International Journal of Management Science
Vol 16, No 3, November 2010

Dynamic Customer Population Management Model at Aggregate Level*

Geonha Kim**
Dept. of Business Administration, Myongji University

(Received: March 24, 2010 / Revised: May 20, 2010 / Accepted: May 20, 2010)

ABSTRACT

Customer population management models can be classified into three categories: the first category includes the models that analyze the customer population at cohort level; the second one deals with the customer population at aggregate level; the third one has interest in the interactions among the customer populations in the competitive market.
Our study proposes a model that can analyze the dynamics of customer population in consumer-durables market at aggregate level. The dynamics of customer population includes the retention curves from the purchase or at a specific duration time, the duration time expectancy at a specific duration time, and customer population growth or decline including net replacement rate, intrinsic rate of increase, and the generation time of customer population.
For this study, we adopt mathematical ecology models, redefine them, and restructure interdisciplinary models to analyze the dynamics of customer population at aggregate level. We use the data of previous research on dynamic customer population management at cohort level to compare its results with those of ours and to demonstrate the useful analytical effects which the precious research cannot provide for marketers.

Keywords: Dynamic Customer Population Management, CLV, CRM, Replacement Behavior of Customer, Customer Churn, Customer Retention, Consumer Durables

1. Introduction

Researches on customer management have been developed in the fields of CRM (Customer Relationship Management) and CLV (Customer Lifetime Value). They focus on the analyses of retention, making customers heavy users, and leading cash

* This work was supported by 2008 Research Fund of Myongji University.
** Corresponding author, E- mail: ghkim@mju.ac.kr

flows into good conditions especially in subscription service industry. In spite of sophisticated mathematical models on CRM and CLV, mathematical models that analyze the dynamics of customer population are not yet developed fully.

Marketers' interest in the customer population dynamics can be summarized into three categories. The first category may include the rate of customer population increase, the customer population distribution at stable state, the distribution of replacement rates, and the sensitivity and elasticity of transition matrix of customer population. The second category may be related with the retention curve, the duration time expectancy, the growth/decline of customer population, and its intrinsic rate. The third category may focus on the increase rate, the carrying capacity, and the competition coefficients among customer populations. The difference between the first and the second resides in the analysis level of customer population. The models from the first category analyze the dynamics of customer population at cohort level; the second models the dynamics of customer population at aggregate level; and the third models the dynamics among customer populations in the entire market.

Our study proposes a mathematical model that can analyze the dynamics of customer population at aggregate level in consumer-durables market. The dynamics of customer population includes the retention curve, the duration time expectancy, the growth or decline of customer population, and its intrinsic rate.

2. Literature Reviews

Schmittlein *et al*. [9] proposed a framework for counting customers. Fader *et al*. [4] developed the beta-geometric/NBD (BG/NBD) model. These studies predict the future purchasing patterns of customers, which can be useful for calculating CLV. Fader and Hardie [3] proposed "shifted-beta-geometric" model which provides forecasts and diagnostics on customer retention. And Schweidel *et al*. [9] proposed a model which forecasts service churn and understands contributing factors using only the duration times of service subscribers.

But, the previous researches on CRM and CLV are mainly concerned with customers of service providers. Researches on customers of consumer durables are relatively limited, because consumer durables have long product-life-cycles, and thus marketers have difficulties in acquiring trade-in data and the calibration of tracing data. And some characteristics of consumer durables compared with those of services have been neglected. Revenues of consumer durables come from sales of products, repairs or maintenances, and replacements without customers' switching to the other competitors. Revenues of services are mainly composed of contracts, cash flows resulted from service usages during the contract term without churning to the other service providers.

While searching for the previous researches that are fit for the analyses of consumer durables, and that can analyze the dynamics of customer population in consumer durables, we have found in mathematical ecology area some possibilities. We would like to recommend some text books on mathematical ecology for further understanding and applications to marketing fields (e.g., Caswell [1], Donovan and Welden [2], Keyfitz and Caswell [7], Tuljapurkar and Caswell [12], Vandermeer and Goldberg [13]).

And fortunately there is a trial model on customer population adopting this approach of mathematical ecology models. Kim [8] described the dynamics of a customer population composed of stage-structured customer cohorts in the position of maker. Kim classified customer population into several cohorts on the basis of duration time. He estimated the rate of customer population increase, the customer population distribution at stable state, and the distribution of replacement rates. And he also provided sensitivity analysis and elasticity analysis of the transition matrix which can explain the effect of marketing management decision on the rate of population growth.

Kim [8] presented the marketing implications of dynamic customer management at cohort level and the effect of customer change at each cohort on the growth of customer population. But, Kim [8] did not provide us with the dynamics of customer management at aggregate level. Therefore, our present study proposes a dynamic customer management model at aggregate level which can analyze retention probability, duration time expectancy, and customer population growth or decline including net replacement rate, intrinsic rate of increase, and the generation time of customer population.

3. Model Development

We assume a hypothetical company that has produced consumer-durables for a long time, accumulated some databases of customer population, sought to manage well them and link the management of them to CRM and CLV. We adopt the concept of life tables of population dynamics in mathematical ecology, redefine the terminologies of population dynamics for the replacement behavior of consumer durables at aggregate level, propose a marketing model for customer population management at aggregate level, and examine its validity and reliability by simulating it under the typical conditions of numerical experiments.

3.1 Retention Rate

At first, we define the retention rate in consumer durables market, and compare the variables with those of mathematical ecology, CRM and CLV. The proportion of original customers surviving to the beginning of each interval (duration time *t*) is

$$S_t = \frac{N_t}{N_0} \tag{1}$$

where N_t is the number of survivors in each duration time *t*. Equation (1) means the rate that a customer survives from purchase to the beginning of duration time *t*. It begins at a value of one (i.e., N_0/N_0), and decreases with duration time *t*. At the last duration time *k*, N_k is zero. It is equivalent to standardized survivorship schedule in Donovan and Welden's notation [2].

