A RESULT ON GENERALIZED DERIVATIONS WITH ENGEL CONDITIONS ON ONE-SIDED IDEALS

Çağrı Demir and Nurcan Argaç

Abstract. Let $R$ be a non-commutative prime ring and $I$ a non-zero left ideal of $R$. Let $U$ be the left Utumi quotient ring of $R$ and $C$ be the center of $U$ and $k, m, n, r$ fixed positive integers. If there exists a generalized derivation $g$ of $R$ such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in I$, then there exists $a \in U$ such that $g(x) = xa$ for all $x \in R$ except when $R \cong M_2(GF(2))$ and $I[I,I] = 0$.

1. Introduction

Throughout this paper unless specially stated, $R$ always denotes a prime ring with center $Z(R)$, extended centroid $C$, left Utumi quotient ring $U$, and two sided Martindale quotient ring $Q$. For any $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$.

We mean by a derivation of $R$ an additive mapping $d$ from $R$ into itself which satisfies the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A well-known result proved by Posner [21] states that $R$ must be commutative if there exists a nonzero derivation $d$ of $R$ such that $[d(x), x] = 0$ for all $x \in R$. Many related generalizations have been obtained by a number of authors in the literature (e.g., see, [10], [14], [15], [16]).

An additive mapping $g : R \to R$ is called a generalized derivation of $R$ if there exists a derivation $d$ of $R$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$ [9]. Obviously any derivation is a generalized derivation. Moreover, another basic example of a generalized derivation is the mapping of the form $g(x) = ax + xb$ for $a, b \in R$. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [1], [2], [3], [13], [9], [18]).

In [13], T. K. Lee extended the definition of a generalized derivation as follows. By a generalized derivation he means an additive mapping $g : J \to U$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in J$, where $U$ is the right Utumi
quotient ring of $R$, $J$ is a dense right ideal of $R$ and $d$ is a derivation from $J$ to $U$. He also proved that every generalized derivation can be uniquely extended to a generalized derivation of $U$. In fact, there exists $a \in U$ and a derivation $d$ of $U$ such that $g(x) = ax + d(x)$ for all $x \in U$ [13, Theorem 3]. A corresponding form to dense left ideals as follows. Let $I$ be a dense left ideal of $R$ and $U$ be the left Utumi quotient ring of $R$. An additive mapping $g : I \to U$ is called a generalized derivation if there exists a derivation $d : I \to U$ such that $g(xy) = xg(y) + d(x)y$ for all $x, y \in I$. Following the same methods in [13], one can extend $g$ uniquely to a generalized derivation of $U$, which we will also denote by $g$, and $g$ assumes the form $g(x) = xa + d(x)$ for all $x \in U$ and some $a \in U$, where $d$ is a derivation of $U$. Notice that $g(x) = ax + (d - \delta_a)(x)$ for all $x \in U$, where $\delta_a$ denotes the inner derivation induced by the element $a \in U$, i.e., $\delta_a(x) = [a, x]$. Setting $\delta = d - \delta_a$, we may always assume that a generalized derivation of a prime ring is of the form $g(x) = ax + \delta(x)$ for all $x \in U$, where $a \in U$ and $\delta$ is a derivation of $U$.

In [11], C. Lanski proved that if $R$ is a prime ring with derivation $d$, $I$ is a left ideal of $R$, and $k, n$ are positive integers such that $[d(x^k), x^n] = 0$ for all $x \in I$, then either $d = 0$ or $R$ is commutative. In [1], this result extended to generalized derivations.

In [17], T. K. Lee and W. K. Shiue showed that if $R$ is a non-commutative prime ring, $I$ is a nonzero left ideal of $R$ and $d$ is a derivation of $R$ such that $[d(x^m)x^n, x^r]_k = 0$ for all $x \in I$, where $k, m, n, r$ are fixed positive integers, then $d = 0$ except when $R \cong M_2(GF(2))$.

The aim of the present paper is to extend this result to generalized derivations. Precisely, we will prove the following.

