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Exact Algorithms of Transforming Continuous Solutions into Discrete Ones for Bit Loading Problems in Multicarrier Systems*

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ABSTRACT

In this study, we present the exact methods of transforming the continuous solutions into the discrete ones for two types of bit-loading problem, marginal adaptive (MA) and rate adaptive (RA) problem, in multicarrier communication systems. While the computational complexity of existing solution methods for discrete optimal solutions depends on the number of bits to be assigned (R), the proposed method determined by the number of subcarriers (N), making ours be more efficient in most cases where R is much larger than N . Furthermore our methods have some strength of their simpler form to make a practical use.

Keywords: Bit-loading Problem, Multicarrier Communication System, Optimal Integer Solution, Computational Complexity

1. Introduction

In multicarrier communication systems (MCSs), a wideband spectrum is divided into the narrowband subcarriers over which a number of low-rate data streams are transmitted in parallel to achieve a high-rate data stream. In the MCS, a loading problem is used to allocate bits or power to subcarriers. The loading problems are classified into

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two types [1] according to the objective functions they attempt to optimize: margin adaptive (MA) and rate adaptive (RA) problems, and are mathematically formulated as the non-linear discrete convex problems. Most of the existing algorithms for optimal integer solutions are based on an incremental greedy procedure whose computation time strongly depends on the number of bits to be assigned. Note that computational complexity of these algorithms is $O(NR)$, where N is the number of subcarriers and R is the number of bits to be assigned [2-5]. The study [5] has provided a procedure to lessen the computational burden of greedy-type algorithms, but the nature of this procedure also belongs to the greedy-type. The goal of this study is to develop an efficient transforming algorithm with computational complexity $O(N\log N)$ of getting an integer optimal solution from the associated optimal continuous solution which can be obtained by the existing algorithms with $O(N\log N)$ complexity [6].

The rest of the paper is organized as follows. In Section 2, we first introduce two mathematical models for the loading problem in MCS systems. Section 3 and 4 provides the optimal solution for the continuous models and the transformation method to obtain the discrete optimal solutions from continuous ones.

2. Bit Loading Problems

Consider an MCS consisting of N subcarriers. Denote the index set of subcarriers by $S=\{1, 2, \cdots, N\}$. Let γ_i be the gain to noise ratio of subcarrier i and p_i be the amount of power assigned to subcarrier i. The data rate of subcarrier i, x_i (bits/symbol), can be then approximated by

$$x_i=\log_2\left(1+\frac{p_i\gamma_i}{\Gamma}\right), \quad i\in S, \tag{1}$$

where Γ is the SNR(Signal to Noise Ratio) gap that depends on the target BER (Bit Error Rate), the coding, and noise margin [1, 3].

Assuming γ_i and Γ are given, the amount of power required for the data rate

x_i, $f_i(x_i)$, could be obtained from the equation (1) as $f_i(x_i) = a_i\left(2^{x_i} - 1\right)$, $i \in S$, where $a_i = \Gamma / \gamma_i$.

The mathematical problem MA of minimizing the total power required to meet the pre-specified target data rate (R) becomes

Problem (MA):

$$\min \sum_{i \in S} f_i(x_i)$$

$$s.t. \quad \sum_{i \in S} x_i = R, \tag{2}$$

$$x_i \in \{0, 1, 2, \cdots\}, \quad i \in S. \tag{3}$$

And the aim of another mathematical problem RA is to maximize the system throughput while satisfying the constraint on total power limit (P).

Problem (RA):

$$\max \sum_{i \in S} x_i$$

$$s.t. \quad \sum_{i \in S} f_i(x_i) \le P, \tag{4}$$

$$x_i \in \{0, 1, 2, \cdots\}, \quad i \in S. \tag{5}$$

Since the function $f_i(x)$ is a strictly increasing convex function with $f_i(0) = 0$, the problems MA and RA become the non-linear discrete convex problems [8]. It is known that a greedy algorithm can find an optimal discrete solution for each of problems MA and RA [9, 10]. The classical greedy algorithm, in every iteration, allocates a single bit based a criterion of maximizing marginal increase (or decrease) of the objective function, for which a sorting procedure should be done before allocating the bit [2, 3, 11]. So the computational complexity of these algorithms mainly depends on the number of bits to be allocated and the complexity of sorting algorithm. It is worth noting the study [1] which has proposed two new greedy-type algorithms to accelerate the computational speed by allocating bits in groups but one by one.