The rate that an individual who has already survived duration time *t* will survive to duration time *t+1* is

$$g_t = \frac{N_{t+1}}{N_t} \tag{2}$$

It is equivalent to age-specific survivorship schedule in Donovan and Welden's notation [2]. Its meaning is the same as λ_t in Kim's notation [8]. We develop the relation between S_t and g_t in equation (3). And S_t in equation (3) is defined as survivor function in Fader and Hardie [3].

$$S_t = \prod_{i=0}^{t-1} g_i \tag{3}$$

We propose the retention probability in two ways. The first is S_t in equation (1) ; the retention rate from the beginning of duration time *t*, and the second is g_t in equation (2); the retention rate at a specific duration time *t*.

3.2 Retention Time Expectancy

We follow the procedure of model development hereafter as in Donovan and Welden [2]. Retention time expectancy is how long a customer at a given duration time can be expected to retain his/her consumer durables beyond its present duration time. Retention time expectancy is also to be considered as retain probability in service field. Let's put L_t the proportion of survivors at the mid-point of each time interval. L_t is calculated by averaging S_t and S_{t+1} in equation (4).

$$L_t = \frac{S_t + S_{t+1}}{2} \tag{4}$$

We can sum all the L_t values from the duration time of interest (x) up to the oldest duration time k as in equation (5).

$$T_x = \sum_{t=x}^{k} L_t \tag{5}$$

And thus, we can calculate the retention time expectancy as in equation (6). Retention time expectancy means that the expected number of time-intervals remaining to customers at a given duration time.

$$e_x = \frac{T_x}{S_x} \tag{6}$$

3.3 The Growth or Decline of Customer Population

We are interested in whether a customer population can be expected to grow, or decline, or remain stable under the specific rates of retention and replacement. We can tell this by computing the net replacement rate (R). To predict long-term changes in customer population size, we must use this concept of net replacement rate to estimate the intrinsic rate of increase (r). We can calculate net replacement rate (R) by multiplying the retention rate of each duration time (S_t) by its replacement rate (b_t), and summing these products as in equation (7).

$$R = \sum_{t=0}^{k} S_t b_t \tag{7}$$

The net replacement rate is the lifetime replacement potential of the average customer adjusted for retention, where replacement rate (b_t) is the average replacement per customer in a duration time t.

Assuming retention rates and replacement rates remain constant over duration time, if $R > 1$, then the population will grow exponentially. If $R < 1$, the customer population will shrink exponentially, and if $R = 1$, the customer population size will not change over duration time.

Note that the net replacement rate (R) is different from the intrinsic rate of increase (r), because r measures customer population change in absolute units of duration time (t) whereas R measures customer population changes in terms of generation time. To get r, we have to calculate generation time (G), and then adjust R.

$$G = \frac{\sum_{t=0}^{k} S_t b_t t}{\sum_{t=0}^{k} S_t b_t} \tag{8}$$

Usually generation time (G) is greater than one, because it depends on the retention and replacement schedules of customers. In geometric and exponential population models, the size of customer population at a certain time is like equation (9).

$$N_t = N_0 \, e^{rG} \tag{9}$$

If we restate equation (9) in a generation time (G), it is like equation (10).

$$N_G = N_0 \, e^{rG} \tag{10}$$

To calculate the intrinsic rate of increase (r) (Gotelli [5]) of equation (10), we divide both sides by N_0. And we can get equation (11).

$$\frac{N_G}{N_0} = e^{rG} \tag{11}$$

Equation (11) can be transferred to equation (12), because N_G/N_0 is roughly equivalent to R.

$$R \approx e^{rG} \tag{12}$$

Taking the natural logarithm of both sides of equation (12) gives us equation (13).

$$\ln R \approx rG \tag{13}$$

We can estimate *r* from equation (13).

$$r \approx \frac{\ln R}{G} \tag{14}$$

We know that equation (14) is an approximation which is usually within 10% of true value (Stearns [11]). To get an exact value for *r*, we can calculate the Euler equation like equation (15).

$$1 = \sum_{t=0}^{k} e^{-rt} S_t b_t \tag{15}$$

The result of right side of equation (15) should be one only when the customer population has mortality : customer population perishes and the number of customers ends zero at a certain time *t*.

4. Numerical Experiments

4.1 Data

We introduce Kim's artificial data [8] because we want to compare the merits of our proposed model with those of Kim's model in the analyses on the dynamics of customer population. Kim's data [8] suppose a customer population that is composed of four cohorts: C_0, C_1, C_2, C_3. They put the replacement rates (R_0, r_1, r_2, r_3) in the form of bath tub curve (0.66, 0.32, 0.32, 0.66), because the typical replacement behavior shows bath tub curve, just as in death rate curve and failure rate curve. And they set the survival rate ($s_{1,0}$, $s_{2,1}$, $s_{3,2}$) in the form of inverse bath tub curve (0.455, 0.75, 0.45), because the survival behaviors are similar to the inverse of the replacement behaviors of customers. A customer population in Kim's research varies into three types according to three scenarios of population distribution (N(C_0, C_1, C_2, C_3)): flat, decreasing, and increasing. The first customer population starts with flat customer population distribution: N(47, 47, 47, 47). The second customer population starts with decreasing customer population distribution: N(100, 50, 25, 12). And the third customer population starts with increasing customer population distribution: N(12, 25, 50, 100).

