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HOMOTOPY TYPE OF A 2-CATEGORY

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Abstract. The classical group completion theorem states that under a certain condition the homology of ΩBM is computed by inverting $\pi_0 M$ in the homology of M. McDuff and Segal extended this theorem in terms of homology fibration. Recently, more general group completion theorem for simplicial spaces was developed. In this paper, we construct a symmetric monoidal 2-category $\mathcal{A}$. The 1-morphisms of $\mathcal{A}$ are generated by three atomic 2-dimensional CW-complexes and the set of 2-morphisms is given by the group of path components of the space of homotopy equivalences of 1-morphisms. The main part of the paper is to compute the homotopy type of the group completion of the classifying space of $\mathcal{A}$, which is shown to be homotopy equivalent to $\mathbb{Z} \times B\mathrm{Aut}_\infty^+$.

1. Introduction

A 2-category is a special case of double category in which all vertical arrows are identity morphisms. A 2-category consists of objects (0-morphisms), horizontal arrows (1-morphisms) and collapsed squares (2-morphisms) which are morphisms between two 1-morphisms. A Δ-category means a category enriched over simplicial sets. For a 2-category $\mathcal{C}$, let $\mathcal{BC}$ be the Δ-category obtained from $\mathcal{C}$ by taking the nerve of the categories of morphisms $\mathcal{C}(m,n)$. The classifying space $B\mathcal{C}$ of $\mathcal{C}$ is defined to be $B\mathcal{BC}$.

In chapter 3, we construct a new 2-category $\mathcal{A}$ which is a symmetric monoidal 2-category and whose classifying space has a homotopy type of

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an infinite loop space. The 1-morphisms of category $\mathcal{A}$ are generated by three atomic 2-dimensional CW-complexes and the set of 2-morphisms is given by the group of path components of the space of homotopy equivalences of 1-morphisms, i.e., $\pi_0\mathcal{H}(C, C'; \partial)$.

The main part of this article is to investigate the homotopy type of $\Omega B\mathcal{A}$. In this work the generalized group completion theorem for simplicial sets(cf. [8],[9],[10]) plays a key role. The classical group completion theorem states that under a certain condition the homology of ΩBM is computed by inverting $\pi_0 M$ in the homology of M(Theorem 2.1). McDuff and Segal extended this theorem in terms of homology fibration(Theorem 2.2). The main result of this article is the following:

Theorem 3.4. *$\Omega B\mathcal{A} \simeq \mathbb{Z} \times B\mathrm{Aut}_\infty^+$, where $\mathrm{Aut}_\infty = \varinjlim \mathrm{Aut}(F_n)$.*

In the proof of this result the automorphism group of free group with boundary also plays a key role. Let $G_{n,k}$ be the graph of a wedge of $n+k$ circles. Let $A_{n,k}$ be the group of components of homotopy equivalences of $G_{n,k}$ fixing k circles pointwise. $A_{n,k}$ is the automorphism group of free group on n generators with boundary. Recently, Galatius ([1]) proved that

$$\mathbb{Z} \times B\mathrm{Aut}_\infty^+ \simeq \Omega^\infty S^\infty$$

where $\Omega^\infty S^\infty := \lim_{n\to\infty} \Omega^n S^n$.

2. The group completion theorem

The group completion theorem for a topological monoid M was originally stated ([3],[4],[5],[6],[7]) in terms of the relation between homology of M and that of ΩBM; under a certain condition, the homology of the group completion ΩBM of M can be computed by inverting $\pi_0 M$ in the homology of M.

Let $\pi = \pi_0 M$. Regard π as a multiplicative subset of the Pontrjagin ring $H_*(M)$. The map $M \to \Omega BM$ induces a homomorphism of Pontrjagin rings, because $\pi_0 \Omega BM$ is a group. The image of π in $H_*(\Omega BM)$ consists of units.

Theorem 2.1. ([7], Prop. 1)

If π is in the center of $H_*(M)$, then

$$H_*(M)[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

McDuff and May extended this theorem in terms of homology fibration. A map $p : E \to B$ is called a *homology fibration* if for each $b \in B$, the natural map $p^{-1}(b) \to F(p, b)$ is a homology equivalence, where $F(p, b)$ denotes the homotopy fiber at b.

Suppose that a topological monoid M acts on a space X. Let $E_M X$ denote the Borel construction $EM \times_M X$ which is actually the classifying space of the topological category whose space of objects is X and whose space of morphisms is $M \times X$.

