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The Histories of the Mathematical Concepts of Infinity and Limit in a Three-fold Role

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The purpose of this study is to classify a three-fold role of the history of mathematics through epistemological analysis. Based on the history of infinity and limit, the "potential infinity" and "actual infinity" discourses are described using four different historical epistemologies. The interdependence between the mathematical concepts is also addressed. By using these analyses, three different uses of the history of mathematical concepts, infinity and limit, are discussed: *past*, *present*, and *future* use.

I. INTRODUCTION

History of mathematics has great implications for the teaching and learning of mathematics. First and foremost, what history offers the students and teachers of mathematics is motivation (Kleiner, 2001). Moreover, the role of history of mathematics in mathematics education is multifaceted. For instance, the role of history in the teaching of algebra may be different from the role of history in the learning of geometry (Fauvel & van Maanen, 2001). This study attempts to answer a specific question raised from the way history of mathematics can be used in mathematics education. The specific question about the role of history of mathematics is: How can mathematics educators use the history of mathematics? Such investigation may provide insight into the role of history of mathematics with implications for student learning and teacher education.

In order to have a better understanding of how the mathematics education researcher use the history of mathematics, I will describe the interdependence between the mathematical concepts of infinity and limit in the history of mathematics through four different epistemological developments: *intuitive finitism*, *infinitism in the context of infinitesimals*, *infinitism in the context of variables*, and *actual infinitism*. First, I will base intuitive finitism on ancient Greek thinking. Then I will discuss infinitism in the context of infinitesimals, which was anchored in the 17th and 18th centuries, followed by infinitism in the context of variables demonstrated by Cauchy and Weierstrass. Next, actual infinitism expressed by Cantor and Dedekind will be discussed. Finally, on the basis of the four different epistemological analyses, I will present the idea that there are three different types of role of the history of mathematics in mathematics education research.

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II. INTUITIVE FINITISM

The story of infinity begins with the ancient Greeks, and leads to the notion of *intuitive finitism*. The Greek word 'peras' refers to limit or bound (Moore, 1990). The Greek word for infinity is 'apeiron.' The word 'apeiron' means no 'peras,' unbounded, indefinite, or undefined (Boyer, 1949; Moore, 1990; Rucker, 1995). The ancient Greeks used the concept of infinity with observations such as endless time, unendingly subdivided space and time, and unbounded space (Allen, 1999). In the thinking of the ancient Greeks, a finite thing is more highly valued than an infinite thing, because infinity is valueless and cannot be formed, unlike finiteness. For instance, the Pythagoreans considered natural numbers as the key to everything (Moore, 1990; Rucker, 1995). They created the cosmic theory of (natural) numbers based on the principle of 'peras.' Plato (427 BC-347 BC) also thought that the apeiron was the indeterminate beyond a given range. Thus the apeiron was valueless, but became valuable, when the peras was imposed on it in an ordered way (Moore, 1990).

In ancient Greek consciousness, infinity was something to be kept out of mathematics at any cost (Maor, 1987). This avoidance on the part of ancient Greek mathematicians was influenced by the infamous paradoxes of Zeno (Rotman, 1993). One of Zeno's paradoxes is the paradox of the runner. Suppose that a runner wants to move from point A to point B. To get to point B, he must pass the midpoint between the two points, then pass the three-quarter point, and so on ad infinitum. Zeno's argument was that the runner can never finish a race because he always has to reach the midpoint of the race's remaining segment, and then the midpoint of the next remaining segment, and so on. Another effect of Zeno's arguments is Aristotle's distinction between a potential infinity and an actual infinity (Rotman, 1993).

Because Aristotle thought that Zeno's paradox of the runner was based on the actual infinity, he created a possible alternative to actual infinity, the notion of the potential infinite. The potential infinite means that the infinite exists potentially but not in actuality. For instance, the progression of integers is potentially infinite because we can always add one to get a larger number. Since no single infinite set of integers is ever completed, there is only an infinite process trying to create it. Infinity, therefore, existed not as a completed form but as a potential construct in ancient Greek consciousness. This ancient Greek conception became the source of the potential infinite as one of the conceptions in the current perspective of infinity.