**Theorem 1.** Let $R$ be a non-commutative prime ring and $k, m, n, r$ fixed positive integers. If there exists a generalized derivation $g$ of $R$ such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in R$, then there exists an element $a \in U$ such that $g(x) = xa$ for all $x \in R$.

**Theorem 2.** Let $R$ be a non-commutative prime ring, $I$ a non-zero left ideal of $R$ and $k, m, n, r$ fixed positive integers. If there exists a generalized derivation $g$ of $R$ such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in I$, then there exists $a \in U$ such that $g(x) = xa$ for all $x \in R$ except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$.

2. Preliminaries

In what follows, unless stated otherwise, $R$ will be a prime ring. The related object we need to mention is the left Utumi quotient ring $U$ of $R$ (sometimes, as in [4], $U$ is called the maximal left ring of quotients).

The definitions, the axiomatic formulations and the properties of this quotient ring $U$ can be found in [4].

In any case, when $R$ is a prime ring, all we need about $U$ is that

1) $R \subseteq U$;

2) $U$ is a prime ring;
3) The center of $U$, denoted by $C$, is a field which is called the extended centroid of $R$.

We also frequently use the theory of generalized polynomial identities and differential identities (see [4], [10], [12], [20]). In particular we need to recall the following:

**Remark 1 ([6])**. If $R$ is a prime ring and $I$ is a non-zero left ideal of $R$, then $I$, $RI$ and $UI$ satisfy the same generalized polynomial identities.

**Remark 2 ([10])**. Let $R$ be a prime ring, $d$ a nonzero derivation of $R$ and $I$ a nonzero two-sided ideal of $R$. Let $f(x_1, \ldots , x_n, d(x_1), \ldots , d(x_n))$ be a differential identity in $I$, that is

$$f(r_1, \ldots , r_n, d(r_1), \ldots , d(r_n)) = 0$$

for all $r_1, \ldots , r_n \in I$.

Then one of the following holds:

1) Either $d$ is an inner derivation in $Q$, the Martindale quotient ring of $R$, in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$, and $I$ satisfies the generalized polynomial identity

$$f(r_1, \ldots , r_n, [q, r_1], \ldots , [q, r_n])$$

or

2) $I$ satisfies the generalized polynomial identity

$$f(x_1, \ldots , x_n, y_1, \ldots , y_n).$$

3. Results

We need the following lemmas.

**Lemma 1.** Let $R = M_t(F)$, where $F$ is a field, $t \geq 2$ and $a, b \in R$. Suppose that

$$[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0 \quad \text{for all } x \in R,$$

where $k, m, n, r$ are fixed positive integers. Then $a + b \in F$.

**Proof.** Let $e$ be an idempotent element in $R$. Setting $x = e$ in (1) and multiplying left side by $(1 - e)$, we see that $(1 - e)(a + b)e = 0$ for any idempotent element $e$. Thus $a + b$ is a diagonal matrix. Note that $u(a + b)u^{-1}$ must be diagonal for each invertible element $u \in R$, since

$$[(uau^{-1})x^{m+n} + ([ubu^{-1}], x^m)x^n, x^r]_k = 0$$

for all $x \in R$. Write $a + b = \sum_{i=1}^t \beta_i e_{ii}$, where $\beta_i \in F$. Then for each $j > 1$, we see that $\beta_j - \beta_1$, the $(1,j)$-entry of $(1 + e_{1j})(a + b)(1 + e_{1j})^{-1}$, equals 0. That is, $\beta_j = \beta_1$ for $j > 1$ and hence $a + b \in F$. \qed