THEOREM 2.2. ([7], Prop. 2)
Let M be a topological monoid acting on a space X by homology equivalences. Then the canonical map $p : E_M X \to X$ is a homology fibration with fiber X.

Let M be a homotopy commutative topological monoid with $\pi_0 M \cong \mathbb{N}$. Choose a point m in the 0-component of M, then let M_∞ be the telescope $M_\infty = (M \xrightarrow{m} M \xrightarrow{m} M \to \cdots)$ which has the action of M by left multiplication.

COROLLARY 2.3. *There is a homology equivalence $M_\infty \to \Omega BM$.*

We now are going to introduce more general version of the group completion theorem for simplicial sets (cf.[10]). From now on in this chapter a space means a simplicial set. Let Δ^n be the standard n-simplex.

DEFINITION 2.4. Let $p : E \to B$ and σ be an n-simplex in B. Let $E_p(\sigma)$ be the pullback:

$$\begin{array}{ccc} E_p(\sigma) & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow{\sigma} & B \end{array}$$

Then $E_p : \Delta{\downarrow}B \to \mathbf{Top}$ is a functor from the simplex category $\Delta{\downarrow}B$ to the category of spaces.

A map $p : E \to B$ is a *homology fibration* if the natural map $E_p(\sigma) \to F_b(\sigma)$ to the homotopy fiber of p over σ is a homology equivalence.

Let $\mathcal{M}$ be a simplicial category with constant objects. For two objects i, j of $\mathcal{M}$, $\mathcal{M}(i,j)$ is regarded as a space (= simplicial set) rather than a set. A morphism from i to j is regarded as an n-simplex. Taking now the nerve of this simplicial category degree by degree yields a simplicial space $B\mathcal{M}_*$, the *simplical classifying space.*

DEFINITION 2.5. An $\mathcal{M}$-*diagram* is a functor $X : \mathcal{M}^{op} \to \mathbf{Top}$ such that for each pair of objects i, j, there is a natural map (action)

$$\mu_{ij} : \mathcal{M}(i,j) \times X(j) \to X(i)$$

which satisfies the usual associativity and identity conditions.

A typical example of an $\mathcal{M}$-diagram is $\mathcal{M}$ itself :

$$\mathcal{M}(i) = \bigcup_{j \in \mathrm{Obj}(\mathcal{M})} \mathcal{M}(i,j)$$

$\mu_{ij} : \mathcal{M}(i,j) \times \mathcal{M}(j) \to \mathcal{M}(i)$ is defined by composition.

Fix $k \in \mathrm{Obj}(\mathcal{M})$ and define a subdiagram $\mathcal{M}_k$

$$\mathcal{M}_k(i) := \mathcal{M}(i,k)$$

and the action μ_{ij} is also defined by composition.

DEFINITION 2.6. The *bisimplicial Borel construction* of an $\mathcal{M}$-diagram $X : \mathcal{M}^{op} \to \mathbf{Top}$ is the simplicial space $E_{\mathcal{M}}X_*$ whose space of n-simplices is the disjoint union over all n-tuples of objects in $\mathcal{M}$

$$\coprod_{i_0, \cdots, i_n} \mathcal{M}(i_n, i_{n-1}) \times \cdots \times \mathcal{M}(i_1, i_0) \times X(i_0).$$

The degeneracy maps are inclusions. The face map $d_n : E_{\mathcal{M}}X_n \to E_{\mathcal{M}}X_{n-1}$ is the projection on the last n factors. $d_0 = 1 \times \mu_{i_1, i_0}$, and the other d_k's are defined by composition $\mathcal{M}(i_{k+1}, i_k) \times \mathcal{M}(i_k, i_{k-1}) \to \mathcal{M}(i_{k+1}, i_{k-1})$.

Let T be the trivial diagram, i.e., $T(i) = \{i\}$ and any morphism $i \to j$ in $\mathcal{M}$ induces a unique map $\{j\} \to \{i\}$. Then we have

$$E_{\mathcal{M}}T_* = B\mathcal{M}_*.$$

For a diagram $X : \mathcal{M}^{op} \to \mathbf{Top}$, there is a collapse natural transformation $\pi : X \to T$, hence we have a map of simplicial spaces

$$E_{\mathcal{M}}\pi_* : E_{\mathcal{M}}X_* \to B\mathcal{M}_*.$$

Note that the preimage of $\{i\}$ in bisimplicial Borel construction is $F(i)$. Here is the generalized group completion theorem ([10]).