Table 1 shows the replacement rates, survival rates, and initial customer population distribution of scenario 1 of Kim's research [8]. Table 2 shows some of the results

Table 1. Replacement Rates, Survival Rates, and Customer Population Distribution of Scenario 1

	Cohort Class				N
	C0	C1	C2	C3	
P	0.660	0.320	0.320	0.660	47
	0.455	0.000	0.000	0.000	47
	0.000	0.750	0.000	0.000	47
	0.000	0.000	0.450	0.000	47

Table 2. The Change of Cohorts, Customer Population, and b_t of scenario 1

Time	C0	C1	C2	C3	Nt	bt
0	47	47	47	47	188	0.49
1	92	21	35	21	170	0.55
2	93	42	16	16	167	0.54
3	90	42	31	7	171	0.51
4	88	41	32	14	175	0.52
5	91	40	31	14	176	0.52
6	92	41	30	14	177	0.52
7	93	42	31	14	179	0.52
8	93	42	31	14	181	0.52
9	94	42	32	14	183	0.52
10	95	43	32	14	184	0.52
11	96	43	32	14	186	0.52
12	97	44	33	14	188	0.52
13	98	44	33	15	190	0.52
14	99	45	33	15	192	0.52
15	100	45	33	15	193	0.52
16	101	45	34	15	195	0.52
17	102	46	34	15	197	0.52
18	103	46	34	15	199	0.52
19	104	47	35	16	201	0.52
20	105	47	35	16	203	0.52
21	106	48	35	16	205	0.52
22	107	48	36	16	207	0.52
23	108	49	36	16	209	0.52
24	109	49	37	16	211	0.52
25	110	50	37	16	213	0.52

of scenario 1 in Kim's research [8]: the change of each cohorts and the total customer population. We add b_t column on the basis of replacement rates and customer number of each cohorts at a certain duration t. We can calculate b_t by equation (16).

$$b_t = (R_0 C_0(t) + r_1 C_1(t) + r_2 C_2(t) + r_3 C_3(t)) / N_t \tag{16}$$

In similar way, we can get data of scenario 2 and scenario 3 like in Table 3, Table 4, Table 5, and Table 6.

Table 3. Replacement Rates, Survival Rates, and Customer Population Distribution of Scenario 2

	Cohort Class				N
	C0	C1	C2	C3	
P	0.660	0.320	0.320	0.660	100
	0.455	0.000	0.000	0.000	50
	0.000	0.750	0.000	0.000	25
	0.000	0.000	0.450	0.000	12

Table 4. The Change of Cohorts, Customer Population, and b_t in Scenario 2

Time	C0	C1	C2	C3	Nt	bt
0	100	50	25	12	187	0.52
1	98	46	38	11	192	0.51
2	99	45	34	17	194	0.52
3	101	45	33	15	195	0.52
4	102	46	34	15	197	0.52
5	103	46	35	15	199	0.52
6	104	47	35	16	201	0.52
7	105	47	35	16	203	0.52
8	106	48	35	16	205	0.52
9	107	48	36	16	207	0.52
10	108	49	36	16	209	0.52
11	109	49	36	16	211	0.52
12	110	50	37	16	213	0.52
13	111	50	37	17	215	0.52

14	112	51	38	17	217	0.52
15	113	51	38	17	219	0.52
16	114	52	38	17	221	0.52
17	116	52	39	17	224	0.52
18	117	53	39	17	226	0.52
19	118	53	39	18	228	0.52
20	119	54	40	18	230	0.52
21	120	54	40	18	232	0.52
22	121	55	41	18	235	0.52
23	123	55	41	18	237	0.52
24	124	56	41	18	239	0.52
25	125	56	42	19	242	0.52

Table 5. Replacement Rates, Survival Rates, and Customer Population Distribution of Scenario 3

	Cohort Class				N
	C0	C1	C2	C3	
P	0.660	0.320	0.320	0.660	12
	0.455	0.000	0.000	0.000	25
	0.000	0.750	0.000	0.000	50
	0.000	0.000	0.450	0.000	100

Table 6. The Change of Cohorts, Customer Population, and b_t in Scenario 3

Time	C0	C1	C2	C3	Nt	bt
0	12	25	50	100	187	0.52
1	98	5	19	23	145	0.60
2	87	45	4	8	144	0.55
3	79	40	33	2	154	0.50
4	77	36	30	15	157	0.52
5	81	35	27	13	157	0.53
6	82	37	26	12	158	0.52
7	83	37	28	12	160	0.52
8	83	38	28	13	161	0.52
9	84	38	28	13	163	0.52

10	85	38	28	13	164	0.52
11	86	39	29	13	166	0.52
12	87	39	29	13	168	0.52
13	87	39	29	13	169	0.52
14	88	40	30	13	171	0.52
15	89	40	30	13	172	0.52
16	90	41	30	13	174	0.52
17	91	41	30	14	176	0.52
18	92	41	31	14	178	0.52
19	93	42	31	14	179	0.52
20	94	42	31	14	181	0.52
21	94	43	32	14	183	0.52
22	95	43	32	14	185	0.52
23	96	43	32	14	186	0.52
24	97	44	33	15	188	0.52
25	98	44	33	15	190	0.52

4.2 The Results of Numerical Experiments

Putting the Kim's data [8] into our proposed model, we summarize the results of numerical experiments. The numerical experiments include the retention curves of both from the beginning of duration time and at a specific duration time. We also analyze the retention time expectancy at a specific duration time, and the growth or decline of customer population during duration time.

4.2.1 Retention Curves from the Beginning of Duration Time

Table 7 and Figure 1 show the retention rate data and curves of both from the beginning of duration time and at a specific duration time in the case of three scenarios. The retention curve of scenario 1 decreases below one until duration time 2, increases gradually but still below one until duration time 12, and increases gradually over one from duration time 12 to duration time 25. The retention curve of scenario 2 increases over one from the beginning of duration time. After fluctuations below one from duration time 0 to duration time 6, the retention curve of scenario 3 increases below one from duration time 7 to duration time 23 and increases over one from duration time 24.

