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Students Approaches in Constructing Convincing Arguments in Geometry Using Technology: A Case Study[1]

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Mathematically, a proof is to create a convincing argument through logical reasoning towards a given proposition or a given statement. Mathematics educators have been working diligently to create environments that will assist students to perform proofs. One of such environments is the use of dynamic-geometry-software in the classroom. This paper reports on a case study and intends to probe into students' own thinking, patterns they used in completing certain tasks, and the extent to which they have utilized technology. Their tasks were to explore the shape-to-shape, shape-to-part, and part-to-part interrelationships of geometric objects when dealing with certain geometric problem-solving situations utilizing dissection-motion-operation (DMO).

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INTRODUCTION AND BACKGROUND

Reasoning and constructing convincing arguments must take place in every mathematics classroom every day. Through such an environment, students ask and answer such questions as "What's going on here!" and "Why do you think that!" Reasoning and constructing convincing arguments do not have to be an extra burden for teachers being

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under pressure dealing with students who are having a difficult time just to learn the procedures. On the contrary, the construction the reasoning brings forms a fundamental support for understanding and continued learning. Nowadays, many students have difficulty because they find mathematics meaningless. Without the connections that reasoning and convincing arguments provide, a seemingly endless cycle of repeated teaching may occur. With close attention and planning, teachers can hold all students in their mathematics classrooms accountable for personally engaging in reasoning and constructing convincing arguments. As a result, this will lead students to actively experience reasoning and constructing convincing arguments themselves rather than being merely observers. Further, technology has to be used purposely in the mathematics classrooms, particularly in the high school, to help accomplishing this goal (NCTM, 2008, p. 5).

The literature indicates that other research studies were more focused at a certain type of reasoning, namely, visual reasoning. Consequently, visual reasoning has become a prime focus in research (Healy & Hoyles, 1999; Liang & Sedig, 2010; Metaxas & Karagiannidou, 2010; Pettersson, 1989; Rahim & Siddo, 2009; Sedig & Liang, 2006; Spence, 2007; Zimmerman & Cunningham, 1991). Kossyn (1996), described visualization as the creation of a mental image of a given concept. Rahim and Siddo (2009) stated that from the teaching point of view, visualization appears to be a powerful method to utilize for enhancing students' understanding of a variety of concepts in many disciplines such as computer science, chemistry, physics, biology, engineering, applied statistics and mathematics. Further, there are many reasons that substantiate the use of visualization for learning and teaching of mathematics at all levels of schooling, from elementary to university passing through the middle and high school levels (p. 496). The literature also indicates that the activity of 'seeing' differently is not a self-evident, innate process, but something created and learned (Hoffmann, 1998; Whiteley, 2000).

The main purpose of the study was to investigate the students' approaches when dealing with problems on shape-to-shape transforms. The investigation will proceed in analyzing a series of case studies through scrutinizing the approaches adopted by the students working through parallel tasks sequences which integrate concrete manipulation and Geometer's Sketchpad (GSP) software. Finally, we formulate reflections as to how students might be encouraged to use visual reasoning both manipulatively and through the technology.

DISSECTION-MOTION-OPERATIONS (DMO) FRAMEWORK

There are three components that constitute the DMO operation:

(1) Dissection Theory (Eves, 1972, pp. 194–239), through which a polygonal region is dissected into a finite number of smaller certain sub-regions (dissection is an 'inside of' or 'intra' operation), and in which there can only be perpendicular, parallel or oblique type of dissection with respect to a selected side in a given shape;

(2) Motion (translation, rotation, and reflection) through which one or more of the sub-regions are moved to another location without overlapping (motion is an 'among of' or 'inter' operation); and

(3) Recursion, through which the two previous operations or one of them may be repeated in the process of creating another certain shape. Together, these three components constitute the operation called Dissection-Motion-Operation. Under such operation, a polygonal region can be transformed into another polygonal region of equivalent area. Note that DMO is a relation that can take a single polygonal region to several distinct polygonal regions, all the while, the area remains unchanged throughout. This can be accomplished without the specific use of numbers (Rahim, in press).

