\section{Introduction}

Let $\mathcal{A}$ be an algebra with identity and let $\tau$ be an endomorphism of $\mathcal{A}$. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a left (right) centralizer of $\mathcal{A}$ if $f(y) = f(1)y$ ($f(y) = yf(1)$) for any $y \in \mathcal{A}$. If $f$ is a left and right centralizer, then it is to call $f$ a centralizer. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a left (right) Jordan centralizer of $\mathcal{A}$ if $f(x^2) = f(x)x$ ($f(x^2) = xf(x)$) for any $x \in \mathcal{A}$. $f$ is called a Jordan centralizer of $\mathcal{A}$ if $f(xy + yx) = f(x)y + yf(x) = f(y)x + xf(y)$ for any $x, y \in \mathcal{A}$. Albaş [1] shows that under some conditions, a left Jordan $\tau$-centralizer of a semiprime ring is a left $\tau$-centralizer and each Jordan $\tau$-centralizer of a semiprime ring is a $\tau$-centralizer.

A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a left (right) $\tau$-centralizer of $\mathcal{A}$ if $f(y) = f(1)\tau(y)$ ($f(y) = \tau(y)f(1)$) for any $x, y \in \mathcal{A}$. If $f$ is a left and right $\tau$-centralizer, then it is to call $f$ a $\tau$-centralizer. A linear mapping $f : \mathcal{A} \to \mathcal{A}$ is called a left (right) Jordan $\tau$-centralizer of $\mathcal{A}$ if $f(x^2) = f(x)\tau(x)$ ($f(x^2) = \tau(x)f(x)$) for any $x \in \mathcal{A}$. $f$ is called a Jordan $\tau$-centralizer of $\mathcal{A}$ if $f(xy + yx) = f(x)\tau(y) + \tau(y)f(x) = f(y)\tau(x) + \tau(x)f(y)$ for any $x, y \in \mathcal{A}$. Albaş [1] shows that under some conditions, a left Jordan $\tau$-centralizer of a semiprime ring is a left $\tau$-centralizer and each Jordan $\tau$-centralizer of a semiprime ring is a $\tau$-centralizer.

We call $f$ a local left centralizer of $\mathcal{A}$ if for each $x \in \mathcal{A}$, there is a left centralizer $f_x$ of $\mathcal{A}$ such that $f(x) = f_x(x)$. Similarly, we can define local right centralizers.

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centralizer and local centralizer. In [2], Hadwin studies local centralizers on von Neumann algebras and nest algebras.

Recently, Nakajima introduced the following definitions. Let $\mathcal{A}$ be an algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Let $\alpha : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{M}$ be a bilinear mapping. $\alpha$ is called a Hochschild 2-cocycle if
\[
\alpha(x, y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y)z = 0.
\]
A linear mapping $\delta : \mathcal{A} \rightarrow \mathcal{M}$ is called a generalized derivation if there is a 2-cocycle $\alpha$ such that
\[
\delta(xy) = \delta(x)y + x\delta(y) + \alpha(x, y).
\]
We denote it by $(\delta, \alpha)$. In [7], Nakajima shows that the usual generalized derivation, left centralizer and $(\sigma, \tau)$-derivation are also generalized derivations in above sense.

The distribution of this paper is as follows.

In Section 2, we prove that if $A' \cap A = C_I$, and $\tau$ is an epimorphism of $A$, then each Jordan $\tau$-centralizer of $A$ is $\tau$-centralizer. And we also show that if $L$ is a CDCSL on $H$ and $\tau$ is an automorphism of $\text{alg}L$, then each Jordan $\tau$-centralizer of $\text{alg}L$ is $\tau$-centralizer.

We introduce the following definition. We call $f$ a local left $\tau$-centralizer of $A$ if for each $x \in A$, there is a left $\tau$-centralizer $f_x$ of $A$ such that $f(x) = f_x(x)$. Similarly, we can define local right $\tau$-centralizer and local $\tau$-centralizer. In Section 3, we generalize some results of [2] to local $\tau$-centralizer. And we show that if $L$ is a CDCSL on $H$ and $\tau$ is an automorphism of $\text{alg}L$, then each local $\tau$-centralizer of $\text{alg}L$ is a $\tau$-centralizer.

In Section 4, we introduce a new type of local generalized derivations and we show that every local generalized derivation of above type from CDCSL into its dual normal unital Banach $\mathcal{A}$-bimodule is a generalized derivation.

The following notations will be used in our paper.

Let $X$ be a complex Banach space with dual $X^*$ and let $B(X)$ be the set of all bounded linear maps from $X$ into itself. Let $H$ be a complex separable Hilbert space.