Table 7. Duration Time and Retention Curves

Duration Time(t)	N_t: Scenario 1	N_t: Scenario 2	N_t: Scenario 3	S_t: Scenario 1	S_t: Scenario 2	S_t: Scenario 3	g_t: Scenario 1	g_t: Scenario 2	g_t: Scenario 3
0	188	187	187	1.0000	1.0000	1.0000	0.9038	1.0276	0.7734
1	170	192	145	0.9038	1.0276	0.7734	0.9811	1.0104	0.9978
2	167	194	144	0.8867	1.0383	0.7717	1.0272	1.0045	1.0647
3	171	195	154	0.9108	1.0430	0.8217	1.0214	1.0097	1.0229
4	175	197	157	0.9303	1.0531	0.8405	1.0051	1.0108	0.9959
5	176	199	157	0.9350	1.0645	0.8370	1.0073	1.0099	1.0068
6	177	201	158	0.9419	1.0750	0.8427	1.0104	1.0095	1.0123
7	179	203	160	0.9517	1.0852	0.8531	1.0105	1.0097	1.0108
8	181	205	161	0.9616	1.0957	0.8623	1.0096	1.0098	1.0091
9	183	207	163	0.9709	1.1065	0.8702	1.0096	1.0098	1.0095
10	184	209	164	0.9802	1.1173	0.8785	1.0098	1.0097	1.0099
11	186	211	166	0.9897	1.1281	0.8871	1.0098	1.0097	1.0098
12	188	213	168	0.9994	1.1391	0.8958	1.0097	1.0097	1.0097
13	190	215	169	1.0092	1.1502	0.9045	1.0097	1.0097	1.0097
14	192	217	171	1.0190	1.1614	0.9133	1.0097	1.0097	1.0097
15	193	219	172	1.0289	1.1727	0.9222	1.0097	1.0097	1.0097
16	195	221	174	1.0389	1.1842	0.9312	1.0097	1.0097	1.0097
17	197	224	176	1.0491	1.1957	0.9403	1.0097	1.0097	1.0097
18	199	226	178	1.0593	1.2074	0.9495	1.0097	1.0097	1.0097
19	201	228	179	1.0696	1.2191	0.9587	1.0097	1.0097	1.0097
20	203	230	181	1.0800	1.2310	0.9681	1.0097	1.0097	1.0097
21	205	232	183	1.0905	1.2430	0.9775	1.0097	1.0097	1.0097
22	207	235	185	1.1012	1.2551	0.9870	1.0097	1.0097	1.0097
23	209	237	186	1.1119	1.2673	0.9966	1.0097	1.0097	1.0097
24	211	239	188	1.1227	1.2797	1.0063	1.0097	1.0097	1.0097
25	213	242	190	1.1337	1.2921	1.0162			

4.2.2 Retention Curves at a Specific Duration Time

Table 7 and Figure 2 show retention rate data and curves of each scenario at a specific duration time. The retention curve of scenario 1 starts below one and soon approaches to 1.0097 from duration time 12. The retention curve of scenario 2 starts above one and soon approaches to 1.0097 from duration time 10. And the retention curve of scenario 3 starts below one and soon approaches to 1.0097 from duration time 12.