Let MA(C) and RA(C) denote the continuous relaxation problems of the problems MA and RA, respectively.

3. Transform Method for MA Problem

3.1 Optimal Solution for MA(C) Problem

Since MA(C) is a classical convex programming, the Karush-Kuhn-Tucker (KKT) condition gives the necessary and sufficient conditions for the optimal continuous solution [8]. Let v_0 and v_i, $i \in S$ be the Lagrange multipliers associated with constraints (2) and the continuous relaxation of (3), i.e., $x_i \geq 0,\ i \in S,$ respectively.

Assume that $a_1 \leq a_2 \leq \cdots \leq a_N$ without a loss of generality and let

$$A(j) = \frac{R + \sum_{i=1}^{j} \log_2 a_i}{j}, \quad j \in S.$$

We will call the subcarrier $k(\in S)$ satisfying $A(k) - \log_2 a_k > 0$ and $A(k+1) - \log_2 a_{k+1} \leq 0$ a *threshold subcarrier*. Note that $A(j) - \log_2 a_j$ is a decreasing function on j.

The deriving procedure for the optimal continuous solution of MA(C) satisfying KKT conditions is out of scope of this paper, so only main results are provided without proofs.

Theorem 1: If the subcarrier $k \in S$ is a threshold one, the KKT conditions are satisfied by the following values v_0^*, x_i^*, and v_i^*, $i \in S$.

$$v_0^* = 2^{A(k)} \cdot \ln 2.$$
$$x_i^* = A(k) - \log_2 a_i, \quad i = 1, \cdots, k.$$
$$x_i^* = 0, \quad i = k+1, \cdots, N.$$
$$v_i^* = 0, \quad i = 1, \cdots, k.$$
$$v_i^* = -v_0^* + a_i \ln 2, \quad i = k+1, \cdots, N.$$

Based on the results given in the Theorem 1, we now present the algorithm MA(C), which generates an optimal solution of problem MA(C).

[Algorithm MA(C) for the Problem MA(C)]

Input: a_i, $i \in S$

Sort a_i in increasing order (let the indices of a_i be arranged so that $a_1 \le a_2 \cdots \le a_N$ holds).

$k \leftarrow 1$;

while ($k \le N$) do{

$A(k) \leftarrow (R + \sum_{i=1}^{k} \log_2 a_i) / k$

if $(R + \sum_{i=1}^{k} \log_2 a_i) / k - \log_2 a_k < 0$, then exit while;

$k \leftarrow k+1$;

}

Output: $x_i^* = A(k) - \log_2 a_i$, $f_i(x_i^*) = 2^{A(k)} - a_i$, for $i = 1, 2, \cdots, k$.

$x_i^* = 0$, $f_i(x_i^*) = 0$, for $i = k+1, \cdots, N$.

Theorem 2: The algorithm MA(C) gives an optimal solution of the problem MA(C) in $O(N \log N)$ time.

Proof: The sorting procedure requires $O(N \log N)$ time. And the computation time of a round of while loop is $O(N)$. Therefore, the whole computation of the algorithm MA(C) needs $O(N \log N)$ time. □

3.2 Transform Methods for MA Problem

Let $x^* = (x_1^*, \cdots, x_N^*)$ be the optimal continuous solution of problem MA(C). The index k still indicates the threshold subcarrier. Let $S^+ = \{1, \cdots, k\}$ which is constructed by eliminating the indices of subcarriers with zero optimal solution (i.e., $x_i^* = 0$, $i = k+1, \cdots, N$) from S. Let S' denote the set of indices of the non-integer solution, i.e., $S' = \{i \in S^+ : x_i - \lfloor x_i \rfloor > 0\}$.