A subspace lattice on $X$ is a collection $\mathcal{L}$ of subspaces of $X$ with $(0), X$ in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\wedge M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\wedge M_r$ denotes the intersection of $\{M_r\}$ and $\vee M_r$ denotes the closed linear span of $\{M_r\}$. A totally ordered subspace lattice is called a nest. For a subspace lattice $\mathcal{L}$, we define $\text{alg}\mathcal{L}$ by
\[
\text{alg}\mathcal{L} = \{T \in B(H) : TN \subseteq N, \forall N \in \mathcal{L}\}.
\]
For any $L \subseteq X$, $L^\perp = \{f \in X^*, f(x) = 0 \text{ for all } x \in L\}$. Let $x \in X$, $f \in X^*$ be nonzero. The rank one operator $x \otimes f$ is defined by $z \mapsto f(z)x$ for any $z \in X$. For any nonzero $x, y \in H$, the operator $x \otimes y$ is defined by $z \mapsto (z, y)x$ for any $z \in H$. If $\mathcal{L}$ is a subspace lattice of $X$ and $E \in \mathcal{L}$, we define
\[
E_- = \vee\{F \in \mathcal{L}, F \nsubseteq E\} \text{ and } E_+ = \wedge\{F \in \mathcal{L}, F \nsubseteq E\}.
\]
It is well known that \( x \otimes f \in \text{alg} \mathcal{L} \) if and only if there is \( E \in \mathcal{L} \) such that \( x \in E \) and \( f \in (E^-)^\perp \) (equivalently, \( x \in E_+ \) and \( f \in E_\perp \)).

A subspace lattice \( \mathcal{L} \) is said to be completely distributive if for every family \( \{x_{i,j}\}_{i,j \in J} \) of elements in \( \mathcal{L} \),

\[
\bigwedge_{i \in I} \bigvee_{j \in J} x_{i,j} = \bigvee_{j \in J} \bigwedge_{i \in I} x_{i,j}
\]

and

\[
\bigvee_{i \in I} \bigwedge_{j \in J} x_{i,j} = \bigwedge_{i \in I} \bigvee_{j \in J} x_{i,j},
\]

where \( J \) denotes the set of all maps from \( I \) into \( J \).

A Hilbert space subspace lattice \( \mathcal{L} \) is called a commutative subspace lattice \((\text{CSL})\) if it consists of mutually commuting projections. If \( \mathcal{L} \) is a commutative subspace lattice, then \( \text{alg} \mathcal{L} \) is called a CSL algebra. If \( \mathcal{L} \) is a completely distributive commutative subspace lattice \((\text{CDCSL})\), then \( \text{alg} \mathcal{L} \) is called a CDCSL algebra.

Given a subspace lattice \( \mathcal{L} \) on \( X \), put

\[
\mathcal{J}_\mathcal{L} = \{ K \in \mathcal{L} : K = \{0\} \text{ and } K_\perp = X \}.
\]

Call \( \mathcal{L} \) a \( \mathcal{J} \)-subspace lattice on \( X \) if it satisfies the following conditions:

1. \( \bigvee \{ K : K \in \mathcal{J}_\mathcal{L} \} = X \);
2. \( \bigwedge \{ K_\perp : K \in \mathcal{J}_\mathcal{L} \} = \{0\} \);
3. \( K \vee K_\perp = X \) for any \( K \in \mathcal{J}_\mathcal{L} \);
4. \( K \wedge K_\perp = 0 \) for any \( K \in \mathcal{J}_\mathcal{L} \).

In this paper, we suppose that \( \mathcal{A} \) is a unital algebra and \( \mathcal{M} \) is a unital \( \mathcal{A} \)-bimodule.

2. Jordan \( \tau \)-centralizers

Since the proof of the following lemma is analogous to that of \([1, \text{Lemma 3}]\), we omit it.

**Lemma 2.1.** Let \( f \) be a left Jordan \( \tau \)-centralizer of an algebra \( \mathcal{A} \). Then

1. \( f(xy + yx) = f(x)f(y) + f(y)f(x) \) for all \( x, y \in \mathcal{A} \),
2. \( f(xyx) = f(x)f(y)f(x) \) for all \( x, y \in \mathcal{A} \),
3. \( f(xyz + zyx) = f(x)f(y)f(z) + f(z)f(y)f(x) \) for all \( x, y \in \mathcal{A} \),
4. \( D(x, y) = -D(y, x) \), where \( D(x, y) = f(xy) - f(x)f(y) \) for all \( x, y \in \mathcal{A} \).

**Lemma 2.2.** Each left Jordan \( \tau \)-centralizer \( f \) of a unital algebra \( \mathcal{A} \) is a left \( \tau \)-centralizer.

**Proof.** Let \( I \) be the identity in \( \mathcal{A} \). Since \( \tau \) is an endomorphism of \( \mathcal{A} \), it follows that \( \tau(I) = I \). Let \( D(x, y) = f(xy) - f(x)f(y) \) for any \( x, y \in \mathcal{A} \). So \( D(x, I) = f(xI) - f(x)f(I) = 0 \) for all \( x \in \mathcal{A} \). By Lemma 2.1(4), we have that \( D(I, x) = -D(x, I) = 0 \) for all \( x \in \mathcal{A} \). Thus

\[
(3) \quad f(x) = f(Ix) = f(I)f(x)
\]

for all \( x \in \mathcal{A} \).

\[
\square
\]
In what follows, we suppose that $A$ is a unital subalgebra of $B(X)$ such that $A' \cap A = CI$, where $I$ is the identity in $A$, and $\tau$ is an epimorphism of $A$. And we denote by $Z = A' \cap A = CI$ the center of $A$.