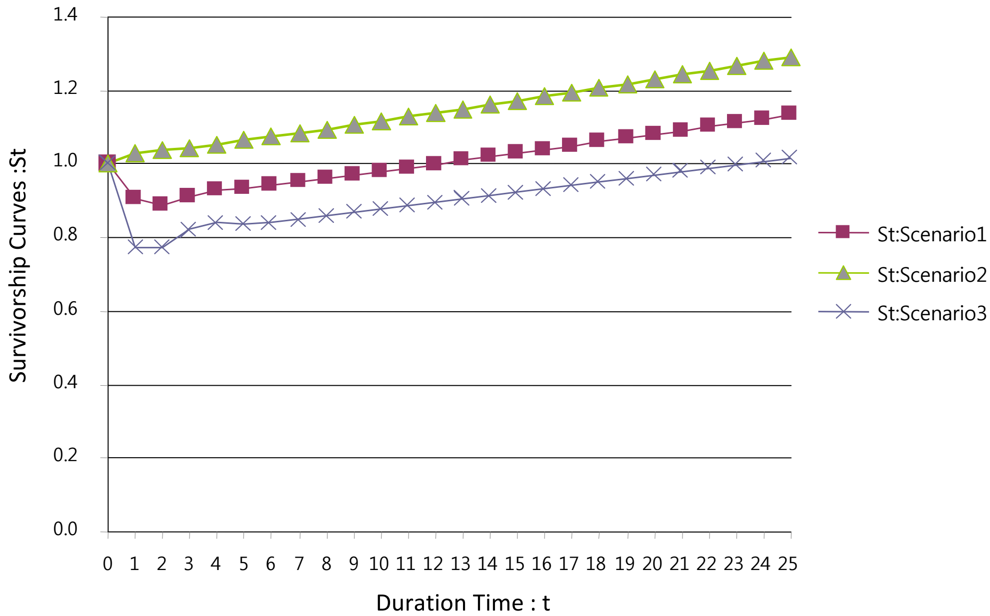

Figure 1. The Retention Curves from the Beginning of Duration Time

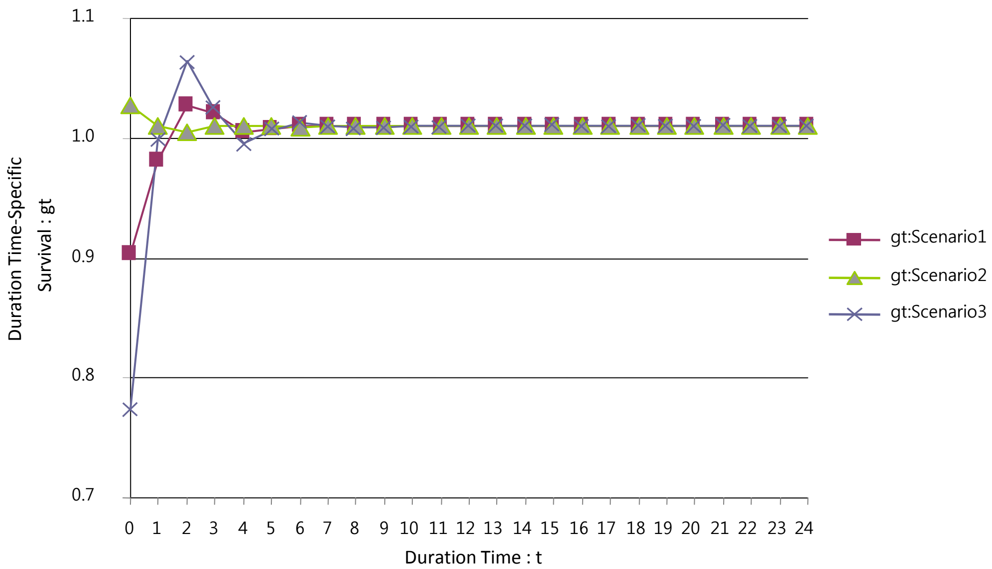

Figure 2. The Retention Curves at Specific Duration Times

4.2.3 Retention Time Expectancy at a Specific Duration Time

Table 8 and Figure 3 show retention time expectancy data and curves of each scenario at a specific duration time. The retention time expectancy curve of scenario 1 starts from 25.2089 and approach approximately to 19.7 after fluctuations from duration time 0 to duration 6. The retention time expectancy curve of scenario 2 starts from 28.6863 and approach approximately to 19.7 without fluctuations from duration time 0 to duration 6. And the retention time expectancy curve of scenario 3 starts from 22.5976 and approach approximately to 19.7 after fluctuations from duration time 0 to duration 6. The retention time expectancies of three scenarios have almost the same values from duration time 7 to duration time 25.

Table 8. Duration Time Expectancy at Specific Duration Time

Duration Time-Specific Life Expectancy	S_t: Scenario 1	S_t: Scenario 2	S_t: Scenario 3	L_t: Scenario 1	L_t: Scenario 2	L_t: Scenario 3	e_t: Scenario 1	e_t: Scenario 2	e_t: Scenario 3
0	1.0000	1.0000	1.0000	0.9519	1.0138	0.8867	25.2089	28.6863	22.5976
1	0.9038	1.0276	0.7734	0.8952	1.0330	0.7726	26.8404	26.9280	28.0712
2	0.8867	1.0383	0.7717	0.8987	1.0407	0.7967	26.3473	25.6564	27.1322
3	0.9108	1.0430	0.8217	0.9205	1.0481	0.8311	24.6630	24.5438	24.5135
4	0.9303	1.0531	0.8405	0.9326	1.0588	0.8387	23.1573	23.3124	22.9761
5	0.9350	1.0645	0.8370	0.9384	1.0697	0.8399	22.0424	22.0695	22.0692
6	0.9419	1.0750	0.8427	0.9468	1.0801	0.8479	20.8852	20.8578	20.9225
7	0.9517	1.0852	0.8531	0.9567	1.0905	0.8577	19.6755	19.6666	19.6737
8	0.9616	1.0957	0.8623	0.9662	1.1011	0.8663	18.4769	18.4825	18.4687
9	0.9709	1.1065	0.8702	0.9755	1.1119	0.8743	17.3061	17.3081	17.3061
10	0.9802	1.1173	0.8785	0.9849	1.1227	0.8828	16.1465	16.1456	16.1481
11	0.9897	1.1281	0.8871	0.9946	1.1336	0.8915	14.9953	14.9948	14.9955
12	0.9994	1.1391	0.8958	1.0043	1.1447	0.9002	13.8549	13.8550	13.8545
13	1.0092	1.1502	0.9045	1.0141	1.1558	0.9089	12.7260	12.7261	12.7260
14	1.0190	1.1614	0.9133	1.0239	1.1671	0.9178	11.6082	11.6081	11.6082
15	1.0289	1.1727	0.9222	1.0339	1.1785	0.9267	10.5010	10.5009	10.5010
16	1.0389	1.1842	0.9312	1.0440	1.1899	0.9358	9.4044	9.4044	9.4044
17	1.0491	1.1957	0.9403	1.0542	1.2015	0.9449	8.3185	8.3185	8.3185
18	1.0593	1.2074	0.9495	1.0644	1.2132	0.9541	7.2431	7.2431	7.2431
19	1.0696	1.2191	0.9587	1.0748	1.2251	0.9634	6.1780	6.1780	6.1780
20	1.0800	1.2310	0.9681	1.0853	1.2370	0.9728	5.1232	5.1232	5.1232
21	1.0905	1.2430	0.9775	1.0959	1.2490	0.9823	4.0786	4.0786	4.0786
22	1.1012	1.2551	0.9870	1.1065	1.2612	0.9918	3.0441	3.0441	3.0441
23	1.1119	1.2673	0.9966	1.1173	1.2735	1.0015	2.0195	2.0195	2.0195
24	1.1227	1.2797	1.0063	1.1282	1.2859	1.0112	1.0049	1.0049	1.0049
25	1.1337	1.2921	1.0162						