**Lemma 2.** Let $R$ be a non-commutative prime ring and $a, b \in R$ such that

$$[ax^{m+n} + [b, x^m]x^n, x^r]_k = 0 \quad \text{for all } x \in R,$$

where $k, m, n, r$ are fixed positive integers. Then $a + b \in Z(R)$.
Proof. Suppose on the contrary that \( a + b \notin C \). Then

\[
f(X) = [(a + b)X^{m+n} - X^m a X^n, X^r]_k
\]

is a nontrivial generalized polynomial identity (GPI) for \( R \). By [6], \( f(X) \) is also a GPI for \( Q \). Denote by \( F \) either the algebraic closure of \( C \) or \( C \) according to the cases where \( C \) is either infinite or finite, respectively. Then, by a standard argument (e.g., see [19, Proposition]), \( f(X) \) is also a GPI for \( Q \otimes_C F \). Since \( Q \otimes_C F \) is centrally closed prime \( F \)-algebra [7, Theorems 2.5 and 3.5], by replacing \( R, C \) with \( Q \otimes_C F, F \), respectively we may assume \( R \) is centrally closed and \( C \) is either finite or algebraically closed. In view of Martindale’s theorem [20], \( R \) is a primitive ring having a non-zero socle \( H \) with \( C \) as its associated division ring.

Since \( a + b \notin C \), we have \( [a + b, h] \neq 0 \) for some \( h \in H \). By Litoff’s theorem [8], there exists an idempotent \( e \in H \) such that \( h, ha, hb, bh \in eRe \). Note that \( ef(eXe)c \) is a GPI for \( R \). Thus, \( [(eae)X^{m+n} + [ebe, X^n]X^n, X^r]_k \) is a GPI for \( eRe \). Since \( eRe \cong M_s(C) \) for some \( s \geq 1 \) then \( eae + ebe \) is central in \( eRe \) by Lemma 1. Then there exists \( c \in C \) such that \( ec = eae + ebe \). Hence \( ch = each + ebeh = cae + ehh = ah + bh = (a + b)h \). Similarly \( he = heae + hebe = hae + hbe = chae + ehbe = ha + hb = h(a + b) \). So \( [a + b, h] = 0 \), a contradiction. Therefore \( a + b \in Z(R) \). \( \square \)

**Corollary 1.** Let \( R \) be a prime ring and \( a \in R \) such that \( [ax^m, x^n]_k = 0 \) for all \( x \in R \), where \( k, m, n \) are fixed positive integers. Then \( a \in Z(R) \).

*Proof of Theorem 1.* As we have already noted that every generalized derivation \( g \) on a dense left ideal of \( R \) can be uniquely extended to \( U \) and assumes the form \( g(x) = ax + d(x) \) for some \( a \in U \) and a derivation \( d \) on \( U \). If \( d = 0 \), then \( [ax^m + d, x^n]_k = 0 \) for all \( x \in R \). By Remark 1, \( U \) satisfies the above generalized identity. Moreover, since \( U \) remains prime by the primeness of \( R \), replacing \( R \) with \( U \), we may assume that \( a \in R \) and \( C \) is just the center of \( R \). By Corollary 1, we have \( a \in Z(R) \). Thus \( g(x) = ax = xa \) for all \( x \in R \). So we may assume that \( d \neq 0 \).

In the light of Remark 2, we divide the proof into two cases:

**Case 1.** Let \( d \) be the inner derivation induced by the element \( b \in U - C \), that is, the \( d(x) = [b, x] \) for all \( x \in U \). Then \( R \) satisfies the nontrivial generalized polynomial identity

\[
[ax^{m+n} + [b, x^n]x^n, x^r]_k.
\]

By Remark 1, \( U \) satisfies the above generalized polynomial identity. Moreover, since \( U \) remains prime by the primeness of \( R \), replacing \( R \) with \( U \), we may assume that \( a, b \in R \) and \( C \) is just the center of \( R \). Then by Lemma 2, we have that \( a + b \in Z(R) \). Therefore \( g(x) = ax + [b, x] = (a + b)x - xb = x(a + b - b) = xa \) for all \( x \in R \).