Within the dynamic geometry software (DGS) environment such as the Geometer's Sketchpad or Cabri, problems of shape-to-shape transforms can easily be handled applying DMO. That is, decomposition (dissection) of an object in 2D (or 3D) into a few parts and composition (motion) the resultant parts to form a distinct new object of same area (or volume) will not be difficult to perform by students through GSP. For example, clicking, dragging, drawing line segments, circles and the like will constitute a rich environment for exploring, constructing and visualizing mathematical 2D or 3D objects.

METHODOLOGY

In this classroom-base research, three tasks were used to investigate students' responses on certain visual tasks. First task was made on the basis of using hands-on manipulation method on concrete models and through the application of the Geometer's Sketchpad software in response to each task. These tasks were as follows:

Task 1: Shape-to- shape transforms using hands-on manipulation, the case of the right angle triangle;

Task 2: GSP construction of certain type of triangles; and

Task 3: Construction of a set of three quadrilaterals (rectangle, parallelogram and trapezoid) and a set of other three polygons (pentagon, hexagon and heptagon) through the use of GSP.

Description of the Tasks

The main purpose of each of the three tasks was to investigate the type of approaches the students would utilize throughout their attempts completing each task.

Task 1: A concrete paper model representing a right triangular region was provided and the student was required to dissect the model into two pieces (one cut) as the student may wish and construct as many distinct shapes of equal area (no overlapping) as possible provided that each of the constructed shape has to be a simple polygon (simple polygonal region means it has no vertex at which more than two external sides meet). This task is totally hands-on manipulative.

Task 2: The student was required to construct a visual representation of each type of the triangles when classified with respect to (a) their sides and (b) their angles. The student was required to use GSP.

Task 3: This task has two folds:

(a) In a computer lab session the student were given a set of quadrilaterals. They were instructed to
  (i) construct and
  (ii) verify each shape of the quadrilaterals using GSP; the shapes were a rectangle, a parallelogram and a trapezoid.

(b) A second set of polygons was given to the student and the students were again asked to
  (i) construct and
  (ii) verify each shape. The shapes were a pentagon, a hexagon and a heptagon using GSP software. The students were instructed to take this task home if the time of the lab wouldn't be sufficient.

RESULTS

Through a series of case studies, we investigated and compared the strategies adopted by the students working through parallel problem-solving situations. Our prime focus has been to identify students' conceptions, strategies and approaches in dealing with the tasks.

Task 1

The collected data indicates that

(a) There have been three types of dissections that the students have used, namely,

(1) Parallel dissection,
(2) Perpendicular dissection, and
(3) Oblique dissection.

Note that the three dissections above have been categorized with respect to a particular side of the right angle triangle. One may choose the hypotenuse or any other side as a reference and make the dissection accordingly.

The data suggests that these dissections performed by the students were encouraging and surprisingly correspond to the three dissections proposed by Eves' Dissection Theory described in his text, A Survey of Geometry, Revised Edition (1972, pp.194–210).

Figure 1 below shows a representation of some dissections emerged through this task. It is useful to note here that the dissections below were made with respect to the hypotenuse of the right angle triangle.

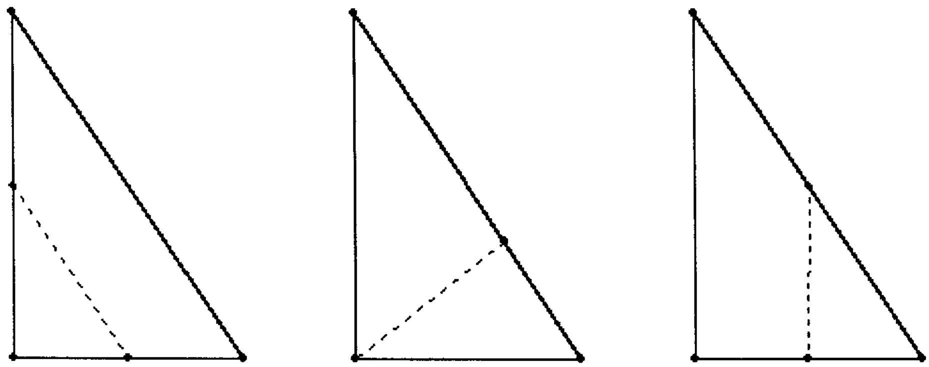


Figure 1. Dissections with respect to the hypotenuse

(b) The constructed shapes in this task formulate a class of simple polygons. However, there were errors in some of the constructed polygons as they do not meet the description of a simple polygon as described above. For example below is a shape with such misconceptions where the resultant shape is a complex polygon because one of the vertices, A, below has more than 2 outer sides passing through it.