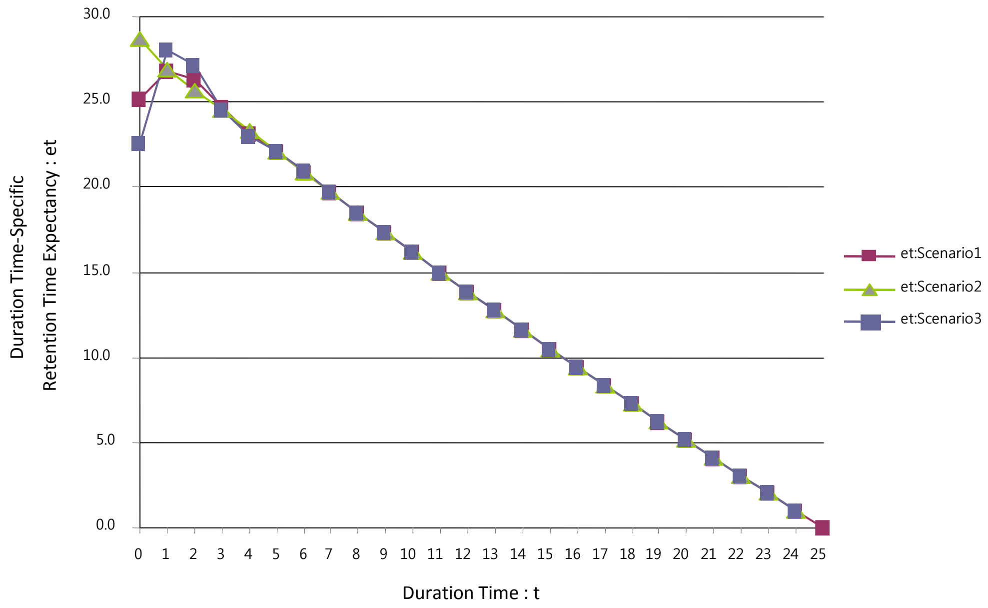

Figure 3. The Duration Time Expectancy Curves at Specific Duration Times

4.2.4 The Growth or Decline of Customer Population

Table 9 and Table 12 show the calculation process and values of net replacement rate, intrinsic rate of increase, and Euler's value of scenario 1. Table 10 and Table 12 show the calculation process of net replacement rate, intrinsic rate of increase, and Euler's value of scenario 2. And also Table 11 and Table 12 show those of scenario 3.

Table 9. Population Growth and Decline of Scenario 1

Duration Time (t)	N_t: Scenario 1	St: Scenario 1	bt: Scenario 1	(St)(bt)	(St)(bt) t	(e^-rt)(St)(bt)
0	188	1.0000	0.4900	0.4900	0.0000	0.4900
1	170	0.9038	0.5467	0.4941	0.4941	0.4039
2	167	0.8867	0.5418	0.4804	0.9608	0.3211
3	171	0.9108	0.5137	0.4678	1.4035	0.2557
4	175	0.9303	0.5185	0.4823	1.9294	0.2155
5	176	0.9350	0.5230	0.4890	2.4450	0.1786
6	177	0.9419	0.5231	0.4927	2.9565	0.1471
7	179	0.9517	0.5217	0.4965	3.4755	0.1212
8	181	0.9616	0.5217	0.5017	4.0137	0.1001
9	183	0.9709	0.5220	0.5068	4.5611	0.0827

10	184	0.9802	0.5220	0.5117	5.1169	0.0683
11	186	0.9897	0.5220	0.5166	5.6827	0.0564
12	188	0.9994	0.5220	0.5217	6.2599	0.0465
13	190	1.0092	0.5220	0.5268	6.8478	0.0384
14	192	1.0190	0.5220	0.5319	7.4464	0.0317
15	193	1.0289	0.5220	0.5371	8.0559	0.0262
16	195	1.0389	0.5220	0.5423	8.6767	0.0216
17	197	1.0491	0.5220	0.5476	9.3088	0.0178
18	199	1.0593	0.5220	0.5529	9.9525	0.0147
19	201	1.0696	0.5220	0.5583	10.6077	0.0122
20	203	1.0800	0.5220	0.5637	11.2748	0.0100
21	205	1.0905	0.5220	0.5692	11.9539	0.0083
22	207	1.1012	0.5220	0.5748	12.6452	0.0068
23	209	1.1119	0.5220	0.5804	13.3487	0.0056
24	211	1.1227	0.5220	0.5860	14.0648	0.0047
25	213	1.1337	0.5220	0.5917	14.7936	0.0038
				13.714	178.276	2.6891

Table 10. Population Growth and Decline of Scenario 2

Duration Time (t)	N_t: Scenario 2	St: Scenario 2	bt: Scenario 2	(St)(bt)	(St)(bt) t	(e^-rt)(St)(bt)
0	187	1.0000	0.5236	0.5236	0.0000	0.5236
1	192	1.0276	0.5132	0.5273	0.5273	0.4273
2	194	1.0383	0.5222	0.5422	1.0845	0.3561
3	195	1.0430	0.5235	0.5460	1.6381	0.2905
4	197	1.0531	0.5222	0.5500	2.2000	0.2371
5	199	1.0645	0.5215	0.5552	2.7758	0.1940
6	201	1.0750	0.5219	0.5611	3.3664	0.1589
7	203	1.0852	0.5221	0.5665	3.9657	0.1300
8	205	1.0957	0.5220	0.5720	4.5758	0.1063
9	207	1.1065	0.5220	0.5775	5.1977	0.0870
10	209	1.1173	0.5220	0.5832	5.8317	0.0712
11	211	1.1281	0.5220	0.5889	6.4775	0.0583
12	213	1.1391	0.5220	0.5946	7.1351	0.0477
13	215	1.1502	0.5220	0.6004	7.8050	0.0390
14	217	1.1614	0.5220	0.6062	8.4873	0.0319
15	219	1.1727	0.5220	0.6121	9.1821	0.0261
16	221	1.1842	0.5220	0.6181	9.8897	0.0214
17	224	1.1957	0.5220	0.6241	10.6102	0.0175
18	226	1.2074	0.5220	0.6302	11.3438	0.0143
19	228	1.2191	0.5220	0.6364	12.0907	0.0117

