DUAL PRESENTATION AND LINEAR BASIS
OF THE TEMPERLEY-LIEB ALGEBRAS

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Abstract. The braid group $B_n$ maps homomorphically into the Temperley-Lieb algebra $TL_n$. It was shown by Zinno that the homomorphic images of simple elements arising from the dual presentation of the braid group $B_n$ form a basis for the vector space underlying the Temperley-Lieb algebra $TL_n$. In this paper, we establish that there is a dual presentation of Temperley-Lieb algebras that corresponds to the dual presentation of braid groups, and then give a simple geometric proof for Zinno’s theorem, using the interpretation of simple elements as non-crossing partitions.

1. Introduction

Since Jones [7, 8] discovered the Jones polynomial for links by investigating representations of braid groups into Hecke algebras and Temperley-Lieb algebras, Temperley-Lieb algebras have played important roles in the quantum invariants of links and 3-manifolds. The Temperley-Lieb algebra $TL_n$ is defined on non-invertible generators $e_1, \ldots, e_{n-1}$ with the relations:

$$e_1 e_j = e_j e_1 \text{ for } |i - j| \geq 2; \quad e_i^2 = e_i; \quad e_i e_{i+1} e_i = \tau e_i$$

along with a complex number $\tau$ (see [8, 9]). Setting $\tau = 1/\delta^2$ and $e_i = (1/\delta) d_i$, we get an equivalent presentation, which is easily understood by diagrams, with non-invertible generators $d_1, \ldots, d_{n-1}$ and defining relations

$$d_i d_j = d_j d_i \text{ for } |i - j| \geq 2; \quad d_i^2 = \delta d_i; \quad d_i d_{i \pm 1} d_i = d_i$$

along with a complex number $\delta$ (see [10, 12]). It is well-known that the dimension of $TL_n$ is the $n$th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Setting $t$ such that $\tau^{-1} = 2 + t + t^{-1}$, and then setting $h_i = (t + 1)e_i - 1$, we get an alternative presentation of $TL_n$ with invertible generators $h_1, \ldots, h_{n-1}$ satisfying the
relations:

\begin{align*}
(1) \quad & h_i h_j = h_j h_i \quad \text{if } |i - j| \geq 2; \\
(2) \quad & h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1}; \\
(3) \quad & h_i^2 = (t-1) h_i + t; \\
(4) \quad & h_i h_{i+1} h_i + h_i h_{i+1} + h_{i+1} h_i + h_i + h_{i+1} + 1 = 0. 
\end{align*}

The braid group $B_n$ is defined by the Artin presentation, where the generators are $\sigma_1, \ldots, \sigma_{n-1}$ and the defining relations are

\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2; \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, n-2.
\end{align*}

The braid group $B_n$ maps homomorphically into the Temperley-Lieb algebra $TL_n$ under $\pi : \sigma_i \mapsto h_i$. There is another presentation [4] with generators $a_{ji}$ ($1 \leq i < j \leq n$) and defining relations

\begin{align*}
a_{ik} a_{ji} &= a_{ji} a_{ik} \quad \text{if } (l-j)(l-i)(k-j)(k-i) > 0; \\
a_{kj} a_{ji} &= a_{ji} a_{ki} = a_{ki} a_{kj} \quad \text{for } i < j < k.
\end{align*}

The generators $a_{ji}$’s are related to the $\sigma_i$’s by

\[ a_{ji} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_{i} \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}. \]

Bessis [1] showed that there is a similar presentation, called the dual presentation, for Artin groups of finite Coxeter type.

Both the Artin and dual presentations of the braid group $B_n$ determine a Garside monoid, as defined by Dehornoy and Paris [6], where the simple elements play important roles. Nowadays, it becomes more and more popular to describe simple elements arising from the dual presentation via non-crossing partitions. Non-crossing partitions are useful in diverse areas [1, 5, 2, 3, 11], because they have beautiful combinatorial structures.