**Lemma 2.3.** Let $a$ be a fixed element in $A$. If $a\tau(x) - \tau(x)a \in Z$ for all $x \in A$, then $a \in Z$.

**Proof.** Since $a\tau(x) - \tau(x)a \in Z$, by [4, Question 182], it follows that $a\tau(x) - \tau(x)a = 0$. Since $\tau$ is surjective, we have that $a \in Z$. \hfill $\Box$

**Lemma 2.4.** Let $a$ be a fixed element in $A$, and $f(x) = a\tau(x) + \tau(x)a$ for any $x \in A$. If $f$ is a Jordan $\tau$-centralizer of $A$, then $a \in Z$.

**Proof.** Since $f$ is a Jordan $\tau$-centralizer of $A$, it follows that $f(xy + yx) = f(x)y + \tau(y)f(x)$ for all $x, y \in A$. Hence

$$a\tau(xy + yx) + \tau(yx + xy)a = (a\tau(x) + \tau(x)a)\tau(y) + \tau(y)(a\tau(x) + \tau(x)a),$$

$$\tau(x)(a\tau(x) + \tau(x)a) = \tau(x)\tau(y) + \tau(y)a\tau(x),$$

for all $x, y \in A$. Since $\tau$ is surjective, we have that $a\tau(y) - \tau(y)a \in Z$. Hence $a \in Z$ by Lemma 2.3. \hfill $\Box$

**Lemma 2.5.** Every Jordan $\tau$-centralizer $f$ of $A$ maps $Z$ into $Z$.

**Proof.** For any $c \in Z$, let $a = f(c)$. Since $f$ is a Jordan $\tau$-centralizer of $A$, we have that

$$2f(cx) = f(cx + xc) = f(c)\tau(x) + \tau(x)f(c) = a\tau(x) + \tau(x)a$$

for all $x \in A$. Let $g(x) = 2f(cx)$. Then

$$g(xy + yx) = 2f(c(xy + yx)) = 2f(cxy + ycx)$$

$$= 2(f(cx)\tau(y) + \tau(y)f(cx)) = g(x)\tau(y) + \tau(y)g(x),$$

$$g(xy + yx) = 2f(c(xy + yx)) = 2f(cxy + cyx)$$

$$= 2(f(cy)\tau(x) + \tau(x)f(cy)) = g(y)\tau(x) + \tau(x)g(y)$$

for any $x, y \in A$. Thus, we have that $g$ is a Jordan $\tau$-centralizer of $A$. By Lemma 2.4, we have $a = f(c) \in Z$ for all $c \in Z$. \hfill $\Box$

**Theorem 2.6.** Each Jordan $\tau$-centralizer $f$ of $A$ is $\tau$-centralizer.

**Proof.** By Lemma 2.5, we have that

$$2f(x) = f(xf + Ix) = f(I)\tau(x) + \tau(x)f(I) = 2f(I)\tau(x) = 2\tau(x)f(I)$$

for all $x \in A$. Thus

$$f(x) = f(I)\tau(x) = \tau(x)f(I)$$

for all $x \in A$. \hfill $\Box$

**Corollary 2.7.** Let $L$ be a nest on $X$ and let $\tau$ be an epimorphism of $\text{alg } L$. Then each Jordan $\tau$-centralizer of $\text{alg } L$ is $\tau$-centralizer.
Proof. Since $\mathcal{L}$ is a nest on $X$, we have that $(\text{alg}\mathcal{L})' = CI$. By Theorem 2.6, we conclude the proof. □

**Definition 2.8.** Let $\mathcal{L}$ be a subspace lattice on $X$ and $L \in \mathcal{L}$. $L$ is said to be a comparable element of $\mathcal{L}$ if for any $M \in \mathcal{L}$, $L \subseteq M$ or $L \supseteq M$.

**Lemma 2.9** ([6, Proposition 2.9]). Suppose that $\mathcal{L}$ is a subspace lattice on $X$ with a nontrivial comparable element $M$. If there is a subspace $N$ of $X$ such that $X = M \oplus N$, then $(\text{alg}\mathcal{L})' = CI$.

By Theorem 2.6 and Lemma 2.9, we can show the following result.

**Corollary 2.10.** Let $\mathcal{L}$ be a subspace lattice on $X$ with a nontrivial comparable element $M$. If there is a subspace $N$ of $X$ such that $X = M \oplus N$ and $\tau$ is a surjective endomorphism of $\text{alg}\mathcal{L}$, then each Jordan $\tau$-centralizer of $\text{alg}\mathcal{L}$ is a $\tau$-centralizer.