20	230	1.2310	0.5220	0.6426	12.8510	0.0096
21	232	1.2430	0.5220	0.6488	13.6250	0.0078
22	235	1.2551	0.5220	0.6551	14.4129	0.0064
23	237	1.2673	0.5220	0.6615	15.2149	0.0052
24	239	1.2797	0.5220	0.6680	16.0311	0.0043
25	242	1.2921	0.5220	0.6745	16.8617	0.0035
				15.566	203.181	2.8867

Table 11. Population Growth and Decline of Scenario 3

Duration Time (t)	N_t: Scenario 3	St: Scenario 3	bt: Scenario 3	(St)(bt)	(St)(bt) t	(e^-rt)(St)(bt)
0	187	1.0000	0.5236	0.5236	0.0000	0.5236
1	145	0.7734	0.6031	0.4664	0.4664	0.3838
2	144	0.7717	0.5454	0.4209	0.8418	0.2849
3	154	0.8217	0.4982	0.4094	1.2281	0.2280
4	157	0.8405	0.5181	0.4355	1.7419	0.1996
5	157	0.8370	0.5260	0.4403	2.2013	0.1660
6	158	0.8427	0.5237	0.4413	2.6480	0.1369
7	160	0.8531	0.5209	0.4444	3.1110	0.1135
8	161	0.8623	0.5216	0.4498	3.5983	0.0945
9	163	0.8702	0.5222	0.4544	4.0895	0.0785
10	164	0.8785	0.5221	0.4586	4.5865	0.0652
11	166	0.8871	0.5219	0.4630	5.0933	0.0542
12	168	0.8958	0.5219	0.4676	5.6110	0.0450
13	169	0.9045	0.5220	0.4722	6.1380	0.0374
14	171	0.9133	0.5220	0.4767	6.6745	0.0311
15	172	0.9222	0.5220	0.4814	7.2208	0.0258
16	174	0.9312	0.5220	0.4861	7.7773	0.0214
17	176	0.9403	0.5220	0.4908	8.3439	0.0178
18	178	0.9495	0.5220	0.4956	8.9208	0.0148
19	179	0.9587	0.5220	0.5004	9.5081	0.0123
20	181	0.9681	0.5220	0.5053	10.1061	0.0102
21	183	0.9775	0.5220	0.5102	10.7148	0.0085
22	185	0.9870	0.5220	0.5152	11.3344	0.0071
23	186	0.9966	0.5220	0.5202	11.9650	0.0059
24	188	1.0063	0.5220	0.5253	12.6069	0.0049
25	190	1.0162	0.5220	0.5304	13.2601	0.0040
				12.385	159.788	2.5750

Table 12. Net Replacement Rates, Generation Time, Intrinsic Rates of Increase, and Euler's Equation of Three Scenarios

	Scenario 1	Scenario 2	Scenario 3
R	13.7140	15.5661	12.3851
G	12.9995	13.0528	12.9016
r est.	0.2014	0.2103	0.1951
Euler equation	2.6891	2.8867	2.5750

5. Discussions

5.1 Replacement Rate (b_t)

As shown in Table 2, Table 4, and Table 6, b_t column has the same value and it approaches to 0.52 after short fluctuations during early duration times. This means that the customer population will not decrease regardless of customer distribution types in each cohort of starting stage, if the replacement rate keeps the level of 0.52. Kim's article [8] made it clear that when the replacement rate of each cohort is (0.66, 0.32, 0.32, 0.66) and the survival rate of each cohorts is (0.455, 0.75, 0.45), the customer population does not decrease regardless of customer distribution types in each cohorts of starting stage. Kim's article [8] showed the objectives of replacement rate and survival rate at cohort level not to decrease the customer population, and this present study shows the total average objective of retention rate at the level of customer population.

5.2 Retention Curves

Table 7 and Figure 1 show us that customer populations of three scenarios increase gradually with the same gap soon after several fluctuations during early duration times. While Kim's article [8] was limited to revealing the dynamics at each cohort of three scenarios in separated graphs without retention rates, our study has the merits to show retention rates and to trace the comparative dynamics of three scenarios in one graph. These retention curves help marketers to identify the retention proportion of customer population from the beginning of duration time *t* to a certain particular duration time *t*. Marketers can also identify and trace the changes of retention rate, and calculate the replacement rate of customer population (b_t) at a certain duration time.

It can be inferred from Kim's article [8] that the distribution types of customer population cause to difference among customer population sizes of three scenarios at stable state, because other conditions including the replacement rate of each cohort and the survival rate of each cohort are the same. But, in this present study, the effects of replacement rate and survival rate at cohort level are summed up to the average replacement rate (b_t) of total customer population, and the distribution types of customer population under the average replacement rate (0.52) cause the difference among customer population sizes of three scenarios at stable state.

5.3 Retention Curves at Specific Duration Times

Figure 2 shows us vividly the retention possibility how many individuals who have already survived during the duration time t will also survive at the duration time $t+1$. It is the same in meaning as λ_t in Kim's article [8]. While Kim's article [8] focused on examining separately the comparative dynamics among cohorts of one scenario in one graph, Figure 2 in this article shows us the retention rates of three scenarios at specific duration times and the traces of them in one graph.

According to the Figure 2, in order to retain their customers under the condition that they do not want to decrease their customer population, marketers of scenario 3 have to make more efforts when their customer population distribution is in scenario 3 than those of scenario 2. The importance of customer population size of early stages was also recognized in Kim's article [8].