Let $P_1, \ldots, P_n$ be the points in the complex plane given by $P_k = \exp(- \frac{2\pi}{n} k) i$. See Figure 1. Recall that a partition of a set is a collection of pairwise disjoint subsets whose union is the entire set. Those subsets (in the collection) are called blocks. A partition of $\{ P_1, \ldots, P_n \}$ is called a non-crossing partition if the convex hulls of the blocks are pairwise disjoint.

A positive word of the form $a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_{k-1},i_k}, i_1 > i_2 > \cdots > i_k$, is called a descending cycle and denoted $[i_1, i_2, \ldots, i_k]$. Two descending cycles $[i_1, \ldots, i_k]$ and $[j_1, \ldots, j_l]$ are said to be parallel if the convex hulls of $\{ P_{i_1}, \ldots, P_{i_k} \}$ and $\{ P_{j_1}, \ldots, P_{j_l} \}$ are disjoint. The simple elements are the products of parallel descending cycles.

We remark that the definition of simple elements depends on the presentations. For example, the simple elements arising from the Artin presentation are in one-to-one correspondence with permutations. Throughout this note, we consider only the simple elements arising from the dual presentation of braid groups as above.
Note that simple elements are in one-to-one correspondence with non-crossing partitions. Our convention is that if a block in a non-crossing partition consists of a single point, then the corresponding descending cycle is the identity (i.e., the descending cycle of length 0). In particular, the number of the simple elements is the $n$th Catalan number $C_n$, which is the dimension of $TL_n$. Zinno [13] established the following result.

**Theorem 1** (Zinno [13]). The homomorphic images of the simple elements arising from the dual presentation of $B_n$ form a linear basis for the Temperley-Lieb algebra $TL_n$.

We explain briefly Zinno’s proof. It is known that the ordered reduced words

$$(h_{j_1} h_{j_1 - 1} \cdots h_{k_1}) (h_{j_2} h_{j_2 - 1} \cdots h_{k_2}) \cdots (h_{j_p} h_{j_p - 1} \cdots h_{k_p}),$$

where $j_i \geq k_i$, $j_{i+1} > j_i$ and $k_{i+1} > k_i$, form a linear basis of $TL_n$, and Zinno showed that the matrix for writing the images of simple elements as the linear combination of the ordered reduced words is invertible. Because the number of the simple elements is equal to the dimension of $TL_n$, this proves the theorem.

In this note, we first establish that there is a dual presentation of $TL_n$. We are grateful to David Bessis for pointing out that the relation (4) in the Temperley-Lieb algebra presentation is equivalent to the forth relation in the dual presentation in the following theorem.

**Theorem 2** (Dual presentation of $TL_n$). The Temperley-Lieb algebra $TL_n$ has a presentation with invertible generators $g_{ji}$ ($1 \leq i < j \leq n$) satisfying the

\[ g_{ji} g_{kj} = g_{kj} g_{ji}, \]

\[ g_{ji} g_{kl} = g_{kl} g_{ji}, \] (for $i < k < j < l$)

\[ g_{ji} g_{kj} g_{jk} = 1, \] (for $i < j < k$)

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relations:

\[ g_{lk}g_{ji} = g_{ji}g_{lk} \quad \text{if} \quad (l - j)(l - i)(k - j)(k - i) > 0; \]
\[ g_{kj}g_{ji} = g_{ji}g_{ki} = g_{ki}g_{kj} \quad \text{for} \quad i < j < k; \]
\[ g_{ji} = (t - 1)g_{ji} + t \quad \text{for} \quad i < j; \]
\[ g_{ji}g_{kj} + tg_{kj}g_{ji} + g_{kj} + g_{ji} + tg_{ki} + 1 = 0 \quad \text{for} \quad i < j < k. \]

The new generators are related to the old ones by

\[ g_{ji} = h_{j-1}h_{j-2} \cdots h_{i+1}h_{i+1}^{-1} \cdots h_{j-2}h_{j-1}. \]

Using the above presentation, we give a new proof of Zinna’s theorem in §3. We exploit non-crossing partitions so as to make the proof easy and intuitive. For the proof, we show that any monomial in the \( h_i^{\pm 1} \)'s can be written as a linear combination of the images of simple elements. Therefore the images of simple elements span \( TL_n \). As a result, they form a linear basis of \( TL_n \) because the number of simple elements is equal to the dimension of \( TL_n \).