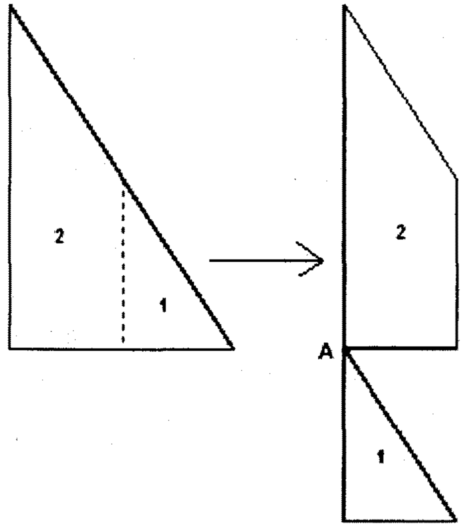


Figure 2. Pieces 1 & 2 were put together to form a non-simple polygon

Task 2

The student's strategies in dealing with this task comprised of the following:
(a) Construction of a scalene triangle through GSP:

This was primarily based on the use of the line segment tool and the length measure from the measure menu. The return responses for drawing the scalene triangle through GSP were initially made through the use of trial and error process with the use of line segment tool and measure menu. The following quote tells it all; it was made by one of our pre-service teachers,

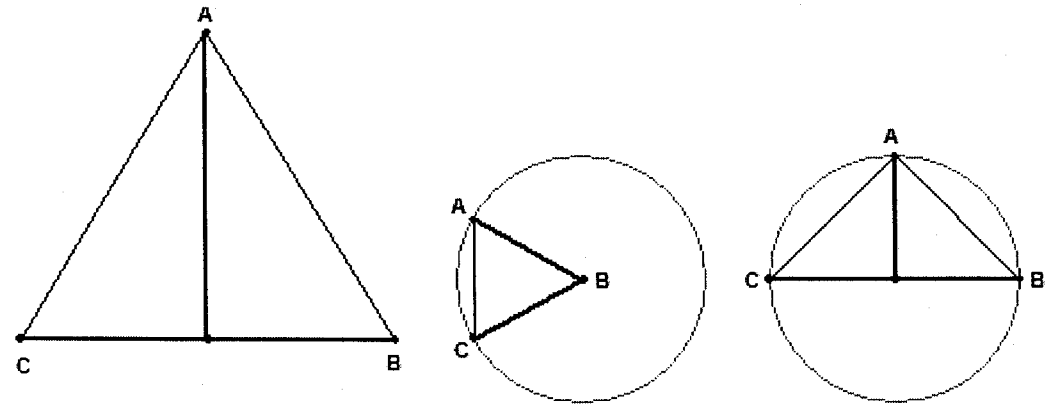
> "Step 1: Last time we'll have to draw a horizontal line, I promise. Actually, this one doesn't even need to be horizontal if you are tired of confirming...
>
> Step 2: Use the line segment tool to draw another segment of any length and angle, go crazy, express yourself, let all of the pent up frustration of adolescent North America life out on GSP. Don't worry, it can take it. It should probably be attached to the initial segment though ....
>
> Step 3: Connect the two line segments together at the free endpoints to form a triangle. Now, unless you have a supernatural gift for eyeballing perfect length and angles (or just couldn't get used to using the line segment tool without the shift key) you should have a triangle with three different side and angles. Measure the sides to make sure, and you're done. Sweet.