**Case 2.** Let now \( d \) be an outer derivation of \( U \). To continue the proof, we set \( G(Y, X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i} \), a non-commuting polynomial in variables...
Let $e$ be idempotent for some $\mu$. In particular, $e_{\mu}c_{e_{\mu}}$ and $e_{\mu}f_{e_{\mu}}$ are idempotents in $I$. Multiplying the first equation by $e_{\mu}$ we get

$$\left[ a x^{m+n} + G(y, x)x^n, x^r \right]_k = 0$$

Using (2) we arrive at the following equations:

$$\left[ a x^{m+n} + G(y, x)x^n, x^r \right]_k = 0$$

Taking $y = 0$ in the above identity, we get

$$\left[ a x^{m+n}, x^r \right]_k = 0 \text{ for all } x \in R.$$ 

So we have

$$\left[ d(x^n)x^n, x^r \right]_k = 0 \text{ for all } x \in R.$$ 

Therefore by [17, Theorem 1], we must have $d = 0$, a contradiction. This proves the result. \hfill \Box

By using almost the same argument in [17], we have the following.

**Lemma 3.** Let $R = M_l(F)$, where $F$ is a field, $l \geq 2$, and $I$ a minimal left ideal of $R$. Suppose $[a x^{m+n} + [b, x^m]x^n, x^r]_k = 0$ for all $x \in I$, where $m, n, r, k$ are fixed positive integers. Then $a + b \in F$ except when $R \cong M_2(GF(2))$.

**Proof.** Suppose that $a + b \notin F$. Since $I$ is a minimal left ideal, it is clear that we may assume $I = Re_{11}$. Let $e = e^2 \in I$. By the hypothesis, we have $[a e + [b, e]e, e]_k = 0$. Left multiplying by $1 - e$, we see that

$$\left( 1 - e \right) \left( a + b \right) e = 0 \text{ for all } e \in I.$$ 

Let $\beta \in F$ and $x \in R$. Then $f = e + (1 - e)xe$ and $g = e + \beta(1 - e)xe$ are idempotents in $I$. Set $c = a + b$, then $c \notin F$ and $(1 - e)cxe = 0$. Thus by (2) we have $(1 - f)cxe = 0 = (1 - g)cxe$. Therefore we see that

$$((1 - e) - (1 - e)xe)c(e + (1 - e)xe) = 0$$

and

$$((1 - e) - \beta(1 - e)xe)c(e + \beta(1 - e)xe) = 0.$$ 

Using (2) we arrive at the following equations:

$$(1 - e)xce - (1 - e)xce - (1 - e)xce(1 - e)xe = 0$$

and

$$\beta(1 - e)xce - \beta(1 - e)xce - \beta^2(1 - e)xce(1 - e)xe = 0.$$ 

Multiplying first equation by $\beta$ and comparing the last two equations we see that

$$\left( \beta^2 - \beta \right) (1 - e)xce(1 - e)xe = 0 \text{ for all } x \in R.$$ 

Then either $\beta \in \{0, 1\}$ or $ce(1 - e) = 0$ for any idempotent $e \in I$. Suppose that the second possibility holds. In particular, $e_{11}c(1 - e_{11}) = 0$. Let $x \in R$. Then we have $xe_{11}e = xe_{11}e_{11}c = \mu xe_{11}$ for some $\mu \in F$. Thus we see that $I(c - \mu) = Re_{11}(c - \mu) = 0$ for some $\mu \in F$. On the other hand, in view of (2) we get $[c, e] = 0$ for any idempotent $e \in I$. Then $0 = [c, e] = [c - \mu, e] = (c - \mu)e$ for all idempotent
Let $c_{11} + (1 - c_{11})xe_{11} \in I$ be an idempotent for every $x \in R$. Assume that $(c - \mu)(1 - e_{11})xe_{11} = 0$ for all $x \in R$. Since $(c - \mu)Re_{11} = 0$, it is clear that $(c - \mu)Re_{11} = 0$. Therefore $c = \mu \in F$, a contradiction. Thus we get $F = GF(2)$.

