpha^k(b)) = \alpha^k(a)\alpha^k(\alpha(b)) = \alpha^k(a\alpha(b)) = \alpha^k(0) = 0$. Since R is strong right α-reversible, we have $0 = \alpha^k(b)\alpha^k(a) = \alpha^k(ba)$ and so $ba = 0$, since α is a monomorphism. Therefore, A is a strong right α-reversible ring.

(4) Suppose that R is a strong right α-reversible ring.

Let $(r_1, s_1), (r_2, s_2) \in D$ with $(r_1, s_1)\bar{\alpha}(r_2, s_2) = 0$. Then $r_1\alpha(r_2) + s_1\alpha(r_2) + s_2r_1 = 0$ and $s_1s_2 = 0$. Since S is a domain, $s_1 = 0$ or $s_2 = 0$. If $s_1 = 0$, then $0 = r_1\alpha(r_2) + s_2r_1$ and so $0 = r_1(\alpha(r_2) + s_2) = r_1\alpha(r_2 + 1 \cdot s_2)$. Since R is strong right α-reversible, $(r_2+s_2)r_1 = 0$. This yields $(r_2, s_2)(r_1, s_1) = 0$. Similarly, let $s_2 = 0$. Then $(r_1 + s_1)\alpha(r_2) = 0$, and so $r_2(r_1 + s_1) = 0$, since R is strong right α-reversible. We obtain $(r_2, s_2)(r_1, s_1) = 0$, and thus the Dorroh extension D is strong right $\bar{\alpha}$-reversible. $\square$

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R, $\bar{\alpha} : T(R, R) \to T(R, R)$ defined by $\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$ is an endomorphism of $T(R, R)$. Since $T(R, 0)$ is isomorphic to R, we can identify the restriction of $\bar{\alpha}$ on $T(R, 0)$ to α.

Note that the trivial extension of a reduced ring is reversible by [7, Proposition 1.6], but we have the following.

PROPOSITION 3.3. *Let R be a reduced ring with an endomorphism α. Then R is strong α-reversible if and only if the trivial extension $T(R, R)$ is a strong $\bar{\alpha}$-reversible ring.*

Proof. It is sufficient to show that the trivial extension $T(R,R)$ of a reduced and strong α-reversible ring R is strong $\bar{\alpha}$-reversible. Assume that $A\alpha(B) = O$ for $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R,R)$. Then $a\alpha(c) = 0$ and $a\alpha(d) + b\alpha(c) = 0$. Since R is strong α-reversible, we get $ca = 0$ and $0 = c(a\alpha(d) + b\alpha(c)) = cb\alpha(c)$, and so $c^2b = 0$. Since R is reduced, $cb = 0$ and $bc = 0$, and so $b\alpha(c) = 0$ by Proposition 2.5. Then $a\alpha(d) = 0$ and $da = 0$. Consequently, we have $BA = O$ and therefore $T(R,R)$ is strong $\bar{\alpha}$-reversible. □

Corollary 3.4. *If R is a reduced ring, then $T(R,R)$ is a reversible ring.*

The condition "R is a reduced ring" in Proposition 3.3 cannot be dropped by the next example.

Example 3.5. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}$ be the strong right α-reversible ring in Example 1.4(1) with α defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. For

$$A = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

in $T(R,R)$, $A\bar{\alpha}(B) = O$ but $BA \neq O$. Thus $T(R,R)$ is not strong right $\bar{\alpha}$-reversible. It is obvious that R is not reduced.

For a ring R and $n \geq 3$, let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

Any endomorphism α of R can be extended to an endomorphism $\bar{\alpha}$ of $S_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. It is natural to ask whether $S_n(R)$ for $n \geq 3$ is strong $\bar{\alpha}$-reversible, when R is a reduced ring. However, the answer is negative by the following example.