For the ancient Greeks, such understandings of infinity dominated mathematical thoughts. Infinity was perceived as an incomplete state and something which is valueless and cannot be experienced in the long run. This epistemology can be called as intuitive finitism, because it was based on geometrical intuition and excluded infinite processes. In intuitive finitism, infinity does not exist in actuality, but rather as a potential construct based on the principle of 'peras.' Thus, there is no concept of limit as the completion of an infinite process in this epistemology. Although there is the notion of bounded processes, there is no concept of limit as a concrete bounding entity. Based on the paradoxes of Zeno and others, skepticism about actual infinity prevented Greek mathematicians from developing the notion of limit.

Because the concept of actual infinity made the ancient Greek mathematicians skeptical, they seemed to reject it in order to maintain the establishment of a firm science, geometry. For instance, to prove a geometric problem, the Greek mathematicians developed the method of exhaustion without carrying out an infinite number of steps. One such example is the idea of Eudoxus (408 BC-355 BC) to escape the usage of infinity. He used the method of exhaustion in the proof of the following proposition: If the areas of the circles are A and A' and their diameters are d and d' respectively, then $A : A' = d^2 : (d')^2$ (Boyer, 1949, p. 34). First of all, he supposed that the proportion was not true. Then, by using reduction to absurdity, he established the truth of the proposition based on a contradiction. He did not consider the concept of infinity in this proof. However, he did not define the area of a circle in the proof and needed to expect the limit value (d^2) in advance. The above shortcomings in his proof originated because the concept of limit was foreign to his epistemology of mathematics.

At the end of ancient times, prevalent philosophy about infinity shifted from finitism to infinitism through the influence of Christianity. In the Middle Ages, based on Christian theology, infinity as a divine property was actively valued, and philosophers valued it more than a finite entity. By considering infinity as an object of recognition, medieval thinkers tried to explain the notion of infinity with logical concepts. However, based on scholastic philosophy, most medieval conceptualizations were not apt to handle the infinitude of any entities other than God (Rucker, 1995). Thus medieval thinkers believed that there was no actually infinite collection in the created world (Rucker, 1995). But, one interesting paradox to the medieval thinkers was the relationship between all points on the circumferences of two small and large circles that share a common center. There are two infinities that are different in the sense of inclusion and the same in the sense of one-to-one correspondence, by drawing a radial segment from the shared center point to the larger circle (Rucker, 1995).

III. INFINITISM IN THE CONTEXT OF INFINITESIMALS

Infinitism in the context of infinitesimals which was anchored in the 17th and 18th centuries involves the notion of the potentially infinite. The limit concept was included but not clarified. With the developments of astronomy and dynamics in the 16th century, there was an urgent need to find the area, volume, and length of a curved figure. In the 17th century, to find the areas of fan-shaped figures and the volumes of solids such as apples, Kepler used infinitesimal methods (Boyer, 1949). For instance, he showed the volumes of solids by considering them as composed of an infinite number of infinitesimal generating elements. Interestingly, his infinitesimal methods were well represented in the subject of the measurement of wine casks to solve uncertainties in measuring their volume (Boyer, 1949). However, Kepler did not seem to distinguish a proof by infinitesimal elements from a proof by the method of exhaustion (Boyer, 1949, p. 109).

While Kepler regarded the infinite as having metaphysical significance, Cavalieri did not argue it explicitly (Boyer, 1949). Cavalieri introduced the word indivisibles. He used the word indivisible to characterize infinitely small lines and areas without thickness for the investigation of an area and a volume respectively (Boyer, 1949; Kleiner, 2001). He did not focus on the total sum of indivisibles. Rather, he focused on the correspondence between two indivisibles of two figures (Boyer, 1949, p. 118). In fact, Cavalieri never explained how the sum of indivisibles without thickness could compose an area or volume (Boyer, 1949, p. 122). Both Kepler and Cavalieri regarded area and volume as geometric concepts rather than numerical values.