Now we prove that $l = 2$. Suppose on the contrary that $l > 2$. Let $i, j$ be two distinct positive integers such that $2 \leq i, j \leq l$. Then $e_{11}, e_{11} + e_{11}, e_{11} + e_{1}$ and $e_{11} + e_{1} + e_{1}$ are idempotents in $I$. In view of (2) we obtain that

$$ce_{11} = e_{11}ce_{11},$$

and

$$c(e_{11} + e_{1}) = (e_{11} + e_{1})c(e_{11} + e_{1}),$$

and

$$c(e_{11} + e_{1} + e_{1}) = (e_{11} + e_{1} + e_{1})c(e_{11} + e_{1} + e_{1}).$$

Using $ce_{11} = e_{11}ce_{11}$ and comparing the other equations in (3), we arrive at $e_{11}ce_{11} + e_{1}ce_{1} = 0$. Set $c = \sum_{1 \leq i, j \leq l} \beta_{ij}e_{ij}$, where $\beta_{ij} \in F$. Then this implies that $\beta_{ii} = 0 = \beta_{1i}$. Hence the second equation in (3) reduces to $ce_{11} = \beta_{11}e_{11}$, and so $\beta_{pp} = 0$ for $p \neq i$ and $\beta_{ii} = \beta_{11}$. Thus we get $c = a + b \in F$, a contradiction. This proves the lemma.

**Lemma 4.** Let $R$ be a prime ring, $I$ a non-zero left ideal of $R$ and $a \in R$ such that $[ax^{m}, x^{n}]k = 0$ for all $x \in I$, where $k, m, n$ are fixed positive integers. Then $a \in Z(R)$ except when $R \cong M_{2}(GF(2))$ and $I[I, I] = 0$.

**Proof.** Assume that $[ax^{m}, x^{n}]k = 0$ for all $x \in I$. Then

$$[(a, x^{n})x^{m}, x^{n}]k = [ax^{m}, x^{n}]k + 1 = 0$$

for all $x \in I$. Now by [17, Lemma 3] we have $a \in Z(R)$ except when $R \cong M_{2}(GF(2))$ and $I[I, I] = 0$.

**Lemma 5.** Let $R$ be a non-commutative prime ring and $I$ a non-zero left ideal and $a, b \in R$ such that

$$[ax^{m+n} + [b, x^{n}]x^{m}, x^{n}]k = 0$$

for all $x \in I$, where $k, m, n, r$ are fixed positive integers. Then $a + b \in Z(R)$ except when $R \cong M_{2}(GF(2))$ and $I[I, I] = 0$.

**Proof.** Assume that $a + b \notin C$. If $I(b - \beta) = 0$ for some $\beta \in C$, then setting $b' = b - \beta$ we have $Ib' = 0$. Moreover by (4) it is clear that

$$[ax^{m+n} + [b', x^{m}]x^{n}, x^{n}]k = 0$$

for all $x \in I$.

Thus we get

$$[(a + b')x^{m+n}, x^{n}]k = 0$$

for all $x \in I$.

By Remark 1, $[(a + b')x^{m+n}, x^{n}]k = 0$ for all $x \in UI$. Moreover $UIb' = 0$ if and only if $Ib' = 0$. Now $I$ and $UI$ satisfy the same basic conditions. Hence
replacing $R$, $I$ with $U$, $UI$, respectively, we may assume that $a, b' \in R$ and $C$ is just the center of $R$. Thus we get the conclusion $R \cong M_2(GF(2))$ and $I[I, I] = 0$ since $a + b' \not\in C$.

So we may assume that $I(b - \beta) \neq 0$ for all $\beta \in C$. Hence, in view of [14, Lemma 3], either $R$ is a PI-ring or there exists an element $u \in I$ such that $ub$ and $u$ are $C$-independent. For the latter case,

$$[a(Xu)^{m+n} + b, (Xu)^m(Xu)^n, (Xu)^r]_k$$

is a non-trivial GPI for $R$.