Example 3.6. Let α be an endomorphism of an α-rigid ring R. Then $\alpha(e) = e$ for $e^2 = e \in R$ by [3, Proposition 5]. In particular $\alpha(1) = 1$. For $S_n(R)$ with $n \geq 3$, $\mathbf{E}_{ij}$ denotes the matrix of $S_n(R)$ whose (i, j) entry is 1 with all other entries 0. Let $A = \mathbf{E}_{23}$ and $B = \mathbf{E}_{12} \in S_n(R)$. Then $A\bar{\alpha}(B) = O$, but $BA \neq O$, concluding that $S_n(R)$ is not strong $\bar{\alpha}$-reversible for $n \geq 3$. Notice that the $n \times n$ matrix ring $Mat_n(R)$ over an α-rigid ring R needs not to be strong $\bar{\alpha}$-reversible by the same method as above.

For an endomorphism of α of a ring R, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall also denote the extended map $R[x] \to R[x]$ by α and the image of $f \in R[x]$ by $\alpha(f)$. The ring of *Laurent* polynomials in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$. For an endomorphism α of R, we define the map $R[x; x^{-1}] \to R[x; x^{-1}]$ by the same endomorphism as in the polynomial ring $R[x]$ above. The next theorem extends [7, Proposition 2.4].

Theorem 3.7. *For a ring R with an endomorphism α, the following are equivalent:*

(1) *$R[x]$ is a strong right (resp., left) α-reversible ring.*

(2) *$R[x; x^{-1}]$ is a strong right (resp., left) α-reversible ring.*

If R is an Armendariz ring, then each of (1) and (2) above is equivalent to

(3) *R is a strong right (resp., left) α-reversible ring.*

Proof. (2)$\Rightarrow$(1)$\Rightarrow$(3) are obvious as subrings. (1)$\Rightarrow$(2) Suppose that $R[x]$ is a strong right α-reversible ring. Let $f, g \in R[x; x^{-1}]$ with $f\alpha(g) = 0$. Then there exists a positive integer n such that $f_1 = fx^n, g_1 = gx^n \in R[x]$ with $f_1\alpha(g_1) = 0$. Since $R[x]$ is strong right α-reversible, we obtain $g_1 f_1 = 0$. Hence $gf = x^{-2n} g_1 f_1 = 0$. Thus $R[x; x^{-1}]$ is strong right α-reversible.

(3)⇒(1) Suppose that R is a strong right α-reversible ring. Let $f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f\alpha(g) = 0$. Then we get $a_i\alpha(b_j) = 0$ for all i and j, and so $b_j a_i = 0$, since R is Armendariz and strong right α-reversible. This yields $gf = 0$, and thus $R[x]$ is strong right α-reversible. □

COROLLARY 3.8. *Let R be an Armendariz ring. The following are equivalent:*

(1) *R is reversible.*

(2) *$R[x]$ is reversible.*

(3) *$R[x; x^{-1}]$ is reversible.*

Observe that the homomorphic image of a reversible ring is not reversible by [5, p.233]. For an ideal I of a ring R, if $\alpha(I) \subseteq I$ then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring R/I.

PROPOSITION 3.9. *Let α be an endomorphism of a ring R.*

(1) *Assume that I is an ideal of R with $\alpha(I) \subseteq I$. If the factor ring R/I is strong right (resp., left) $\bar{\alpha}$-reversible and I is α-rigid as a ring without identity, then R is strong right (resp., left) α-reversible.*

(2) *Assume that R is a reduced ring and n is any positive integer. R is a strong α-reversible ring if and only if $R[x]/\langle x^n \rangle$ is a strong $\bar{\alpha}$-reversible ring, where $\langle x^n \rangle$ is the ideal generated by x^n.*

Proof. (1) Let $a\alpha(b) = 0$ for $a, b \in R$. Since R/I is strong right $\bar{\alpha}$-reversible, we have $ba \in I$. Hence, $ba\alpha(ba) = ba\alpha(b)\alpha(a) = 0$ and so $ba = 0$ since I is α-rigid. Thus R is strong right α-reversible.