(b) Construction of an isosceles triangle through GSP:

The students' strategies reveal that there were three distinct approaches that dominate the construction of an isosceles triangle. They were,

(1) Drawing a perpendicular bisector for a line segment,
(2) Using a single circle and two radii, and
(3) Using a circle; drawing horizontal diameter; and drawing perpendicular radius at the center.

The most popular approach in this task was drawing a perpendicular bisector for a line segment. Line segment, circle tool and construct menu were used frequently throughout the construction. Figure 3 below shows representations of the three strategies identified above.



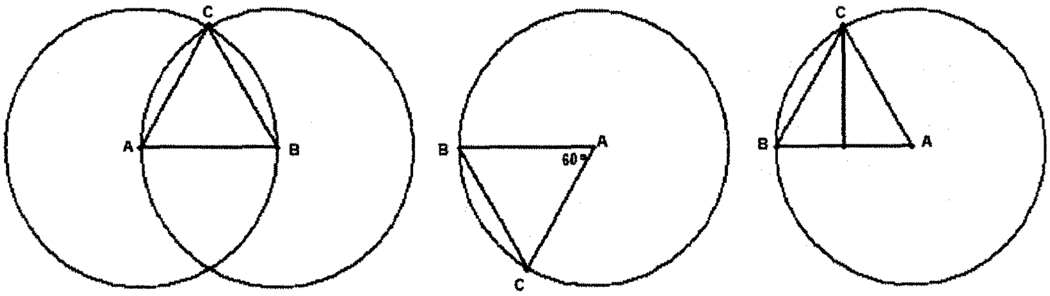
(a) Perpendicular bisector **(b) Two radii** **(c) Horizontal diameter and perpendicular radius**

Figure 3. Construction of an isosceles triangle

(a) Construction of an equilateral triangle though GSP:

The data suggests that there were three distinct approaches in drawing an equilateral triangle through GSP (Figure 4). They were,

(1) Two circle approach,
(2) Transformation approach, and
(3) Perpendicular bisector approach.



(a) Two circles **(b) Transformation** **(c) Perpendicular bisector**

Figure 4. Three approaches of constructing equilateral triangle

The two circles approach was most popular than the others in constructing equilateral triangle.

Task 3

We analyzed the students' strategies using GSP by starting from the position that mathematical meanings are developed by establishing connections between different ways of experiencing and expressing the same mathematical concepts.

The student's strategies while working on this task comprised of,

(a) Constructing a set of quadrilaterals: rectangle, parallelogram and trapezoid.

Students' emerging strategies for constructing these shapes are described as follows:

(1) Using the line segment tool; drawing parallel lines, perpendicular lines, intersection points using the construct menu (Figure 5),

(2) Measuring angles and lengths through the measure menu (Figure 6), and

(3) Applying translation, rotation and or reflection (Figure 7).