On the other hand we have $[a x^{m+n} + [b, x^m] x^n, x^r]_k = 0$ for all $x \in QI$ by [6]. Thus applying the same argument in Lemma 2 and replacing $I$ by $QI$, we may assume that $R$ is a centrally closed prime ring having a non-zero socle $H$, with $C$ as its associated division ring and $I = IC$. Moreover $C$ is either algebraically closed or finite. If $R$ contains no non-trivial idempotents, then $R$ is a division ring and $I = R$. Then by the proof of Theorem 1 we obtain that $a + b \in C$, a contradiction. So we may assume that $R$ contains a non-trivial idempotent. On the other hand we have

- $I[I, I] = 0$ if and only if $HI[H1, H1] = 0$ by [6],
- $I(b - \mu) = 0$ if and only if $HI(b - \mu) = 0$ for some $\mu \in C$.

So replacing $I$ by $HI$ we may assume $I \subseteq H$. Suppose first that $I[I, I] \neq 0$. Then $I$ always contains an idempotent with rank 2 or greater that 2. Let $e$ be such an idempotent in $I$.

Now choose $x$ as $exe$ in (4), then

$$[a(exe)^{m+n} + [b, (exe)^m](exe)^n, (exe)^r]_k = 0$$

and left-side multiplying by $e$ yields

$$[(exe)(exe)^{m+n} + [ebe, (exe)^m](exe)^n, (exe)^r]_k = 0$$

for all $x \in R$. Left-side multiplying by $e$ yields that

$$eexe = eexe - eexe \in Ce.$$  

Since $ex = xe \in C$, we get $[e, c]xe \in Ce$ for all $x \in R$. Suppose for the moment that $[e, c] \neq 0$. Choose $x_0 \in R$ such that $[e, c]x_0e = \beta e \neq 0$ for some $\beta \in C$. Then we have $\beta exe = [e, c]x_0exe \in Ce$. Therefore $eexe = Ce$, because $\beta \neq 0$. But $eexe = Ce$ implies that rank$(e) = 1$, a contradiction. Hence $[e, c] = 0$. Now since $I$ is completely reducible left $H$-module, each element of $I$ is contained in $Hf$ for some $f^2 = f \in I$ with rank$(f) \geq 2$. But $fc = cf \in Cf$. 


Let $x \in I$. Then $x = hf$ for some $h \in H$. We see that $xc = hfc \in Chf = Cx$, and so $[xc, x] = 0$ for all $x \in I$. Linearizing this last equation, we get

$$[xc, y] + [yc, x] = 0 \quad \text{for all } x, y \in I.$$  

Replacing $y = e$ in (6) and using the fact that $[e, c] = 0$, we obtain

$$0 = [xc, e] + [ec, x] = e[c, x] \quad \text{for all } x \in I.$$  

Hence we have $0 = e[c, xy] = ex[c, y]$ for all $x, y \in I$. Therefore we get $eRI[c, I] = (0)$, and so $I[c, I] = 0$. In particular, $[x[c, x], x] = 0$ for all $x \in I$. So in view of [17, Lemma 3(ii)] one obtains $I(c - \lambda) = 0$ for some $\lambda \in C$.

Let $x \in R$, then it is clear that $f = e + (1 - e)xe \in I$ is an idempotent with $\text{rank}(f) = \text{rank}(e) \geq 2$. Since $[e, c] = 0$ for all $e = e^2 \in I$ with $\text{rank}(e) \geq 2$, in particular we have

$$[c - \lambda, e + (1 - e)xe] = 0 \quad \text{for all } x \in R.$$  

Hence we get $(c - \lambda)e + (c - \lambda)(1 - e)xe = 0$. On the other hand we have

$$(c - \lambda)e = [c - \lambda, e] = 0. \quad \text{So } (c - \lambda)xe = 0 \text{ for all } x \in R. \quad \text{Thus the primeness of } R \text{ implies that } c = \lambda \in C, \quad \text{and hence } a + b = c \in Z(R), \text{ a contradiction. This proves that } I[I, I] = 0.$$