The above epistemology is called as infinitism in the context of infinitesimals. In infinitism in the context of infinitesimals, infinitesimals mean infinitely small quantities as tools to be used in calculus. In other words, they were used to find the areas and the volumes as made up of an infinite number of small quantities. These epistemologists concentrated on the development of the infinitesimal calculus pragmatically. Although the concept of limits was involved in the infinitesimal calculus, it had not been clarified.

Later, Pascal demonstrated the method of infinitesimals as the concept of the total sum. In other words, he established that the total sum of infinitesimal lines or areas is the figure's or shape's area or volume, respectively. These concepts of infinitesimals became the basis of calculus. Leibniz developed the calculus based on infinitesimals (Kleiner, 2001; Moore, 1990). He also developed a theory of infinite series. In addition, Leibniz changed infinitesimal geometry to infinitesimal analysis as a type of symbolic mathematics. However, Leibniz failed to construct a foundation in the analysis of infinitesimals (Boyer, 1949; Kleiner, 2001). In other words, he recognized mathematical truth, but could not establish the formal foundations. Because mathematicians found the useful infinitesimal calculus during the 17th and 18th centuries, they had no compelling reason to construct firm theoretical foundations for their subject (Kleiner, 2001).

Throughout the 18th century, no certain conceptual bases of the calculus emerged (Edwards, 1979; Kleiner, 2001). Thus, criticism and controversy existed in the process of the development of calculus. For instance, Euler achieved a conceptual development by making the notion of function central in calculus around the mid-18th century (Kleiner, 2001). He also pointed out inconsistencies in infinite series. As an example, if we consider the infinite sum $(1-x)/(1+x)^2 = 1-3x+5x^2-7x^3+...$ and substitute 1 instead of x, then the result $1-3+5-7+...= 0$ (Boyer, 1949). Another example is the result, $1-1+1-1... = 1/2$, if the infinite sum $1/(1+x) = 1-x+x^2-x^3+...$ is considered when x equals 1.

At the end of the 18th century, there were also inconsistencies and absurdities about infinitesimal magnitudes (Rotman, 1993). There was a lack of rigor to make interpretations of calculus in terms of infinitesimals. For instance, Berkeley criticized the use of infinitesimals in calculus. He pointed out an inconsistency in finding the derivative $nx^{n-1}$of x^n from the increment, $(x+d)^n = nx^{n-1}d + 1/2n(n-1)x^{n-2}d^2 + ... + d^n$, by first dividing by d, supposing that d is non-zero and then setting d equal to zero (Edwards, 1979, p. 294). Therefore, mathematicians faced a serious crisis within mathematics due to the inconsistencies and absurdities in calculus.

IV. INFINITISM IN THE CONTEXT OF VARIABLES

In *infinitism in the context of variables* by Cauchy and Weierstrass, the notion of limit emerged as the underlying concept of calculus needed to remedy uncertainties and make the infinitesimal calculus more rigorous. Infinity was also the potential infinite in this epistemology. Cauchy and Weierstrass were pioneers for a movement toward a rigorous calculus in the beginning of the 19th century (Boyer, 1949; Kleiner, 2001; Moore, 1990; Rotman, 1993).

It was during the 19th century that the concept of limit became the basis of calculus (Kleiner, 2001). In *Cours d'analyse*, Cauchy defined the limit concept as follows: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others (Boyer, 1949, p. 272)." Even though Cauchy did not define the limit using the modern epsilon-delta definition, he used epsilon-delta arguments in the various proofs involving limits (Kleiner, 2001). According to Cauchy, the limit is a number, whereas a derivative, an integral, and an infinite series are the limits (Hairer and Wanner, 1995). With the basis of the arithmetical definition of limit, Cauchy considered infinitesimals as no more than variables based on the function concept of relations between variables (Boyer, 1949). Cauchy defined an infinitesimal as a variable whose limit is zero (Edwards, 1979; Kleiner, 2001).