Theorem 2.7. *Let $\mathcal{M}$ be a simplicial category and $X : \mathcal{M}^{op} \to \mathbf{Top}$ be an $\mathcal{M}$-diagram. Assume that any morphism $f : i \to j$ induces an isomorphism $H_*(X(j);\mathbb{Z}) \to H_*(X(i);\mathbb{Z})$. Then for each object i in $\mathcal{M}$, the map $X(i) \to \mathrm{Fib}_i(\pi_{\mathcal{M}})$ to the homotopy fiber of $\pi_{\mathcal{M}}$ over i is a homology equivalence.*

In this theorem the map $\pi_{\mathcal{M}} : E_{\mathcal{M}}X \to B\mathcal{M}$ is induced by $\pi : X \to T$. Recall the subdiagram $\mathcal{M}_k$ for $k \in \mathrm{Obj}(\mathcal{M})$:

$$\mathcal{M}_k(i) := \mathcal{M}(i,k).$$

Lemma 2.8. *$E_{\mathcal{M}}\mathcal{M}_k$ is contractible.*

Proof. This is obviously true if $\mathcal{M}$ is a monoid, i.e., has only one object, because we have $\mathcal{M}_k = \mathcal{M}$ and $E_{\mathcal{M}}\mathcal{M} \simeq *$.

Suppose first that $\mathcal{M}$ is a constant simplicial category, i.e. an ordinary category. Then $\mathcal{M}(i,j)$ is just a hom set. Let $k/\mathcal{M}$ be the category whose objects are (i,v), where $i \in \mathrm{Obj}(\mathcal{M})$ and $v \in \mathcal{M}(i,k)$ and a morphism $(i,v) \to (i',v')$ is a morphism $\omega : i \to i'$ such that $v'_0\omega = v$. Then $k/\mathcal{M}$ has a final object $(k, 1_k)$, hence $B(k/\mathcal{M}) \simeq *$. Note that

$$E_{\mathcal{M}}\mathcal{M}_k = B(k/\mathcal{M}).$$

The simplicial case follows by the fact that for a map $f : X_n \to X'_n$ is a weak equivalence for each n, then so is f. □

3. A monoidal 2-category

Definition 3.1. A *2-category* consists of objects (0-morphisms or 0-cells), horizontal arrows between objects (1-morphisms or 1-cells) and morphisms between morphisms (2-morphisms or 2-cells). Each hom set in a 2-category carries a structure of a category. It is a category enriched over **Cat** (the category of categories).

For a 2-category $\mathcal{C}$, let A, B be objects on $\mathcal{C}$. Then $\mathcal{C}(A,B)$ form a category. Between two objects $f : A \to B$, $g : A \to B$ in this category, we have a morphism (2-morphism) $\alpha : f \Rightarrow g$. There is a functor $\circ_h : \mathcal{C}(B,C) \times \mathcal{C}(A,B) \to \mathcal{C}(A,C)$, called horizontal composition. There is also vertical composition, denoted by $\circ_v$.

For composable 2-morphisms $\alpha, \beta, \gamma, \delta$, we have an interchange law:

$$(\alpha \circ_h \beta) \circ_v (\gamma \circ_h \delta) = (\alpha \circ_v \gamma) \circ_h (\beta \circ_v \delta).$$

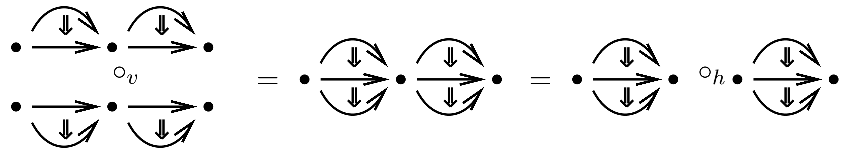

FIGURE 1.

We define a 2-category $\mathcal{A}$ modifying Tillmann's surface category. The category $\mathcal{A}$ is a categorical extension of the concept of the automorphism groups of free groups. It has both monoidal and symmetric structure, hence it gives rise to an infinite loop space through nerve constructions.

The construction of $\mathcal{A}$.

The objects of $\mathcal{A}$ are nonnegative integers. An object $n \in \mathbb{N}$ may be regarded as a disjoint union of n circles.