5.4 Retention Time Expectancy at a Specific Duration Time

Figure 3 shows us vividly the retention time expectancy at a specific duration time. Retention time expectancy means how long a customer at a given duration time can be expected to survive beyond the given time. This is one of our article's merits which Kim's article [8] could not provide. Retention time expectancy is also to be considered as retain probability in service field. But, the managerial meaning of retention time expectancy in consumer durables is to be somewhat different from that in the area of service subscribers. The longer the retention time expectancy of service subscribers is, the more revenues will come from the survived subscribers. We cannot say that the long retention time expectancy will lead to the replacement behaviors of customers in consumer durables. And marketers can stimulate customers to replace their consumer durables between the time that the retention curves at specific duration times stabilize in Figure 2 (after duration time 5) and the times of the generation time of customer population in Table 12 (about 13 duration times), because the curves of retention time expectancy gradually decrease after the duration time 5 in Figure 3.

5.5 Net Replacement Rates, Generation Time, Intrinsic Rates of Increase, and Euler's Equation of Three Scenarios

Table 9, Table 10, and Table 11 demonstrate the dynamics of customer population growth and decline of three scenarios. Table 12 is a table summarizing comparative results of the three scenarios in each category of net replacement rates, generation time, intrinsic rates of increase, and Euler's equation. These are also some merits of this present article that Kim's article [8] could not provide. The order of size of net replacement rates is that of scenario 2 (decreasing distribution), scenario 1 (flat distribution), and scenario 3 (increasing distribution). And this order of scenario 2, scenario 1 and scenario 3 is also applied to the order in each of the rest categories of generation time, intrinsic rate of increase, and Euler's equation value. But, the cause of this order and differences of variables will not be explained without the help of Kim's article [8].

It is natural that the net replacement rates (*R*) are bigger than 1, because replacement rates and survival rates are manipulated not to decrease the customer population in Kim's article [8]. It is inferred that the order of three scenarios of *R*s (15.5661, 13.7140, 12.3851) results from the effects of S_t in early duration times shown in Table 9, Table 10, and Table 11.

The generation time (*G*) is about duration time 13. *G* value is very important to marketers, because their customers want to replace their goods in use to new ones near the point of *G* value. It takes time for customers of early stages of cohorts to reach the duration time of replacing their goods. Therefore, *G* value of scenario 2 is bigger than that of scenario 3 under the condition that their customer population sizes are the same.

The difference of *r*s among three scenarios are seems to be small. But the big differences among customer population sizes of three scenarios at stable state result from these small differences, because the other conditions of three customer populations except their distribution types are same.

Euler's equation value of three scenarios should be one only when the customer population ends zero at observed duration time t, but three customer populations in our data are increasing. Therefore, the values of Euler equation in table 12 are bigger than one.

6. Conclusions

Our proposed model is proper enough to explain the dynamics of customer population at aggregate level. Compared with Kim's article [8], the model proposed in this article provides the three dynamic analytical points at customer population level that Kim's article [8] cannot articulate: the first point is a retention curve from the purchase; the second is a retention curve at a specific duration time; and the third is a retention expectancy curve at a specific duration time.

And also our proposed model adds dynamic and mathematical analyses at customer population level to the previous article (Kim [8]) in four aspects: net replacement rate, generation time, the intrinsic rate of replacement, and Euler's equation value.

Though using same data, this present study succeeds in producing different outcomes at aggregate level from those of Kim's article [8] at cohort level. In order for marketers to analyze a customer population completely, the two models proposed respectively in Kim's article [8] and this present study should be combined. The two models are complementary to each other; the former presents the information on customer population at cohort level, and the latter at aggregate level.

But, our proposed model has some limitations. The first comes from data acquisition, because it is not easy for a researcher to get trade-in data of consumer durables. The future study, being based on the real data through collaboration with companies, will verify the validity of our proposed model. The second comes from the simplicity of model proposition, whose proposition of the function of customer population growth is exponential type. Other function like logistic type will be challengeable for the better fitness.

References

[1] Caswell, H., *Matrix Population Models-Construction, Analysis, and Interpretation, 2nd ed.*, Sinauer Associates, Inc., 2001.

[2] Donovan, T. M. and C. W. Welden, *Spreadsheet Exercises in Ecology and Evolution*, Sinauer Associates, Inc., 2002.

[3] Fader, P. S. and B. G. S. Hardie, "How to Project Customer Retention," *Journal of Interactive Marketing* 21, 1 (2007), 76-90.

[4] Fader, P. S., B. G. S. Hardie, and K. L. Lee, "Counting Your Customers the Easy Way: An Alternative to the Pareto/NBD Model," *Marketing Science* 24, 2 (2005), 275-284.

[5] Gotelli, N. J., *A Primer of Ecology 3rd ed.*, Sinauer, 2001.

[6] Murray, J. D., *Mathematical Biology I: An Introduction 3rd ed.*, Springer, 2002.

[7] Keyfitz, N. and H. Caswell, *Applied Mathematical Demography 3rd ed.*, Springer, 2005.

[8] Kim, G., "Dynamic Customer Management on Duration Time Structure," *Journal of Korean Marketing Association* 25, 1 (2010), 95-115.

[9] Schmittlein, D. C., D. G. Morrison, and R. Colombo, "Counting Your Customers: Who They Are and What Will They Do Next?," *Management Science* 33, 1 (1987), 1-24.

[10] Schweidel, D. A., P. S. Fader, and E. T. Bradlow, "Understanding Service Retention Within and Across Cohorts Using Limited Information," *Journal of Marketing* 72 (2008), 82-94.

[11] Stearns, S. C., *The Evolution of Life Histories*, Oxford University Press, Oxford, 1992.

[12] Tuljapurkar, S. and H. Caswell, *Structured-Population Models in Marine, Terrestrial, and Freshwater Systems*, Chapman and Hall, 1997.

[13] Vandermeer, J. H. and D. E. Goldberg, *Population Ecology-First Principles*, Princeton University Press, 2003.