If we denote $\delta_i = x_i^* - \lfloor x_i^* \rfloor$, $i \in S'$ and $\sum_{i \in S'} \delta_i = r$, then r should be an integer because the constraint $\sum_{i \in S^+} x_i^* = R$ would not be satisfied otherwise. And $r < |S'|$ due to $0 < \delta_i < 1$. To avoid the trivial case $r = 0$ ($S' = \varnothing$) in which all solutions x_i^*, $i \in S^+$ are integers, we assume that $r \ge 1$. For the convenience of description, the values of δ_i, $i \in S'$ are assumed to be listed in a descending order, i.e.,

$\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{|S'|}$.

With T denoting a subset of S' with $|T| = r$, we consider the following transform method (which is called TF(MA)) of converting the continuous solutions x_i^*, $i = 1, \cdots, k$ into the integer solutions $y_i(T)$.

$$y_i(T) = \begin{cases} x_i^* - \delta_i + 1, & i \in T, \\ x_i^* - \delta_i, & i \in S' - T, \\ x_i^*, & i \in S - S'. \end{cases} \tag{6}$$

Lemma 1: Let $F_{S'}(T)$ be the increase in objective function when the continuous solutions x_i^* are replaced with $y_i(T),\ i \in S$ by the TF(MA) with $T(\subset S')$, i.e., $F_{S'}(T) = \sum_{i \in S} f(y_i(T)) - \sum_{i \in S} f(x_i^*)$. Then the subset $T^* = \{1, \cdots, r\}$ of S' achieves the minimum value of $F_{S'}(T)$, i.e., $F_{S'}(T^*) = \min_{T \in S', |T| = r} \{F_{S'}(T)\}$.

Proof: From $a_i 2^{x_i^*} = 2^{A(k)},\ i = 1, \cdots, k$ and (6)

$$\begin{aligned} F_{S'}(T) &= \sum_{i \in T} (f_i(x_i^* - \delta_i + 1) - f_i(x_i^*)) - \sum_{j \in S' - T} (f_j(x_j^*) - f_j(x_j^* - \delta_j)) \\ &= \sum_{i \in T} a_i 2^{x_i^*} (2^{1-\delta_i} - 1) - \sum_{j \in S' - T} a_j 2^{x_j^*} (1 - 2^{-\delta_j}) \\ &= 2^{A(k)} \left[\sum_{i \in T} 2^{1-\delta_i} - |S'| + \sum_{j \in S' - T} 2^{-\delta_j} \right] \\ &= 2^{A(k)} \left[\sum_{i \in T} 2^{-\delta_i} - |S'| + \sum_{j \in S'} 2^{-\delta_j} \right]. \end{aligned} \tag{7}$$

Note that the two terms $|S'|$ and $\sum_{j \in S'} 2^{-\delta_j}$ in (7) are constant. The minimum of $F_{S'}(T)$ would be then attained when the remaining term $\sum_{i \in T} 2^{-\delta_i}$ has the least value. By $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{|S'|}$, we can conclude $T^* = \{1, \cdots, r\}$. □

Remark: The integer solution $y_i(T^*)$, $i \in S$ holds the constraint $\sum_{i \in S} y_i(T^*) = R$ because

$$\sum_{i\in S} y_i(T^*) = \sum_{i=1}^{r}(x_i^* - \delta_i + 1) + \sum_{i>r, i\in S'}(x_i^* - \delta_i) + \sum_{i\in S-S'} x_i^* = \sum_{i\in S} x_i^* - \sum_{i\in S'} \delta_i + r = R .$$

Theorem 3: Consider a continuous optimal solution $x^* = (x_1^*, \cdots, x_N^*)$ of the problem MA(C). And assume that $\sum_{i\in S'} \delta_i = r$. Then the integer solution $y(T^*) = (y_1(T^*), \cdots, y_N(T^*))$ obtained from the transform method TF(MA) is the optimal integer solution for the problem MA.

Proof: To prove this theorem, it suffices to show that any one-bit transfer from a subcarrier to the other in the integer solution $y(T^*) = (y_1(T^*), \cdots, y_N(T^*))$ would lead an increase of objective function. For the compact exposition, the notation $y_i(T^*)$ is abbreviated to y_i if no confusion arises.