To make the basis of the calculus more rigorous, Weierstrass considered a precise algebraic limit definition instead of Cauchy's intuitive conception (Kleiner, 2001). Weierstrass's definition of the limit of a function is as follows: "The number L is the limit of the function f(x) for $x = x_0$ if, given any arbitrarily small number, epsilon, another number, delta, can be found such that for all values of x differing from x_0 by less than delta, the value of f(x) will differ from that of L by less than epsilon"(Boyer, 1949, p. 287). Weierstrass replaced infinitesimals with his epsilon-delta formulation in calculus (Boyer, 1949; Kleiner, 2001). According to the epsilon-delta definition of Weierstrass, the limit of a continuous variable became the fundamental concept of the calculus without infinitesimals. In fact, Cauchy's and Weierstrass's fundamental ideas were based on the *method of exhaustion*, because they thought of a curved figure as the limit of a sequence of polygons (Moore, 1990).

To Cauchy and Weierstrass, the infinite indicated the potential infinite of Aristotle (Boyer, 1949). The concept of limit became the fundamental concept of calculus based on the epsilon and delta variables in *infinitism in the context of variables*. Limits, in this epistemology, became an arithmetical concept rather than a geometrical concept (as they were in *infinitism in the context of infinitesimals*). The concept of the limit which is based on variables became the rigorous underlying concept of calculus in the former epistemology, while uncertainties and inconsistencies of infinitesimals are the basis of the infinitesimal calculus in the latter epistemology.

V. ACTUAL INFINITISM

In the epistemology of *actual infinitism* demonstrated best by Cantor and Dedekind, infinity is the actual infinite rather than a potential construct, unlike the three previous epistemologies. Limit is also the fundamental concept of calculus. In the view of Cauchy and Weierstrass, an infinitesimal was a variable whose limit is zero and the limit concept involved only the definition of number (Boyer, 1949). However, this definition implicitly presupposes the existence of infinite sets (Boyer, 1949).

In order to complete Weierstrass' foundations of arithmetic, Dedekind and Cantor developed the theory of infinite sets (Boyer, 1949). In other words, during the last thirty years of the 19th century, the construction of the real number system was the main progress for the rigorization of calculus (Edwards, 1979). Without a full understanding of the real number system, it was impossible to construct firm foundations of calculus. One such example is an infinite sequence of rational numbers which can have the limit as an irrational number. Dedekind thought that the difficulty in the limit concept would be solved by understanding irrational numbers arithmetically (Boyer, 1949; Moore, 1990). Therefore, the existence of the limit was an important issue in 1872 (Boyer, 1949).

To complete the definition of real numbers, Dedekind defined that "A system S is said to be infinite when it is similar to a proper part of itself; in the contrary case S is said to be a finite system" (Boyer, 1949, p. 296). In this definition, Dedekind handled the actual infinite instead of the potential infinite. Based on the continuity property of the real number system, Dedekind constructed the real numbers, whereas Cantor based his construction on the concept of limit (Edwards, 1979). Cantor and Dedekind considered the set of rational numbers as the starting point for the construction of the set of all real numbers (Edwards, 1979).

However, Cantor was not satisfied with only defining infinite sets. With the developments of the equivalence of two infinite sets and degrees of infinity, Cantor comprehended infinite sets as something actually existing in front of his eyes and compared two infinite sets based on degrees of infinity. Beyond our intuition, Cantor opposed the idea that there is no actual infinity. Interpretation of infinity as a potentiality rather than an actuality dominated mathematics until the Cantorian revolution in the 19th century. Though Cantor discovered the property of actual infinity, the result was shocking to him. In his letter to Dedekind, Cantor expressed his difficulties well. He exclaimed, "I see it, but I don't believe it!" (Aczel, 2000; Maor, 1987).