We remark that it seems possible to prove the linear independence of the images of the simple elements directly from the relations in the dual presentation of \( TL_n \) (without using the fact that the dimension of \( TL_n \) is the same as the number of simple elements), but that would be beyond the scope of this note because it would require repeating all the arguments used in the proof for the embedding of the positive braid monoid in the braid group.

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### 2. Dual presentation of the Temperley-Lieb algebras

Let \( D^2 \) be the disc in the complex plane with radius 2 and \( P_1, \ldots, P_n \) be the points in \( D^2 \) given by \( P_j = \exp(-\frac{2\pi j}{n}i) \). Let \( D_n = D^2 \setminus \{ P_1, \ldots, P_n \} \). The braid group \( B_n \) can be regarded as the group of self-homeomorphisms of \( D_n \) that fix the boundary pointwise, modulo isotopy relative to the boundary. The generators \( \sigma_i \) and \( a_{ji} \) correspond to the positive half Dehn-twists along the arcs \( P_iP_{i+1} \) and \( P_iP_j \), respectively.
Let $L_n = \{ P_i P_j \mid i \neq j \}$ be the set of line segments as in Figure 2 (a). We say that a pair $(u, v) \in L_n^2$ is parallel if $u$ and $v$ are disjoint as in Figure 2 (b), and admissible if $u = P_i P_j$ and $v = P_j P_k$ for some pairwise distinct points $P_i$, $P_j$, and $P_k$ which are in counterclockwise order on the unit circle as in Figure 2 (c). A triple $(u, v, w) \in L_n^3$ is said to be admissible if so are all the pairs $(u, v)$, $(v, w)$ and $(w, u)$. For $u = P_i P_j$ with $i < j$, let $a_u$ denote the generator $a_{ji}$ of the dual presentation of $B_n$. Then the dual presentation of $B_n$ can be written as follows:

$$B_n = \left\{ \left( a_u \mid a_u a_w = a_w a_u \text{ if } u \text{ and } v \text{ are parallel} \right) \left( a_u a_v = a_v a_u = a_w a_u \text{ if } (u, v, w) \text{ is admissible} \right) \right\}.$$  

It is easy to see the following: (i) if $u \cap v = \{ P_i \}$ for some $P_i$ in the unit circle, then exactly one of $(u, v)$ and $(v, u)$ is admissible; (ii) if $(u, v)$ is admissible, then $a_u a_v$ can be written in three ways as in the presentation, but $a_v a_u$ is not equivalent to any other positive word on the $a_u$'s; (iii) $a_u a_v$ is a simple element if and only if $(u, v)$ is parallel or admissible.

Now we prove Theorem 2. For $u = P_i P_j$ with $i < j$, let $g_u$ denote the generator $g_{ji}$ of the presentation in that theorem. Then the presentation can be reformulated as follows. Its proof is elementary. However, we present it for completeness.

**Theorem 3** (Dual presentation of $TL_n$—reformulated). $TL_n$ has a presentation with invertible generators $g_u (u \in L_n)$ satisfying the relations:

\begin{align*}
(5) & \quad g_u g_v = g_v g_u \text{ if } u \text{ and } v \text{ are parallel;} \\
(6) & \quad g_u g_w = g_w g_u \text{ if } (u, v, w) \text{ is admissible;} \\
(7) & \quad g_u^2 = (t - 1) g_u + t \text{ for } u \in L_n; \\
(8) & \quad g_u g_u + t g_u g_v + g_u + g_v + t g_w + 1 = 0 \text{ if } (u, v, w) \text{ is admissible.}
\end{align*}

**Proof.** From the results on the dual presentation of $B_n$ in [4], it follows that the relations (1) and (2) are equivalent to the relations (5) and (6).