A morphism from m to n is a 2-dimensional CW-complex (with a coloring on the boundary) generated by gluing the following three types of atomic 2-dimensional CW-complexes:

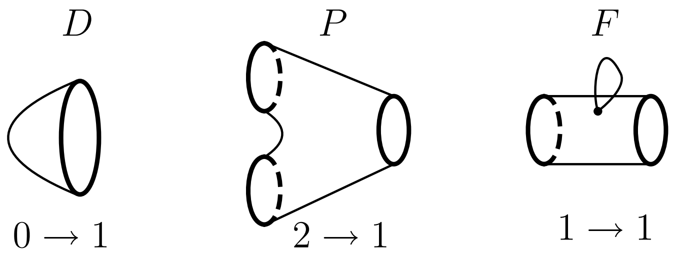

FIGURE 2.

The disk D is thought as a 1-morphism from 0 to 1. The pair of pants is thought as a morphism from 2 to 1. an 1-morphism $F : 1 \to 1$ is obtained by attaching a loop on a cylindrical surface. 1-morphisms are generated by these three atomic CW-complexes by gluing along incoming and outgoing circles and disjoint unions. we also give an ordering (called coloring) on both incoming and outgoing boundaries of surfaces, respectively. The following is an example of a 1-morphism.

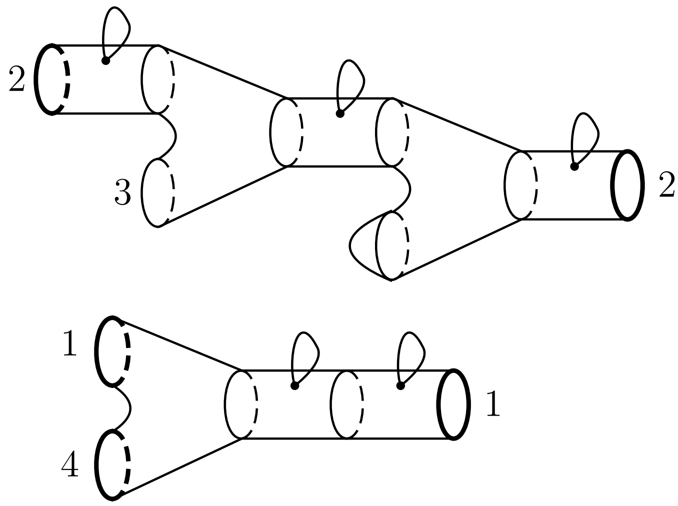

FIGURE 3. A 1-morphism from 4 to 2.

A circle may be thought as a morphism from 1 to 1 with a coloring on both sides. Note that a 1-morphism $m \to n$ has exactly n connected components.

Let $\mathcal{A}(m,n)$ be the category of morphisms in $\mathcal{A}$. Let C and C' be 1-morphisms from m to n, i.e., objects in $\mathcal{A}(m,n)$, then the set of 2-morphisms from C to C' is given by $\pi_0\mathcal{H}(C,C';\partial)$, where $\mathcal{H}(C,C';\partial)$ means the space of homotopy equivalences from C to C' which fix the boundary pointwise. That is, there is a 2-morphism from C to C' only when C and C' have the same homotopy type. $\pi_0\mathcal{H}(C,C';\partial)$ is the group of path components of the space $\mathcal{H}(C,C';\partial)$. It is known that two homotopy equivalences $f,g \in \mathcal{H}(C,C';\partial)$ are in the same path component if and only if they are homotopic relative to the boundary.

DEFINITION 3.2. Let $G_{n,k}$ be the graph which is a wedge of $n+k$ circles. Let $A_{n,k}$ be the group of components of homotopy equivalences of $G_{n,k}$ fixing k circles pointwise.

A 1-morphism C from k to 1 in $\mathcal{A}$ is a connected 2-dimensional CW-complex which is homotopy equivalent to $G_{n,k}$ for some n. Therefore, the group of components of homotopy equivalences of C fixing the boundary, denoted by $\pi_0\mathcal{H}(C;\partial)$ is isomorphic to $A_{n,k}$.

Note that the disjoint union $C_1 \coprod C_2$ induces a monoidal structure on $\mathcal{A}$. Moreover, the symmetries are given by the block transposition $m+n \to n+m$ in $\mathcal{A}(m+n, n+m)$.