Dedekind and Cantor considered the limit of a continuous variable as the fundamental concept of calculus, like Weierstrass (Boyer, 1949, p. 298). However, they developed the theory of the infinite sets as an actuality rather than a potentiality to complete Weierstrass' foundations of calculus. This suggests an epistemic stance of actual infinitism because infinity is seen as not only the potential infinite but also the actual infinite. Since infinity in infinitism in the context of variables is always related to the concept of variables, it is an infinite in the process of production, that is, a potential infinity. Therefore, the concept of the actual infinity does not exist in this epistemology. To handle the difficulties in the limit concept such as the existence of the limit, the theory of the actual infinity was developed in actual infinitism. The epistemology by Cauchy and Weierstrass was augmented by actual infinitism, based on Dedekind's continuum and Cantor's theory of the actual infinite (Rotman, 1993).

VI. CONCLUSION

Through the four different epistemologies - *intuitive finitism*, *infinitism in the context of infinitesimals*, *infinitism in the context of variables*, and *actual infinitism* - the interconnectedness between infinity and limit and their historical development was discussed. In the first three of these epistemologies, infinity as a process can be seen as it is considered a potential construct in the process of production. In actual infinitism, the actual infinite can be observed as an object: an unproblematic completed infinity. If we take infinity as the potential infinite, the idea of a limited process can be examined. If we consider infinity as the actual infinite, the limit of these infinite processes can be scrutinized as something actually existing. These developments not only constructed firm foundations of calculus, but also completed the definition of real numbers.

The historical analysis of infinity and limit can be presented with a three-fold role of *past*, *present*, and *future* for the mathematics education researcher. First, the concepts of infinity and limit need to be seen from the perspective that focuses on where the words came from and how they have been developed for the *past* use. For instance, the first use of the term "infinity" is generally credited to the ancient Greeks and they kept it out of mathematics for consistency until the Middle Ages. Then, the concept of infinity has been developed through the four different stages in conjunction with the concept of limit. Although it is a simple statement without a mathematical concept, some students and teachers gain some good insights: mathematics is an ongoing field; some mathematical concepts were rejected in order to maintain the establishment of mathematics in the past, but they are now accepted in a more rigorous development of mathematics. In addition, the different historical developments of understanding infinity and limit help students and teachers to make more sense of mathematics, humanize the subject, recognize the continuous and continuing development of mathematics, and foster an appreciation of the multicultural inheritance and culturally dependent nature of the subject. The four different developmental stages of infinity and limit can also show how difficult it has been to expand mathematical concepts as they are through epistemological analysis. Students and teachers are often faced with the problems and the results without having been given time to think about difficulties behind the problems in the history of mathematics.

Generally, numerous researchers have noted the past use of the history of mathematics through experimental data for the field of mathematics education. For instance, mathematics history lessons helped students to humanize the subject and make it more accessible to them (Bartolome, 1994). In addition, the history of mathematics helped students develop more positive attitudes toward mathematics as well as learning mathematics (McBride & Rollins, 1977; Furinghetti, 1997; Troutman & McCoy, 2008). Students also tended to have a deeper conceptual understanding of the topics with knowledge of the history of mathematics which helped them understand how mathematical thinking patterns behind the discipline were developed (Bartolome, 1994). However, how to make a plan for introducing the history of mathematics in a teaching sequence including steps from informal to formal mathematics is still a big question unanswered to teachers and mathematics education researchers, especially for advanced mathematical concepts such as infinity and limit.

Second, in order to utilize the history of mathematics for the *present* use of mathematics education research, it may provide insight into current mathematical structures. For example, in the development of the discourse on infinity, two different types of discourse have been appeared in the history of mathematics. The "potential infinity"discourse which is grounded in the potential infinite as a process is one thing. The "actual infinity" discourse which is derived from the actual infinite as an object is another. The concept of potential infinity dominated mathematics until the Cantorian revolution in the 19th century because actual infinity had been banned from mathematics to preserve the consistency of our usual logic. As evolved in the history of mathematics, the "potential infinity" discourse is developed with our intuitions and general intellectual development in the stages of learning infinity. However, the "actual infinity" discourse is beyond our intuitions because the concept of actual infinity conflicts with our intuitive understanding of infinity. The historical dimension described above to the learning of infinity brings out two important developmental stages in a dialectical way. This kind of analysis of mathematical structure from the history of mathematics can shed an additional light on current perspectives on mathematical generalizations and abstractions.