(2) It suffices to show the necessity. Assume that R is a reduced and strong α-reversible ring. Recall that R is a reduced ring if and only if $a^2b = 0$ implies $ab = 0$ for any $a, b \in R$, and every reduced ring is reversible. We freely use these facts in the following. Let $S = R[x]/\langle x^n \rangle$. If $n = 1$, then $S \cong R$. If $n = 2$, then S is strong $\bar{\alpha}$-reversible by Proposition 3.3, since $S \cong T(R, R)$. Now, we assume $n \geq 3$. Let $f = a_0 + a_1\bar{x} + \cdots + a_{n-1}\bar{x}^{n-1}, g = b_0 + b_1\bar{x} + \cdots + b_{n-1}\bar{x}^{n-1} \in S$ with $f\bar{\alpha}(g) = 0$, where $\bar{x} = x + \langle x^n \rangle$. We claim that $b_j a_i = 0$ for all i and j. If $i + j \geq n$, then $a_i\alpha(b_j)\bar{x}^{i+j} = 0$ from $f\bar{\alpha}(g) = 0$. We proceed by induction on $i + j \leq n - 1$. From $f\bar{\alpha}(g) = 0$, we obtain $a_0\alpha(b_0) = 0$ and so $b_0 a_0 = 0$, proving for $i + j = 0$. Now assume that our claim is true for $i + j \leq k - 1 (< n - 1)$. For $i + j = k \leq n - 1$, we have

$$a_0\alpha(b_k) + a_1\alpha(b_{k-1}) + \cdots + a_{k-1}\alpha(b_1) + a_k\alpha(b_0) = 0. \tag{1}$$

Multiplying Eq.(1) by b_0 on the left-hand side, $b_0 a_k \alpha(b_0)$ by induction hypothesis, and hence $b_0^2 a_k = 0$ and so $b_0 a_k = 0$ and $a_k \alpha(b_0) = 0$ by the assumption. Hence Eq.(1) becomes

$$a_0\alpha(b_k) + a_1\alpha(b_{k-1}) + \cdots + a_{k-1}\alpha(b_1) = 0. \tag{2}$$

Multiplying Eq.(2) by b_1 on the left-hand side, we have $b_1 a_{k-1}\alpha(b_1) = 0$, and so $b_1 a_{k-1} = 0$ and $a_{k-1}\alpha(b_1) = 0$. Continuing this process, we get $b_j a_i = 0$ for $i + j = k$. Consequently, $b_j a_i = 0$ for all i and j, and therefore $gf = 0$, concluding that S is strong right $\bar{\alpha}$-reversible. □

The following results in [7, Proposition 1.12 and Theorem 2.5] are obtained directly from Proposition 3.9.

Corollary 3.10. (1) *Suppose that R/I is a reversible ring for some ideal I of a ring R. If I is reduced, then R is reversible.*

(2) *If R is a reduced ring, then $R[x]/\langle x^n \rangle$ is a reversible ring for any positive integer n.*

The next example shows that the condition "I is α-rigid for an ideal I of a ring R" in Proposition 3.9(1) is not superfluous.

Example 3.11. Let F be a field and consider a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$ and an endomorphism α of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $A\alpha(B) = O$, but $BA \neq O$. Thus R is not strong right α-reversible. For an ideal $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, the factor ring R/I is strong right $\bar{\alpha}$-reversible, since $R/I = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I \mid a, c \in F \right\}$ is reduced and $\bar{\alpha}$ is an identity map on R/I. Clearly, I is not α-rigid as a ring.

For a ring R and $n \geq 2$, let

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, \ldots, a_n \in R \right\}.$$

Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha}$ of $V_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ is an endomorphism of $V_n(R)$. The next corollary which directly follows from Proposition 3.9(2) can be compared with Example 3.6.

Corollary 3.12. *Assume that R is a reduced ring with an endomorphism α. R is a strong α-reversible ring if and only if $V_n(R)$ is a strong $\bar{\alpha}$-reversible ring for any $n \geq 2$.*

Proof. Note that $V_n(R) \cong R[x]/\langle x^n \rangle$ by [9]. □

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