**Remark 2.11.** Let $\mathcal{L} = \{(0), K, L, M, X\}$ be a pentagonal lattice on $X$. Then $(\text{alg}\mathcal{L})'$ is trivial. Hence, by Theorem 2.6, we have that each Jordan $\tau$-centralizer of $\text{alg}\mathcal{L}$ is $\tau$-centralizer.

In the following, we give a result of an algebra $\mathcal{A}$ such that the center of $\mathcal{A}$ $\neq CI$.

**Lemma 2.12.** Suppose that $\mathcal{L}$ is a CDCSL on $H$ and $\tau$ is an automorphism of $\text{alg}\mathcal{L}$. Then every Jordan $\tau$-centralizer of $\text{alg}\mathcal{L}$ maps $I$ into the center $Z$.

**Proof.** Let $e = e^2 \in \text{alg}\mathcal{L}$. Since $\tau$ is an automorphism of $\text{alg}\mathcal{L}$, it follows that $P = \tau^{-1}(e)$ such that $P = P^2 \in \text{alg}\mathcal{L}$. Since $f$ is a Jordan $\tau$-centralizer, it follows that

$$2f(P) = f(P + IP) = f(I)\tau(P) + \tau(P)f(I),$$

$$2f(P) = f(P^2 + P^2) = f(P)\tau(P) + \tau(P)f(P).$$

Thus

$$\tau(P)f(P)f(P)\tau(P) = \tau(P)f(I)\tau(P),$$

$$f(P)\tau(P) = \tau(P)f(P) = \tau(P)f(P)\tau(P).$$

By (4), (5), (6), (7), we have that

$$f(I)\tau(P) = 2f(P)\tau(P) - \tau(P)f(I)\tau(P) = \tau(P)f(P)\tau(P),$$

$$\tau(P)f(I) = 2\tau(P)f(P) - \tau(P)f(I)\tau(P) = \tau(P)f(P)\tau(P).$$

It follows that $f(I)\tau(P) = \tau(P)f(I)$. Thus $f(I)e = ef(I)$ for any $e = e^2 \in \text{alg}\mathcal{L}$. By [3, Lemma 2.3], for any $x \otimes y \in \text{alg}\mathcal{L}$, $x \otimes y \in \text{span}\{e \in \text{alg}\mathcal{L}, e = e^2\}$. We have that

$$f(I)(x \otimes y) = (x \otimes y)f(I).$$

Let $\mathcal{R}_4(\text{alg}\mathcal{L})$ be the algebra generated by all of rank one operators of $\text{alg}\mathcal{L}$. By [5, Theorem 3],

$$\mathcal{R}_4(\text{alg}\mathcal{L})^{SOT} = \text{alg}\mathcal{L}. $$
It follows that $f(I)T = Tf(I)$ for any $T$ in $\text{alg} \mathcal{L}$. So $f(I) \in Z$. \hfill \Box

**Theorem 2.13.** If $\mathcal{L}$ is a CDCSL on $H$ and $\tau$ is an automorphism of $\text{alg} \mathcal{L}$, then each Jordan $\tau$-centralizer of $\text{alg} \mathcal{L}$ is $\tau$-centralizer.

**Proof.** Let $f$ be a Jordan $\tau$-centralizer of $\text{alg} \mathcal{L}$. We have that

$$2f(x) = f(Ix + xI) = f(I)\tau(x) + \tau(x)f(I).$$

By Lemma 2.12, $f(I) \in Z$, it follows that $f(I)\tau(x) = \tau(x)f(I)$. Thus $f(x) = f(I)\tau(x) = \tau(x)f(I)$. \hfill \Box

### 3. Local $\tau$-centralizer

In this section, we suppose that $\mathcal{R}$ is a commutative ring with identity, $\mathcal{A}$ is an algebra with identity over $\mathcal{R}$, and $\tau$ is an endomorphism of $\mathcal{A}$.

**Proposition 3.1.** Suppose $\varphi : \mathcal{A} \to \mathcal{A}$ is a linear mapping and $\tau : \mathcal{A} \to \mathcal{A}$ is an endomorphism such that for any $e = e^2 \in \mathcal{A}$, $\varphi(e) \in \mathcal{A}\tau(e)$ (respectively, $\varphi(e) \in \tau(e)\mathcal{A}$). Then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any $a$ in the linear span of all idempotents in $\mathcal{A}$.

**Proof.** Suppose that $e = e^2 \in \mathcal{A}$. Since $I - e = (I - e)^2 \in \mathcal{A}$, it follows that there are $c, d$ in $\mathcal{A}$ such that $\varphi(e) = c\tau(e)$ and $\varphi(I - e) = d\tau(I - e)$. Hence $\varphi(I) = \varphi(e) + \varphi(I - e) = c\tau(e) + d\tau(I - e)$. Multiplying by $\varphi(e)$, we have that $\varphi(I)\tau(e) = c\tau(e)\varphi(e) + d\tau(I - e)\varphi(e) = c\tau(e^2) + d\tau((I - e)e) = c\tau(e) = \varphi(e)$. Thus $\varphi(a) = \varphi(I)\tau(a)$ for any $a$ in span$\{e \in \mathcal{A}, e = e^2\}$.