For instance, some research focused on cognitive obstacles of learning infinity and limit in order to approach student difficulties of learning on these concepts. Some of those (Fischbein, Tirosh, & Hass, 1979; Tall, 1992 Tirosh, 1992) emphasized the importance of intuition. Tall's work on intuition of infinity (Tall, 1992) and the work of Fischbein and colleagues work on infinity (Fischbein et al., 1979) are representative. Their main point is that a main source of difficulties is the intuitions students use. According to Fischbein and colleagues (1979), intuition is "direct, global, self-evident forms of knowledge". One of the intuitions of infinity is that the whole includes its parts because it contains the part. Another example is a concept of infinity as endlessness. Fischbein and colleagues address that the resistance of those intuitions continues through age and teaching influences, starting with age 12. They believe that the intuition of infinity is affected by the schema at the stage of intellectual development.

Finally, as for the *future* use of mathematics education, it should be noted that the histories of the mathematical concept of infinity and limit have been interwoven since their beginning. In spite of the mutual interdependence of the concepts of limit and infinity, there has been little research to examine students' understandings of and difficulties with both concepts simultaneously. To more fully understand the notions of infinity and limit and their relationships, more research on both infinity and limit together is needed. Thus, the interconnectedness between infinity and limit in the history of mathematics justifies the need for an additional approach to study the learning of infinity and limit simultaneously.

There is a limited number of studies on the interconnectedness in learning these concepts. Some researchers have stressed that the concept of infinity appears when we consider limits (Weller et al. 2004), as it is difficult to deal with the concept of infinity without the concept of limit (Monaghan, 2001). More precisely, when we contemplate the concept of infinity, we can get to the notion of limit (Mamona-Downs, 2001). The concept of limit exists through infinite processes. In other words, the concept of limit can be considered as the infinite processes which are conceptualized as an ultimate result (Lakoff et al., 2001). For instance, the limit concept in the notion of derivative implies the potential infinity with its geometric conception and a result through infinite processes. Historically, mathematical infinity and limit have also been interwoven with and dependent upon each other in mathematical contexts. Therefore, not only the concepts of limit and infinity but also their relationships are important in the curriculum. This additional approach to the concepts of infinity and limit may enlighten us about the ways students and teachers deal with these concepts as well as other notions such as continuity, differentiability, and integration in calculus. Likewise, an additional support for empirical evidence can be drawn from the history of mathematics.

It should be noted that combinations of historical and psychological perspectives need to deserve much attention.

The role of history of mathematics in mathematics education has been discussed through the three different uses of the history of mathematics. On the basis of the history of mathematics, information about where mathematical concepts came from, how their mathematical structures have been developed, and how they have evolved historically would enable students and teachers to understand that the problems they are doing lead to something bigger and to use what they learn to solve new kinds of problems they will inevitably face in the future. The new perspectives offered by the history of mathematics may improve students' motivation, resolve their learning difficulties through epistemological analysis, and influence the future research in mathematics education which value combinations of historical and psychological perspectives.

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세 가지 역할과 관련된 무한과 극한의 수학사

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이 연구의 목적은 인식론 분석을 통해 수학사의 세 가지 역할을 분류하는 것이다. 무한과 극한에 대한 수학사를 바탕으로 네 가지의 다른 인식론들을 통해 "잠재적 무한" 과 "실제적 무한" 담화를 묘사한다. 무한과 극한 개념의 상호 의존성을 또한 제시한다. 이러한 분석들을 이용하여 무한과 극한에 대한 수학사의 세 가지 다른 사용을 보이고자 한다 : 과거, 현제, 그리고 미래사용.

* **Key Words** : 수학사(History of mathematics), 잠재적 무한(potential infinity), 실제적 무한(actual infinity), 극한(limit), 인식론(epistemology)

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