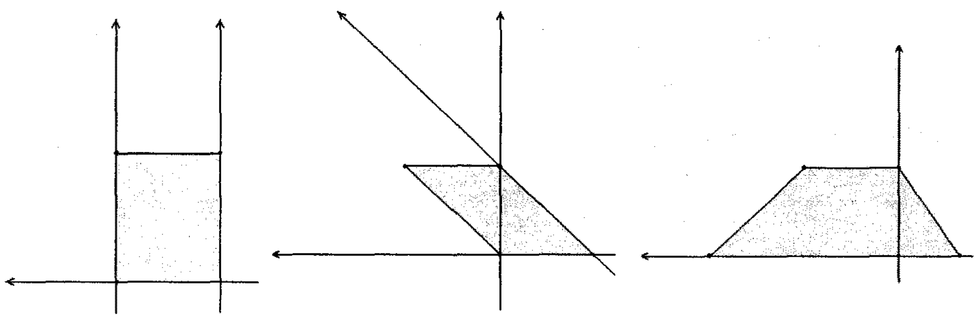


Figure 5. Using intersect points, line segments, parallel & perpendicular lines

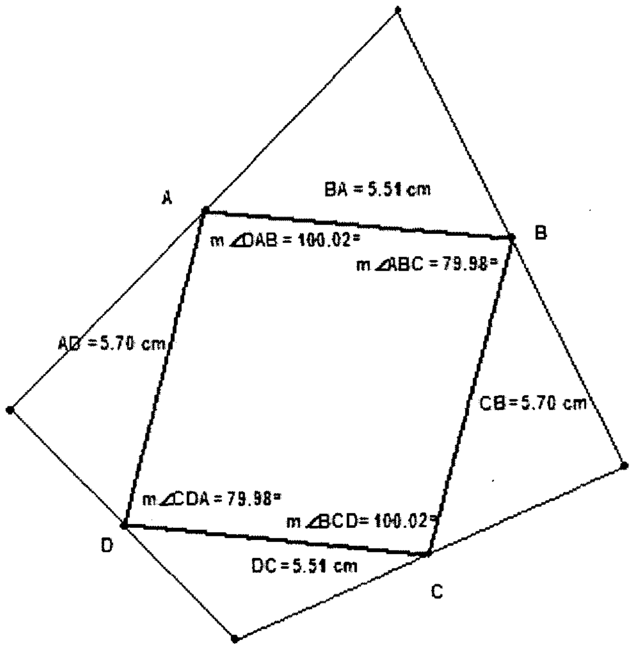


Figure 6. Using midpoints, line segments and measures

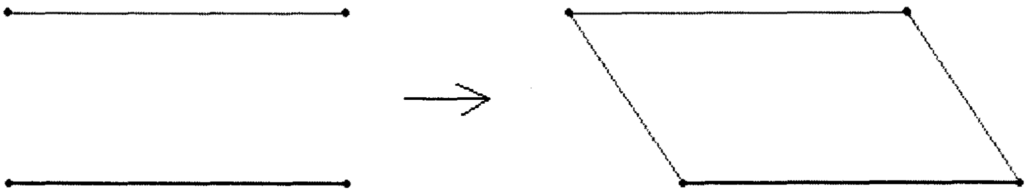


Figure 7. Using slide, line segments, parallel lines

(b) Constructing a set of polygons: pentagon, hexagon and heptagon.

In this part of the investigation, we observed some interesting and elegant students' strategies for constructing these polygons. For example, the pentagon was produced in the following attractive way of composing identical squares as shown in Figure 8 below.

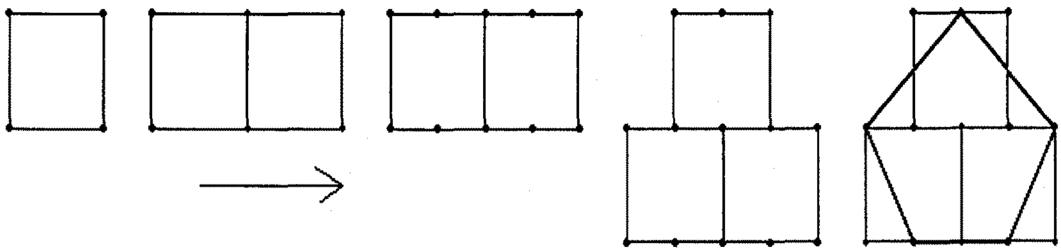


Figure 8. Constructing a pentagon through certain composition of three congruent squares

As well, a hexagon was produced in another interesting manner through the use of three congruent circles. In particular, Figure 9 below demonstrates this strategy.