The proof of the other case is similar. \hfill \Box

**Proposition 3.2.** Suppose that $\varphi : \mathcal{A} \to \mathcal{A}$ is a linear mapping and $\tau : \mathcal{A} \to \mathcal{A}$ is an endomorphism such that for any $e = e^2 \in \mathcal{A}$, $\varphi(\mathcal{A}e) \subseteq \mathcal{A}\tau(e)$ (respectively, $\varphi(\mathcal{A}e) \subseteq \tau(e)\mathcal{A}$). Then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any $a$ in the algebra generated by all idempotents in $\mathcal{A}$.

**Proof.** We first show that for any idempotents $e_1, \ldots, e_n$ in $\mathcal{A}$,

$$\varphi(e_1 \cdots e_n) = \varphi(I)\tau(e_1 \cdots e_n). \tag{8}$$

If $n = 1$, by Proposition 3.1, $\varphi(e_1) = \varphi(I)\tau(e_1)$.

Suppose that if $n = k$, (8) is true. For $n = k + 1$, by assumption, there are $c, d$ in $\mathcal{A}$ such that

$$\varphi(e_1 \cdots e_ke_{k+1}) = c\tau(e_{k+1}), \varphi(e_1 \cdots e_k(I - e_{k+1})) = d\tau(I - e_{k+1}).$$

Hence

$$\varphi(e_1 \cdots e_k) = c\tau(e_{k+1}) + d\tau(I - e_{k+1}).$$

Multiplying by $\tau(e_{k+1})$, we have that

$$\varphi(e_1 \cdots e_k)\tau(e_{k+1}) = c\tau(e_{k+1}) = \varphi(e_1 \cdots e_{k+1}),$$

and therefore

$$\varphi(e_1 \cdots e_{k+1}) = \varphi(I)\tau(e_1 \cdots e_k) = \varphi(I)\tau(e_1 \cdots e_{k+1}) = \varphi(I)\tau(e_1 \cdots e_{k+1}).$$
Thus \( \varphi(a) = \varphi(I)\tau(a) \) for any \( a \) in the algebra generated by all idempotents in \( \mathcal{A} \). \( \square \)

We call a left (right) ideal \( \mathcal{T} \) of \( \mathcal{A} \) a separating left (right) set, if for any \( a \) in \( \mathcal{A} \), \( a\mathcal{T} = \{0\} \) \( (\mathcal{T}a = \{0\}) \) implies \( a = 0 \). If \( \mathcal{T} \) is both a separating left set a separating right set then we call it a separating set.

**Proposition 3.3.** Suppose \( \mathcal{A} \) has a left (right) ideal \( \mathcal{T} \) that is contained in the algebra generated by all idempotents in \( \mathcal{A} \). If \( \varphi : \mathcal{A} \to \mathcal{A} \) is a linear mapping and \( \tau : \mathcal{A} \to \mathcal{A} \) is an endomorphism of \( \mathcal{A} \) such that \( \tau(\mathcal{T}) \) is a separating left (right) set of \( \mathcal{A} \) and \( \varphi(\mathcal{A}e) \subseteq \mathcal{A}\tau(e) \) (respectively, \( \varphi(e\mathcal{A}) \subseteq \tau(e)\mathcal{A} \)) for any \( e \in \mathcal{A} \). Then \( \varphi(a) = \varphi(I)\tau(a) \) (respectively, \( \varphi(a) = \tau(a)\varphi(I) \)) for any \( a \in \mathcal{A} \).

**Proof.** We only prove the case that \( \mathcal{T} \) is a left ideal and \( \tau(\mathcal{T}) \) is a separating left set of \( \mathcal{A} \), the other case is similar.

We first show that for any idempotents \( e_1, \ldots, e_n \) in \( \mathcal{A} \), \( a \) in \( \mathcal{A} \),

\[
\varphi(a)e_1, \ldots, e_n) = \varphi(ae_1, \ldots, e_n).
\]

If \( n = 1 \), since \( \varphi(\mathcal{A}e_1) \subseteq \mathcal{A}\tau(e_1) \), \( \varphi(\mathcal{A}(I - e_1)) \subseteq \mathcal{A}\tau(I - e_1) \), we know that there are \( c_1 \) and \( d_1 \) in \( \mathcal{A} \) such that \( \varphi(ae_1) = c_1\tau(e_1), \varphi(a(I - e_1)) = d_1\tau(I - e_1) \). So

\[
\varphi(a) = \varphi(ae_1) + \varphi(a(I - e_1)) = c_1\tau(e_1) + d_1\tau(I - e_1).
\]

Thus \( \varphi(a)e_1) = c_1\tau(e_1) = \varphi(ae_1) \).