Consider the case that a single bit of subcarrier j is transferred to the subcarrier i ($y_i \to y_i + 1,\ \ y_j \to y_j - 1$). While a bit augment to the subcarrier i, ($y_i \to y_i + 1$), would yield the increase of objective value as much as

$$f_i(y_i + 1) - f_i(y_i) = a_i(2^{y_i+1} - 1) - a_i(2^{y_i} - 1) = a_i 2^{y_i} = \begin{cases} 2^{A(k)-\delta_i+1}, & \text{for } i \in T^*, \\ 2^{A(k)-\delta_i}, & \text{for } i \in S' - T^*, \\ 2^{A(k)}, & \text{for } i \in S - S', \end{cases}$$

And one bit reduction from the subcarrier j, ($y_j \to y_j - 1$), would decrease the objective value as much as

$$f_j(y_j) - f_j(y_j - 1) = a_j(2^{y_j} - 1) - a_j(2^{y_j-1} - 1) = a_j 2^{y_j-1} = \begin{cases} 2^{A(k)-\delta_j}, & \text{for } T^*, \\ 2^{A(k)-\delta_i-1}, & \text{for } j \in S' - T^*, \\ 2^{A(k)-1}, & \text{for } i \in S - S'. \end{cases}$$

Note that $f_i(y_i + 1) - f_i(y_i)$ has the smallest value when subcarrier i is in $S'-T^*$, while $f_j(y_j) - f_j(y_j - 1)$ has the largest one when subcarrier j is in T^*. The amount of the net increase $\Delta f = f_i(y_i + 1) - f_i(y_i) - [f_j(y_j) - f_j(y_j - 1)]$ in the objective function by the one-bit transfer would be minimized when $i \in S' - T^*$ and $j \in T^*$.

Therefore, for our purpose, it is enough to show that the one-bit transfer ($y_i \to y_i + 1,\ y_j \to y_j - 1$), $i \in S' - T^*$ and $j \in T^*$ yields $\Delta f \geq 0$. Since $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{|S'|}$ and $T^* = \{1, \cdots, r\}$, it is easily seen that $\delta_i \leq \delta_j$, $i \in S' - T^*$, $j \in T^*$ which means that $\Delta f = 2^{A(k)}(2^{-\delta_i} - 2^{-\delta_j}) \geq 0$. □

Remark: To transform the continuous optimal solution into the optimal integer solution by the transform method TF(MA), only a sorting procedure (i.e., $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{|S'|}$) is just required, so the computational complexity of TF(MA) becomes $O(N \log N)$.

4. Transform Method for RA Problem

4.1 Optimal Solution for RA(C) Problem

Some notations introduced in the previous section need to be restated for RA(C) problem. First the threshold subcarrier is redefined as follows: The threshold subcarrier $k (\in S)$ satisfies: $B(k) - a_k > 0$ and $B(k+1) - a_{k+1} \leq 0$, where $B(j) = (P + \sum_{i=1}^{j} a_i)/j$. Note that the optimal continuous solution of RA(C) always consumes the entire amount of power resource, and we can replace the inequality constraint (4) with equality constraint $\sum_{i \in S} f_i(x_i) = P$. Let v_o and v_i, $i \in S$ denote the Lagrangean multipliers associated with constraints (4) and the continuous relaxation of (5), i.e., $x_i \geq 0$, $i \in S$, respectively.

Theorem 4: If the subcarrier $k \in S$ is a threshold one, the KKT conditions are satisfied by the following values v_0^*, x_i^*, and v_i^*, $i \in S$.

$$v_0^* = \frac{1}{B(k) \cdot \ln 2}.$$

$$x_i^* = \log_2 B(k) - \log_2 a_i, \quad i = 1, \cdots, k.$$

$$x_i^* = 0, \quad i = k+1, \cdots, N.$$

$$v_i^* = 0, \quad i = 1, \cdots, k.$$

$$v_i^* = v_0^* \cdot a_i \ln 2 - 1, \quad i = k+1, \cdots, N.$$

[Algorithm RA(C) for the Problem RA(C)]

Input: a_i, $i \in S$

Sort a_i in increasing order (let the indices of a_i be arranged so that $a_1 \le a_2 \cdots \le a_N$ holds).