Assume the relations (1), (2), and hence (5), (6).

(7) $\Rightarrow$ (3) It is clear since (3) is a special case of (7).

(3) $\Rightarrow$ (7) It is clear since each $g_u$ is conjugate to $h_i$ (for some $i$) by a monomial in the $h_j$'s.

Now assume the relations (1), (2), (3), and hence (5), (6), (7)

(8) $\Rightarrow$ (4) Let $u = P_{i+2} P_{i+1}$, $v = P_{i+1} P_i$ and $w = P_i P_{i+2}$. Then $g_u = h_{i+1}$, $g_v = h_i$ and $(u, v, w)$ is admissible. Since $g_u g_v g_w = g_v g_u g_w = (t - 1) g_v + 1 g_v g_w = (t - 1) g_u g_v + t g_w = (t - 1) g_u g_v + t g_w$,

$$h_i h_{i+1} + h_{i+1} h_i + h_{i+1} h_i + h_i = h_{i+1} + 1$$

$$= ((t - 1) g_u g_v + t g_u) + g_v g_u + g_u g_v + g_v + g_u + 1$$

$$= g_v g_u + t g_u g_v + g_u + g_v + t g_w + 1 = 0.$$
Note that for each admissible triple \((u', v', w')\), there is a self-homeomorphism of \(D_n\) sending \((u', v', w')\) to \((u, v, w)\). Therefore, there is a monomial \(x\) in the \(h_i\)'s such that \(xg_{u'}x^{-1} = g_u\), \(xg_{v'}x^{-1} = g_v\) and \(xg_{w'}x^{-1} = g_w\), simultaneously. Let \(u' = P_{i+1}P'_{i+1}, v' = P_{i+1}P_i\) and \(w' = P_iP_{i+2}\). In the same way as in \((8) \Rightarrow (4)\), we obtain
\[
g_v g_u + t g_v g_w + g_u + g_v + t g_w + 1 = x(g_v g_{u'} + t g_v g_{w'} + g_{v'} + g_{v'} + t g_{w'} + 1)x^{-1} = \pi(a_u) - \pi(a_v) - t \pi(a_w) - 1.\]

Before starting the proof of Zinno’s theorem, let us observe the relations written as a linear combination of the images of simple elements. Therefore the relations can be interpreted as instructions for converting a product of two generators into a linear combination of the images of simple elements.

Generalizing this idea, we will show in Proposition 4 that for a simple element \(A\) and an Artin generator \(\sigma_i\), the homomorphic image \(\pi(A\sigma_i)\) in \(TL_n\) can be written as a linear combination of the images of simple elements.

Recall that the simple elements are in one-to-one correspondence with non-crossing partitions. For a simple element \(A\), take union of the convex hulls of the blocks in the non-crossing partition of \(A\), and then remove those containing only one point. The resulting set is called the underlying space of \(A\) and denoted \(\bar{A}\).

It is known that for a simple element \(A\) and \(u \in L_n\), \(Aa_u\) is a simple element if and only if for any \(w \in L_n\) with \(w \subset \bar{A}\), the product \(a_w a_u\) is a simple element, in other words, \((w, u)\) is parallel or admissible [4, Corollary 3.6]. Figure 3 shows typical cases of \((\bar{A}, u)\) such that \(Aa_u\) becomes a simple element, and Figure 4 shows some cases of \((\bar{A}, u)\) such that \(Aa_u\) is not a simple element.

It is easy to see that if \(\bar{A}\) and \(u\) satisfy one of the following conditions, then \(Aa_u\) is a simple element and its underlying space is the union of the convex hulls of components of \(A \cup u\).