For a 2-category $\mathcal{C}$, we may apply the nerve construction to the morphism categories. Since the nerve construction commutes with products, this gives a category $\mathcal{BC}$ enriched over simplicial sets, the category of simplicial sets. In other words, $\mathcal{BC}$ is a simplicial category with constant objects. Applying the nerve construction yields a bisimplicial set. The realization of this bisimplicial set is the classifying space $B\mathcal{C}$ of $\mathcal{C}$ that is,

$$B\mathcal{BC} = B\mathcal{C}.$$

LEMMA 3.3. *$\mathcal{A}$ is a monoidal symmetric 2-category. Therefore, $B\mathcal{A}$ has the homotopy type of an infinite loop space.*

We are now going to determine the homotopy type of (the group completion of) the classifying space $B\mathcal{A}$.

THEOREM 3.4. $\Omega B\mathcal{A} \simeq \mathbb{Z} \times B\mathrm{Aut}_\infty^+$, *where* $\mathrm{Aut}_\infty = \varinjlim \mathrm{Aut}(F_n)$.

Proof. Let $\mathcal{BA} = \mathcal{M}$. Then $\mathcal{M}_1$ is an $\mathcal{M}$-diagram with $\mathcal{M}_1(k) := \mathcal{M}(k,1)$ and $\mathcal{M}_\infty$ is also an $\mathcal{M}$-diagram defined by gluing $F \in \mathcal{A}(1,1)$ on the right:

$$\mathcal{M}_\infty(k) := \mathrm{hocolim}_F \mathcal{M}(k,1).$$

That is, $\mathcal{M}_\infty(k)$ is the telescope $(\mathcal{M}(k,1) \xrightarrow{F} \mathcal{M}(k,1) \xrightarrow{F} \cdots)$. Since $\mathcal{M}(k,1)$ is homotopy equivalent to $\bigcup_{n\geq 0} BA_{n,k}$, there is a homotopy equivalence

$$\mathcal{M}_\infty(k) \simeq \mathbb{Z} \times BA_{\infty,k}.$$

The vertices of $B\mathcal{M}$ act on $\mathcal{M}_\infty$ by homology isomorphisms ([2]). Hence we can apply the generalized group completion theorem to the diagram

$$\begin{array}{ccc} \mathcal{M}_\infty(k) & \longrightarrow & E_{\mathcal{M}}\mathcal{M}_\infty \\ \downarrow & & \downarrow \\ k & \longrightarrow & B\mathcal{M} \end{array}$$

Note that $E_{\mathcal{M}}\mathcal{M}_\infty$ is contractible, because for each k, $E_{\mathcal{M}}\mathcal{M}_k$ is contractible (Lemma 2.8) and $E_{\mathcal{M}}\mathcal{M}_\infty$ is homotopy equivalent to the telescope $(E_{\mathcal{M}}\mathcal{M}_1 \to E_{\mathcal{M}}\mathcal{M}_1 \to \cdots)$. Hence we have a homology equivalence

$$\Omega B\mathcal{A} \to \mathcal{M}_\infty(0) \simeq \mathbb{Z} \times B\mathrm{Aut}_\infty.$$

By the Whitehead theorem for simple spaces, this gives a homotopy equivalence after plus construction, that is, we have

$$\Omega B\mathcal{A} \simeq \mathbb{Z} \times B\mathrm{Aut}_\infty^+.$$

□

References

[1] S, Galatius, *Stable homology of automorphism groups of free groups*, to appear in Ann. of Math.

[2] A. Hatcher and N. Wahl, *Stabilization for the automorphisms of free groups with boundaries*, Geom. Topol. **9**(2005), 1295–1336.

[3] C. Jensen and N. Wahl, *Automorphisms of free groups with boundaries*, Algebr. Geom. Topol. **4**(2004), 543–569.

[4] J. P. May, *E_∞-Spaces, group completions and permutative categories*, London Math. Soc. Lecture Note Ser. **11**(1974), 61–95.

[5] J. P. May, *Classifying space and fibrations*, Mem. Amer. Math. Soc. **1**(1,155):xiii+98, 1975.

[6] D. McDuff, *Configuration spaces of positive and negative particles*, Topology **14**(1975), 91–107.

[7] D. McDuff and G. Segal, *Homology fibrations and the group completion theorem*, Invent. Math. **31**(1976), 279–284.

[8] W. Pitsch and J. Scherer, *Homology fibrations and group completion, revisited*, Homology, Homotopy Appl. **6**(1) (2004), 153–166.

[9] Y. Song and U. Tillmann, *Braid groups, mapping class groups and categorical delooping*, Math. Ann. (2007), 377–393.

[10] U. Tillmann, *On the homotopy of the stable mapping class groups*, Invent. Math. **130**(1997), 257–275.

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