Suppose that if \( n = k \), (9) is true. For \( n = k + 1 \), by assumption, there are \( c_{k+1}, d_{k+1} \) in \( \mathcal{A} \) such that

\[
\varphi(ae_1, \ldots, e_k e_{k+1}) = c_{k+1}\tau(e_{k+1}), \varphi(ae_1, \ldots, e_k(I - e_{k+1})) = d_{k+1}\tau(I - e_{k+1}),
\]

and therefore

\[
\varphi(ae_1, \ldots, e_k) = \varphi(ae_1, \ldots, e_k e_{k+1}) + \varphi(ae_1, \ldots, e_k(I - e_{k+1})) = c_{k+1}\tau(e_{k+1}) + d_{k+1}\tau(I - e_{k+1}).
\]

It follows that

\[
\varphi(ae_1, \ldots, e_k)e_{k+1}) = c_{k+1}\tau(e_{k+1}) = \varphi(ae_1, \ldots, e_{k+1}).
\]

Thus

\[
\varphi(ae_1, \ldots, e_{k+1}) = \varphi(ae_1, \ldots, e_k)e_{k+1}) = \varphi(a)e_{k+1}).
\]

Hence \( \varphi(at) = \varphi(a)\tau(t) \), where \( t \) in the algebra generated by idempotents in \( \mathcal{A} \). In particular, \( \varphi(at) = \varphi(a)\tau(t) \) for any \( a \) in \( \mathcal{A} \), \( t \) in \( \mathcal{T} \). Since \( \mathcal{T} \) is a left ideal, it follows that

\[
\varphi(at) = \varphi(I)\tau(at) = \varphi(I)\tau(a)\tau(t).
\]

Thus \( \varphi(at) = \varphi(I)\tau(a)\tau(t) \). Since \( \tau(\mathcal{T}) \) is a separating left set, it follows that \( \varphi(a) = \varphi(I)\tau(a) \) for any \( a \in \mathcal{A} \). \( \square \)
Corollary 3.4. Suppose that $A$ has a separating left (right) set $T$ that is contained in the algebra generated by all idempotents in $A$. If $\varphi : A \to A$ is a linear mapping and $\tau : A \to A$ is an automorphism such that for any $e = e^2 \in A$, $\varphi(\mathcal{A}e) \subseteq \mathcal{A}\tau(e)$ (respectively, $\varphi(eA) \subseteq \tau(e)A$), then $\varphi(a) = \varphi(I)\tau(a)$ (respectively, $\varphi(a) = \tau(a)\varphi(I)$) for any $a \in A$.

Corollary 3.5. Suppose that a subspace lattice $\mathcal{L}$ satisfies one of the following conditions:

1. $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on a Banach space $X$,
2. $\mathcal{L}$ is CD CSL on a separable Hilbert space $H$,
3. $\mathcal{L}$ satisfies $0_+ \neq \{0\}$, $X_- \neq X$,

and $\tau$ is an automorphism of $\mathcal{L}$.

If $\varphi : \mathcal{A}\mathcal{L} \to \mathcal{A}\mathcal{L}$ is a local $\tau$-centralizer, then $\varphi$ is a $\tau$-centralizer.

**Proof.** Case 1. $\mathcal{L}$ satisfies Condition (1). Let $I = \text{span}\{T : T \in \mathcal{A}\mathcal{L}, \text{rank } T = 1\}$. Then $I$ is an ideal of $\mathcal{A}\mathcal{L}$. By [3, Lemma 2.10], $I$ is contained in the linear span of the idempotents in $\mathcal{A}\mathcal{L}$. By [3, Lemma 2.11], $I$ is a separating set of $\mathcal{A}\mathcal{L}$.

Case 2. $\mathcal{L}$ satisfies Condition (2). Let $I = \text{span}\{T : T \in \mathcal{A}\mathcal{L}, \text{rank } T = 1\}$. Then $I$ is an ideal of $\mathcal{A}\mathcal{L}$. By [3, Lemma 2.3], $I$ is contained in the linear span of the idempotents in $\mathcal{A}\mathcal{L}$. It follows from [5, Theorem 3] that $I$ is a separating set of $\mathcal{A}\mathcal{L}$.

Case 3. $\mathcal{L}$ satisfies Condition (3). Let $I = \text{span}\{x \otimes f_0, x_0 \otimes f : x \in X, f_0 \in (X_-)^\perp, x_0 \in 0_+, f \in X^*\}$. Then $I$ is an ideal of $\mathcal{A}\mathcal{L}$ and $I$ is a separating set of $\mathcal{A}\mathcal{L}$. For any $x \in X$, $0 \neq f_0 \in (X_-)^\perp$, then $x \otimes f_0 \in \mathcal{A}\mathcal{L}$. If $f_0(x) \neq 0$, then $\frac{1}{f_0(x)} x \otimes f_0$ is an idempotent in $I$. If $f_0(x) = 0$, choose $x_1 \in X$ such that $f_0(x_1) = 1$, we have that $x \otimes f_0 = \frac{1}{2}(x_1 + x) \otimes f_0 - \frac{1}{2}(x_1 - x) \otimes f_0$, both $(x_1 + x) \otimes f_0$ and $(x_1 - x) \otimes f_0$ are idempotents. The case of $x_0 \otimes f$ is similarly. Thus $I$ is contained in the algebra generated by the idempotents in $\mathcal{A}\mathcal{L}$.