If now $H \cong M_l(C)$ for some $l \geq 2$, then in view of Lemma 3, we are done. Thus we may assume $H \not\cong M_l(C)$ for all $l \geq 2$. Since $c \notin C$, it is clear that $ch \neq hc$ for some $h \in I$. It follows from Litoff’s theorem [8] that there exists $e = e^2 \in H$, $\text{rank}(e) \geq 3$, such that $ch, hc, h \in eHe$. Note that $ece \notin Ce$. Indeed, if $ece \in Ce$, then $eche \neq eceh$, and hence $ch = hc$, a contradiction. On the other hand, $0 \neq h \in I \cap eRe$. Since $R$ is centrally closed, $IC = I$ and $I[I, I] = 0$, it is clear that $I$ is a minimal left ideal of $R$ by [5, Lemma 5.1]. We also have that $I \cap eRe$ is still a minimal left ideal of $eRe$ and $eRe \cong M_l(C)$, where $l = \text{rank}(e) \geq 3$. Indeed, if $J$ is a left ideal of $eRe$ such that $J \not\subseteq I \cap eRe$, then $RJ \not\subseteq RJ \subseteq I$. Using the fact that $RJ$ is a left ideal of $R$ such that $RJ \not\subseteq I$ and $I$ is a minimal ideal of $R$, we get $RJ = 0$. Hence $J = 0$ by the primeness of $R$. Now by the hypothesis, we have

$$[(eae)(exe)^{m+n} + [ebe, (exe)^m](exe)^n, (exe)^r]_k = 0 \quad \text{for all } x \in R,$$

and so

$$[(eae)x^{m+n} + [ebe, x^m]x^n, x^r]_k = 0 \quad \text{for all } x \in I \cap eRe.$$  

In view of Lemma 3 this yields that $eRe \cong M_2(GF(2))$, a contradiction. This proves the result. \qed

**Example 1.** Let $R = M_s(F), s > 1$, the $s \times s$ matrices over a field $F$ and $I = Re_{11}$. If we set $a = 1 - e_{11}$ and $b = e_{11}$, then $[ax^{m+n} + [b, x^n]x^n, x^r]_k = 0$ for all $x \in I$, where $k, m, n, r$ are fixed positive integers and $a + b \in Z(R)$.

**Proof of Theorem 2.** As we have already noted that every generalized derivation $g$ on a dense left ideal of $R$ can be uniquely extended to $U$, we may assume that $g$ has the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation $d$ on $U$. If $d = 0$, then $[ax^{m+n}, x^r]_k = 0$ for all $x \in I$. Then by Lemma 4 we have $a \in C$
except when \( R \cong M_2(GF(2)) \) and \( I[I, I] = 0 \). If \( a \in C \), then \( g(x) = ax = xa \) for all \( x \in R \). So we may assume that \( d \neq 0 \).

In the light of Remark 2, we divide the proof into two cases:

**Case 1.** Let \( d \) be the inner derivation induced by the element \( b \in U - C \), that is, \( d(x) = [b, x] \) for all \( x \in U \). Then \( I \) satisfies the nontrivial generalized polynomial identity

\[
[aX^m+n + [b, X^m]X^n, X^r]_k.
\]

By Remark 1, \( RI \) satisfies the above generalized identity. Since by [4], \( R \) and \( U \) satisfy the same GPIS, we have that \( UI \) satisfies above identity. Then applying Lemma 5 to \( UI \), we have that \( a + b \in C \) except when \( U \cong M_2(GF(2)) \) and \( UI[UI, UI] = 0 \). Moreover as in the proof of Lemma 5 we may replace \( R, I \) by \( U, UI \), respectively. Then in particular, \( a + b \in C \) except when \( R \cong M_2(GF(2)) \) and \( I[I, I] = 0 \). If \( a + b \in C \), then \( g(x) = ax + [b, x] = (a + b)x - xb = x(a + b - b) = xa \) for all \( x \in R \).