- \(A\) and \(u\) are disjoint.
- \(A\) and \(u\) intersect at the boundary of \(u\) as in the left hand sides of Figure 3. Intuitively, when we stand at an intersection point, with \(u\) on the right and the component of \(\bar{A}\) containing the intersection point on the left, we are facing towards the inside of the unit circle.
Figure 3. In the left hand sides $\bar{A}$ and $u$ are depicted as shaded regions and dotted lines. The right hand sides show the underlying spaces of $A\sigma_u$'s.

Figure 4. The shaded regions and the dotted lines represent the underlying space of a simple element $A$ and an element $u \in \mathcal{L}_n$, respectively. In this case, $A\sigma_u$ is not a simple element.

**Proposition 4.** For a simple element $A$ and an Artin generator $\sigma_i$, $\pi(A\sigma_i)$ can be expressed as a linear combination of the images of simple elements.

**Proof.** Let $u = P_iP_{i+1}$. Then $\sigma_i = a_u$ and $\pi(\sigma_i) = g_u$. We prove the assertion in three cases.

**Case 1.** If $u \subset \bar{A}$, then $\bar{A}$ and $u$ are as in Figure 5 (a). Let $B$ be the simple element whose underlying space is as in Figure 5 (b). More precisely, the non-crossing partition of $B$ is obtained from that of $A$ by making $\{P_i\}$ a new block.
Figure 5. \( A = Ba_u \) if \( \bar{A}, u \) and \( B \) are as above.

Then \( A = Ba_u \) and

\[
\pi(Aa_u) = \pi(Ba_u^2) = \pi(B)g_u^2 = \pi(B)((t-1)g_u + t) = (t-1)\pi(Ba_u) + t\pi(B) = (t-1)\pi(A) + t\pi(B).
\]

Case 2. If \( u \not\subset \bar{A} \) and \( P_i \not\in \bar{A} \), then \( \bar{A} \) and \( u \) are as in Figure 6. In this case, \( Aa_u \) itself is a simple element.

Case 3. If \( u \not\subset \bar{A} \) and \( P_i \in \bar{A} \), then \( \bar{A} \) and \( u \) are as in either (a) or (b) of Figure 7, depending on whether \( P_{i+1} \) belongs to \( \bar{A} \) or not. Let \( v \) be the line segment containing \( P_i \) such that \( Ba_v = A \) for some simple element \( B \) as in (c) and (d) of Figure 7. (More precisely, \( v = P_iP_j \) for some \( P_j \) such that \( P_iP_j \subset \bar{A} \) and the interior of \( P_{i+1}P_j \) does not intersect \( A \)). Let \( w \) be the line segment connecting the endpoints of \( u \) and \( v \) other than \( P_i \). Then \((u,v,w)\) is admissible and

\[
\pi(Aa_u) \overset{\text{Case 3}}{=} \pi(Ba_u a_v) = \pi(B)g_v g_u
= -\pi(B)(tg_v g_u + g_u + g_v + tg_w + 1)
= -t\pi(Ba_u a_v) - \pi(Ba_u) - \pi(Ba_v) - t\pi(Ba_w) - \pi(B).
\]

Note that \( Ba_v a_v, B a_u, B a_v \) and \( B a_w \) are simple elements.

Proof of Theorem 1. Let \( V_n \) be the subspace (of \( TL_n \)) spanned by the images of simple elements. Since the number of simple elements is equal to the dimension
of $TL_n$, the images of simple elements form a linear basis of $TL_n$ if we show that $V_n = TL_n$ (i.e., every monomial in the $h_i^{±1}$'s belongs to $V_n$).

Observe that $h_i^{-1} = t^{-1}h_i + t^{-1} - 1$ and $h_i = \pi(\sigma_i)$ for all $i$. Therefore, it suffices to show that the images of monomials in the $\sigma_i$'s belong to $V_n$. Use induction on the word length of monomials in the $\sigma_i$'s. By Proposition 4, it is easy to get the desired result. \(\square\)

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