$k \leftarrow 1$;

while ($k \le N$) do{

$B(k) \leftarrow (P + \sum_{i=1}^{k} a_i) / k - a_k$

if $(P + \sum_{i=1}^{k} a_i) / k - a_k < 0$, then exit while;

$k \leftarrow k+1$;

}

Output: $x_i^* = \log_2 B(k) - \log_2 a_i$, $f_i(x_i^*) = B(k) - a_i$, for $i = 1, 2, \cdots, k$.

$x_i^* = 0$, $f_i(x_i^*) = 0$, for $i = k+1, \cdots, N$.

Theorem 5: The algorithm RA(C) correctly computes an optimal solution of the problem RA(C) in $O(N \log N)$ time.

4.2 Transform Methods for RA Problem

Let $x^* = (x_1^*, \cdots, x_N^*)$ be the optimal continuous solution of problem RA(C). Note that $x_i^* > 0$, $i = 1, \cdots, k$ and $x_i^* = 0$, $i = k+1, \cdots, N$. Let r, ξ be the integer and fractional parts of $\sum_{i \in S'} \delta_i$, i.e., $r = \lfloor \sum_{i \in S'} \delta_i \rfloor$, $\xi = \sum_{i \in S'} \delta_i - r$. Notations of δ_i, S^+ and S' have the same meaning as in the previous subsection.

Theorem 6: $(\sum_{i \in S} \lfloor x_i^* \rfloor + r)$ becomes an upper bound on the optimal objective value of problem RA.

Proof: Since the optimal objective value of problem RA is less than or equal to the one of problem RA(C), $\lfloor \sum_{i \in S} x_i \rfloor$ becomes obviously an upper bound on the objective value of problem RA. The following then completes the proof:

$$\left\lfloor \sum_{i\in S} x_i \right\rfloor = \left\lfloor \sum_{i\in S^+} (x_i - \delta_i) + \sum_{i\in S^+} \delta_i \right\rfloor = \sum_{i\in S^+} \lfloor x_i \rfloor + \left\lfloor \sum_{i\in S^+} \delta_i \right\rfloor$$
$$= \sum_{i\in S^+} \lfloor x_i \rfloor + r + \lfloor \xi \rfloor = \sum_{i\in S^+} \lfloor x_i \rfloor + r = \sum_{i\in S} \lfloor x_i \rfloor + r. \qquad \square$$

From the above theorem, we can first see that if $r = 0$, $(\lfloor x_1^* \rfloor, \cdots, \lfloor x_k^* \rfloor, x_{k+1}^*, \cdots, x_N^*)$ becomes an optimal integer solution of the problem RA. Hence assume that $r \geq 1$ and the values of δ_i, $i \in S'$ are listed in descending order as in the previous subsection. Let $T(\kappa)$ be a subset of S' whose cardinality $(|T(\kappa)|)$ is $\kappa (1 \leq \kappa \leq r)$ and $y(T(\kappa)) = (y_1(T(\kappa)), \cdots, y_N(T(\kappa)))$ denotes the integer solution induced from the continuous solution by the following transformation (which is called TF(RA)):

$$y_i(T(\kappa)) = \begin{cases} x_i^* - \delta_i + 1, & i \in T(\kappa), \\ x_i^* - \delta_i & i \in S' - T(\kappa), \\ x_i^*, & i \in S - S'. \end{cases}$$

Lemma 2: Let $G_{S'}(T(\kappa))$ denote the change in the amount of the required power when the optimal continuous solution x^* is replaced with the integer solution $y(T(\kappa))$, i.e., $G_{S'}(T(\kappa)) = \sum_{i\in S} f_i(y_i(T(\kappa))) - \sum_{i\in S} f_i(x_i^*)$. Then the subset $T^*(\kappa) = \{1, \cdots, \kappa\}$ of S' achieves the minimum value of $G_{S'}(T(\kappa))$, i.e., $G_{S'}(T^*(\kappa)) = \min_{T(\kappa)\in S'} \{G_{S'}(T(\kappa))\}$.