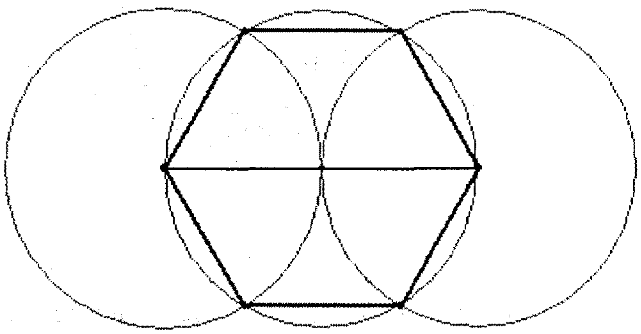
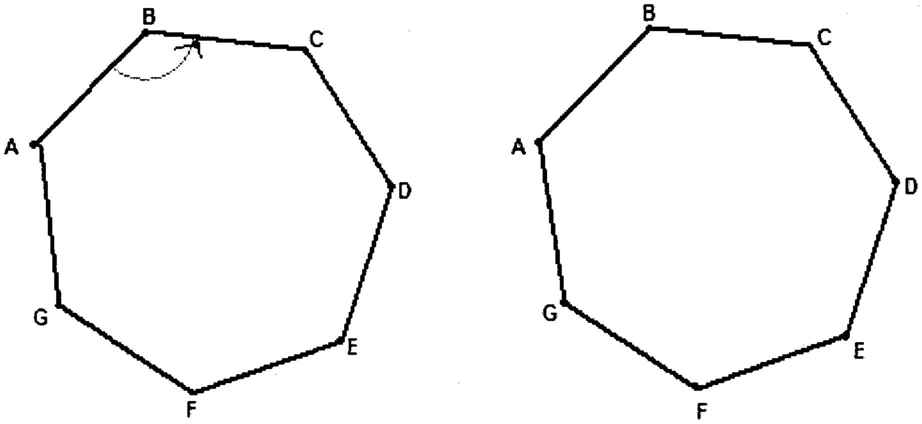


Figure 9.

Finally, a strategy based on the calculation of vertex angles of any regular n-sided figure, with $n \geq 3$, was applied to construct a regular heptagon. As such, in one of the utilized strategies by one of the students, the formula

$$\text{Vertex angle of a regular polygon} = [(n-2)180]/n$$

where n is the number of sides, was applied; this formula exists in several mathematics texts; *e.g.*, O' Daffer and Clemens (1992, p. 67). Then transformation rotation was applied on a selected line segment rotating it successively by an angle equals $[(n-2)180]/n = [(7-2)180]/7 = 128.57$, until a closed shape, in this case a regular heptagon, was formed (Figure 10).



(a) Heptagon ABCDEFG with a marginal error at A (b) ABCDEFG without error

It should be noted that in Figure 10 part (a) the marginal error that is apparent at the image of vertex A was not due to a conceptual error on the part of the student, rather it was due to the approximation in the calculation of the vertex angle. The value for the vertex angle (using TI-83) in a regular heptagon is $[(n-2)180]/n=[(7-2)180]/7$ $=128.5714286$ □ while the students has used 128.57 □ and this difference caused the imperfect meetings at vertex A.

CONCLUDING REMARKS

From observing these student patterns, through a series of case studies, one would come to think that such innovative strategies through the use of computers and GSP dynamic software were unheard of just a few decades ago. The technological advances we are witnessing nowadays have opened up new ways and valuable possibilities for visual expressions and the process of mathematical reasoning. Visual representations and images that can be produced through computer software will deliver more precise and explicit information that were previously hidden properties and structures. Visual images made through the computer are clear and ready for investigation as an object of reflection which can serve as a building block in the construction of convincing arguments and proofs. Healy and Hoyles (1999) stated that "once visual representations are constructed on the computer, images are manipulated: They can be debugged, reconstructed, transformed, **separated, or combined together**, following sets of procedures with something like reproducibility and rigor previously limited to symbolic representations" [bold added] (p. 59). Incidentally, Healy and Hoyles' statement echoes Rahim and Sawada

(1986; 1990) ideas of "Dissection" (decomposition) and "Motion" (composition) operations. Rahim and Sawada (1986; 1990) have introduced "Dissection-Motion-Operations" approach explicitly as a way to explore properties of geometric shapes through shape-to-shape, shape-to-part and part-to-part interrelationships; and to deal with the dilemma of focusing too early on quantitative knowing at the expense of qualitative understanding (Piaget, 1962; Wirszupe, 1976).

Finally, based on our observations reported throughout this work on the use of DMO both through concrete manipulation and technology in constructing and preparing convincing arguments and proving, specifically through the use of dynamic software such as GSP, we recommend that pre-service and in-service teachers would have to consider using the problem solving environment in the classrooms – there is more than meet the eyes there!

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