Thus, by Cases 1, 2 and 3, if $\mathcal{L}$ satisfies one of above conditions, $\mathcal{A}\mathcal{L}$ has an ideal $I$ which is contained in a subalgebra of $\mathcal{A}\mathcal{L}$ generated by its idempotents and $I$ separates $\mathcal{A}\mathcal{L}$.

Since $\varphi$ is a local $\tau$-centralizer, we have that for each $x$ in $\mathcal{A}\mathcal{L}$, there is a $\tau$-centralizer $\varphi_x$ such that $\varphi(x) = \varphi_x(x)$. It follows that for any $e = e^2 \in \mathcal{A}\mathcal{L}$, $a \in \mathcal{A}\mathcal{L}$,

$$\varphi(ae) = \varphi_{ae}(ae) = \varphi_{ae}(a)\tau(e) \in (\mathcal{A}\mathcal{L})\tau(e).$$

By Corollary 3.4, $\varphi(a) = \varphi(I)\tau(a)$ for any $a \in \mathcal{A}\mathcal{L}$. Thus $\varphi$ is a left $\tau$-centralizer. Similarly, $\varphi$ is also a right $\tau$-centralizer. Hence $\varphi$ is a $\tau$-centralizer. \hfill \Box

4. Generalized derivations associate with Hochschild 2-cocycles

In this section, we suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule.
Motivated by Nakajima [7], we introduce a new type of local generalized derivation. A map \((\delta, \alpha)\) is called a local generalized derivation if for any \(x \in A\), there is a generalized derivation \((\delta_x, \alpha)\) such that \(\delta(x) = \delta_x(x)\). If \(\alpha = 0\), then \(\delta\) is a local derivation.

**Lemma 4.1.** Let \(\delta\) be a linear mapping from \(A\) into \(M\) and \(\alpha : A \times A \rightarrow M\) be a Hochschild 2-cocycle bilinear mapping. Then the following relations are equivalent

(i) \(P^\perp \delta(PAQ)Q^\perp = P^\perp \alpha(PA, Q)Q^\perp\),
(ii) \(\delta(PAQ) = \delta(PA)Q + P\delta(AQ) - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q)\), where \(P = P^2, Q = Q^2, A \in A\).

**Proof.** It is obvious that (ii) implies (i).

Suppose that (i) is true. Let \(h(x, y) = \delta(xy) - \alpha(x, y)\). Then
\[
P^\perp h(PA, Q)Q^\perp = 0,
\]
\[
Ph(A, Q)Q^\perp = Ph(PA, Q)Q^\perp = (I - P^\perp)h(PA, Q)Q^\perp = h(PA, Q)Q^\perp.
\]
Therefore, we have that
\[
h(PA, Q) - Ph(A, Q) = (h(PA, Q) - Ph(A, Q))Q
= h(PA, I)Q - h(PA, Q^\perp)Q - Ph(A, Q)Q
= h(PA, I)Q - Ph(A, Q^\perp)Q - Ph(A, Q)Q
= h(PA, I)Q - Ph(A, I)Q.
\]

Then
\[
\delta(PAQ) - \alpha(PA, Q) - P\delta(AQ) + P\alpha(A, Q)
= \delta(PA)Q - \alpha(PA, I)Q - P\delta(A)Q + P\alpha(A, I)Q.
\]
Thus
\[
\delta(PAQ) = P\delta(AQ) + \delta(PA)Q - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q)
- \alpha(PA, I)Q + P\alpha(A, I)Q.
\]
Since \(\alpha\) is Hochschild 2-cocycle, we have that
\[
P\alpha(A, I) - \alpha(PA, I) + \alpha(P, A) - \alpha(P, A) = 0.
\]
Hence
\[
\delta(PAQ) = P\delta(AQ) + \delta(PA)Q - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q).
\]