The proof is similar to the case of Lemma 1, and is omitted here.

Theorem 7: Consider a continuous optimal solution $x^* = (x_1^*, \cdots, x_N^*)$ of the problem RA(C). Assume that $\kappa^* (1 \leq \kappa^* \leq r)$ satisfies $G_{S'}(T^*(\kappa^*)) \leq 0$ and $G_{S'}(T^*(\kappa^* + 1)) > 0$, where $T^*(\kappa^*) = \{1, \cdots, \kappa^*\}$, then the integer solution $y(T^*(\kappa^*)) = (y_1(T^*(\kappa^*)), \cdots, y_N(T^*(\kappa^*)))$ obtained by transform TF(RA) with $T^*(\kappa^*)$ is the optimal integer solution of the problem RA.

Proof: Note that with transform TF(RA), increasing the cardinality of $T(\kappa)$, κ, would improve the objective function value of the problem RA. Since for a fixed κ, the minimum of $G_{S'}(T(\kappa))$ is attained at $T^*(\kappa) = \{1, \cdots, \kappa\}$ by Lemma

2, the maximum value of κ satisfying the power constraint (4) would be limited to κ^*. Therefore, the greatest value of the objective function of the problem RA achieved by the transform method TF(RA) is when $T^*(\kappa^*) = \{1, \cdots, \kappa^*\}$.

Now, we shall show that the integer solution $y(T^*(\kappa^*)) = (y_1(T^*(\kappa^*)), \cdots, y_N(T^*(\kappa^*)))$ becomes indeed an optimal integer solution of the problem RA. From now on, the notation $y_i(T^*(\kappa^*))$ will be abbreviated to y_i for the convenience of exposition. Note that one-bit transfer from on subcarrier to another does not need to be considered because no improvement on objective function value occurs.

We first explore the possibility of increasing the objective function value of the problem RA through any single bit addition in the integer solution given in (8). Note that the single bit addition to the subcarrier $i \in S'$ (i.e., $y_i \to y_i + 1,\ i \in S'$) leads to the violation of the power constraint, which is due to the definition of κ^*.

Now consider the case of $i \in S - S'$. The marginal power required for the bit addition to the subcarrier $i \in S - S'$ is as follows:

$$f_i(y_i + 1) - f_i(y_i) = f_i(x_i^* + 1) - f_i(x_i^*) = a_i 2^{x_i^*} = B(k),\ i \in S - S'.$$

Hence the total amount of the required power after adding a single bit to the subcarrier $i(\in S - S')$ becomes $P + G_{S'}(T^*(\kappa^*)) + B(k)$. And since

$$\begin{aligned} G_{S'}(T^*(\kappa^*+1)) - G_{S'}(T^*(\kappa^*)) &= f_{\kappa+1}(x^*_{\kappa^*+1} - \delta_{\kappa^*+1} + 1) - f_{\kappa+1}(x^*_{\kappa^*+1} - \delta_{\kappa^*+1}) \\ &= a_i(2^{x^*_{\kappa^*+1} - \delta_{\kappa^*+1} + 1} - 1) - a_i(2^{x^*_{\kappa^*+1} - \delta_{\kappa^*+1}} - 1) = a_i 2^{x^*_{\kappa^*+1} - \delta_{\kappa^*+1}} = B(k) 2^{-\delta_{\kappa^*+1}} \end{aligned}$$

and $0 < \delta_{\kappa^*+1} < 1$, the following holds

$$\begin{aligned} G_{S'}(T^*(\kappa^*)) + B(k) &> G_{S'}(T^*(\kappa^*)) + B(k) 2^{-\delta_{\kappa^*+1}} \\ &= G_{S'}(T^*(\kappa^*+1)) > 0. \end{aligned}$$

We can conclude that adding a single bit to the subcarrier $i(\in S - S')$ also violates the power constraint, from which we conclude that any additional assignment of single bit in the integer solution given in (8) is not allowed.