**Case 2.** Let now \( d \) be an outer derivation of \( U \). To continue the proof we set \( G(Y, X) = \sum_{i=0}^{m-1} X^iYX^{m-1-i} \), a non-commuting polynomial in variables \( X \) and \( Y \). Note that \( d(x^n) = G(d(x), x) \). Since

\[
[aX^m+n + G(d(x), x)x^n, x^r]_k
\]

is an identity for \( I \), then for any \( u \in I - C \)

\[
[a(xu)^{m+n} + G(d(xu), xu)(xu)^n, (xu)^r]_k
\]

is an identity for \( R \). Thus \( R \) satisfies the following

\[
[a(xu)^{m+n} + G(d(xu), xu)(xu)^n, (xu)^r]_k.
\]

From Remark 2, since \( d \) is an outer derivation \( R \) satisfies the following identity

\[
[a(xu)^{m+n} + G(yu + xd(u), xu)(xu)^n, (xu)^r]_k.
\]

Taking \( y = 0 \) in (7) we get

\[
[a(xu)^{m+n} + G(xd(u), xu)(xu)^n, (xu)^r]_k = 0.
\]

By the linearity of \( g(Y, X) \) on \( Y \), subtracting equation (7) from (8) yields that \( R \) satisfies

\[
[G(yu, xu)(xu)^n, (xu)^r]_k = 0,
\]

which means that \( R \) satisfies

\[
0 = \left[ \sum_{i+j=m-1} (xu)^i(yu)(xu)^{j+n}, (xu)^r \right]_k = \sum_{i+j=m-1} (xu)^i[(yu), (xu)^r]_k(xu)^{j+n}.
\]

Clearly (9) is a nontrivial GPI for \( R \), since \( u \notin C \). So \( RC \) is a primitive ring with a non-zero socle \( H \) ([20]). \( J = HI \) is a non-zero left ideal of \( H \). Note that \( H \) is simple, \( J = HJ \) and \( J \) satisfies the same basic conditions as \( I \) ([12]). Now
replace $R$ by $H$ and $I$ by $J$, then, without loss of generality, $R$ is simple and equal to its own socle and $RI = I$. Let $e^2 = e$ be some non-trivial idempotent in $I$. So for all $x, y \in R$, we have
\[ \sum_{i+j=m-1} (xe)^i[(ye),(xe)^r]_k(xe)^{j+n} = 0 \]
and choosing $y = (1-e)r \in R$ we get
\[ (1-e)(re)(xe)^{kr+m+n-1} = 0. \]
This leads to the contradiction that either $e = 0$ or $e = 1$. Thus any idempotent element of $I$ is trivial, that is, $I = R$. Therefore we have to consider the condition
\[ \sum_{i+j=m-1} x^i[y,x]_k x^{j+n} = 0 \]
for all $x, y \in R$, which is a polynomial identity. From Lemma 2 in [11], it follows that there exists a suitable field $F$ such that $R \subseteq M_s(F)$, the ring of all $s \times s$ matrices over $F$, and moreover $M_s(F)$ satisfies the same polynomial identity of $R$. In particular $M_s(F)$ satisfies
\[ \sum_{i+j=m-1} x^i[y,x]_k x^{j+n} = 0. \]
Suppose $s \geq 2$ and choose $x = e_{11}$ and $y = e_{21}$ in (10). Then we have $e_{21} = 0$. Thus $s = 1$ and $R$ is commutative, a contradiction. □

The following example shows our results do not hold in semiprime rings:

**Example 2.** Let $F$ be any field. Consider the semiprime ring
\[ R = \begin{pmatrix} GF(2) & GF(2) & 0 \\ GF(2) & GF(2) & 0 \\ 0 & 0 & F \end{pmatrix}. \]
Let
\[ I = \begin{pmatrix} GF(2) & 0 & 0 \\ GF(2) & 0 & 0 \\ 0 & 0 & F \end{pmatrix} \]
be the left ideal of $R$. If $a = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ for $\alpha \in F$ fixed, one can easily see that $[ax^2 + [b,x]x,x] = 0$ for all $x \in I$, since $uv(u + v) = 0$ for all $u, v \in GF(2)$. Then $g(x) = ax + [b,x] = (a+b)x - xb$ is a generalized derivation such that $[g(x)x,x] = 0$ for all $x \in I$. But
\[ a + b \notin C = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \mid \lambda \in GF(2), \mu \in F \right\}. \]
References


cagri.demir@ege.edu.tr