Next, consider the case of 3-bit rearranging between two subcarriers (i.e., $y_i \to y_i + 2$ and $y_j \to y_j - 1$, $i, j \in S$). While the two-bit addition to the subcarrier i, ($y_i \to y_i + 2$), would increase the amount of power required as much as

$$\Delta_1 = f_i(y_i + 2) - f_i(y_i) = a_i 2^{y_i} \cdot 3 = \begin{cases} B(k) \cdot 2^{-\delta_i + 1} \cdot 3, & \text{for } i \in T^*(\kappa^*), \\ B(k) \cdot 2^{-\delta_i} \cdot 3, & \text{for } i \in S' - T^*(\kappa^*), \\ B(k) \cdot 3, & \text{for } i \in S - S', \end{cases}$$

the one bit reduction from the subcarrier j, ($y_j \to y_j - 1$), would decrease the amount of power required as much as

$$\Delta_2 = f_j(y_j) - f_j(y_j - 1) = a_j 2^{y_j - 1} = \begin{cases} B(k) \cdot 2^{-\delta_j}, & \text{for } j \in T^*(\kappa^*), \\ B(k) \cdot 2^{-\delta_j - 1}, & \text{for } j \in S' - T^*(\kappa^*), \\ B(k) \cdot 2^{-1}, & \text{for } j \in S - S'. \end{cases}$$

The net increase $\Delta f = \Delta_1 - \Delta_2$ in the amount of power required for 3-bit rearranging would be minimized when Δ_1, Δ_2 have the least and the greatest values, respectively, which would be obtained when $i \in S' - T^*(\kappa^*)$ and $j \in T^*(\kappa^*)$. Therefore, to complete the proof, it is sufficient to show that the 3-bit rearranging ($y_i \to y_i + 2$, $i \in S' - T^*(\kappa^*)$ and $y_j \to y_j - 1$, $j \in T^*(\kappa^*)$) violates the power constraint, making eventually the integer solution given by TF(RA) be optimal.

We first note the total amount of the required power after 3-bit rearranging is $P + G_{S'}(T^*(\kappa^*)) + \Delta f$, where Δf is expressed as follows:

$$\Delta f = B(k) \cdot 2^{-\delta_i} (3 - 2^{-\delta_i - \delta_j}),\ i \in S' - T^*(\kappa^*),\ j \in T^*(\kappa^*).$$

Since $0 < \delta_i < \delta_j < 1$ and $0 < \delta_i \le \delta_{\kappa^* + 1} < 1$, the following can be easily obtained

$$2^{-\delta_i} (3 - 2^{\delta_i - \delta_j}) > 2^{-\delta_i} \cdot 2 = 2^{1 - \delta_i} \ge 2^{1 - \delta_{\kappa^* + 1}}. \tag{9}$$

The relation (9) gives

$$\begin{aligned} G_{S'}(T^*(\kappa^*)) + \Delta f &= G_{S'}(T^*(\kappa^*)) + B(k) \cdot 2^{-\delta_i}(3 - 2^{-\delta_i - \delta_j}) \\ &> G_{S'}(T^*(\kappa^*)) + B(k) \cdot 2^{-\delta_{\kappa^*+1}} \\ &= G_{S'}(T^*(\kappa^* + 1)) > 0, \end{aligned}$$

which implies that the 3-bit rearranging violates the power constraint. □

Remark: To get the optimal integer solution by TF(RA), it suffices to rearrange δ_i, $i \in S'$ as $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{|S'|}$. So the computational complexity of TF(RA) becomes $O(N \log N)$.

5. Conclusion

In this study, we have developed the methods of transforming the optimal continuous solutions into the optimal discrete ones for both MA and RA problems. The computational complexity $O(N \log N)$ of the proposed methods depends only on the number of subcarriers. It is expected that the strength of much simple form of our algorithms not only exhibits an outstanding implementation efficiency compared to the existing algorithms, but also makes a key role of a subroutines to develop some heuristics for the more complex problems.

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