Let \(\delta\) be a linear mapping from \(A\) into \(M\) and \(\alpha : A \times A \rightarrow M\) be a Hochschild 2-cocycle bilinear mapping. We say that \((\delta, \alpha)\) satisfies the condition \((\ast)\) if
\[
\delta(PAQ) = P\delta(AQ) + \delta(PA)Q - P\delta(A)Q + \alpha(PA, Q) - P\alpha(A, Q)
\]
and \(\delta(I) = -\alpha(I, I)\) hold for each \(A \in A\) and any idempotents \(P, Q\) in \(A\).
Lemma 4.2. Suppose that \( \delta \) is a linear mapping from \( A \) into \( M \) and \( \alpha : A \times A \to M \) is a Hochschild 2-cocycle bilinear mapping satisfying the condition (*). Then

\[
\delta(P_1 \cdots P_n AQ_1 \cdots Q_m) = \delta(P_1 \cdots P_n A)Q_1 \cdots Q_m + P_1 \cdots P_n \delta(AQ_1 \cdots Q_m)
\]

\[
- P_1 \cdots P_n \delta(A)Q_1 \cdots Q_m + \alpha(P_1 \cdots P_n A, Q_1 \cdots Q_m)
\]

(10)

for any idempotents \( P_1, \ldots, P_n, Q_1, \ldots, Q_m \) in \( A \) and any \( A \) in \( A \).

Proof. We first show that for any positive integer \( n \),

\[
\delta(P_1 \cdots P_n AQ) = \delta(P_1 \cdots P_n A)Q + P_1 \cdots P_n \delta(AQ) - P_1 \cdots P_n \delta(A)Q + \alpha(P_1 \cdots P_n A, Q) - P_1 \cdots P_n \alpha(A, Q).
\]

(11)

If \( n = 1 \), by the condition (*), (11) is obvious.

Suppose that if \( n = k \), (11) is true. For \( n = k + 1 \), by the condition (*), it follows

\[
\delta(P_1 \cdots P_{k+1} AQ) = \delta(P_1 \cdots P_{k+1} A)Q + P_1 \delta(P_2 \cdots P_{k+1} A)Q
\]

\[
+ \alpha(P_1 \cdots P_{k+1} A, Q) - P_1 \delta(P_2 \cdots P_{k+1} A, Q)
\]

\[
- P_1 \delta(P_2 \cdots P_{k+1} A, Q) - P_1 \cdot P_{k+1} \delta(AQ) + \alpha(P_1 \cdots P_{k+1} A, Q)
\]

\[
+ \alpha(P_1 \cdots P_{k+1} A, Q) - P_1 \cdot P_{k+1} \alpha(A, Q).
\]

Now we show that (10) is true.

If \( m = 1 \), by (11), we have that (10) is true.

Suppose that if \( m = k \), (10) is true. For \( m = k + 1 \), by the condition (*) and (11), we have

\[
\delta(P_1 \cdots P_n AQ_1 \cdots Q_{k+1}) = \delta(P_1 \cdots P_n A)Q_1 \cdots Q_{k+1} + P_1 \cdots P_n \delta(AQ_1 \cdots Q_{k+1})
\]

\[
- P_1 \cdots P_n \delta(AQ_1 \cdots Q_k, Q_{k+1}) + \alpha(P_1 \cdots P_n A, Q_1 \cdots Q_k, Q_{k+1})
\]

\[
- P_1 \cdots P_n \alpha(AQ_1 \cdots Q_k, Q_{k+1}) + \alpha(A, Q_1 \cdots Q_k)Q_{k+1}
\]

\[
= \delta(P_1 \cdots P_n A)Q_1 \cdots Q_{k+1} + P_1 \cdots P_n \delta(AQ_1 \cdots Q_{k+1})
\]

\[
- P_1 \cdots P_n \delta(A)Q_1 \cdots Q_{k+1} + \alpha(P_1 \cdots P_n A, Q_1 \cdots Q_k, Q_{k+1})
\]

\[
+ \alpha(P_1 \cdots P_n A, Q_1 \cdots Q_k, Q_{k+1}) - P_1 \cdots P_n \alpha(AQ_1 \cdots Q_k, Q_{k+1}) + \alpha(A, Q_1 \cdots Q_k, Q_{k+1})
\]

(10)
Let $I$ be an ideal of $A$. We say that $I$ is a separating set of $M$ if for any $m, n \in M$, $mI = \{0\}$ implies $m = 0$ and $In = \{0\}$ implies $n = 0$.

**Theorem 4.3.** Let $I$ be a separating set of $M$. Suppose that $I$ is contained in the algebra generated by the idempotents in $A$. If $\delta$ is a linear mapping from $A$ into $M$ and $\alpha : A \times A \to M$ is a Hochschild 2-cocycle bilinear mapping satisfying the condition (*), then $(\delta, \alpha)$ is a generalized derivation.

**Proof.** Since $I$ is contained in the algebra generated by the idempotents in $A$, by Lemma 4.2, for any $S$ and $T$ in $I$,

$$
\delta(ST) = \delta(ST) + S\delta(T) - S\delta(I)T + \alpha(S, T) - S\alpha(I, T)
$$

$$
= \delta(ST) + S\delta(T) + \alpha(S, T) + S\alpha(I, T) - S\alpha(I, T)
$$

$$
= \delta(ST) + S\delta(T) + \alpha(S, T).
$$

Let $A$ belongs to $A$. Since $I$ is an ideal of $A$, it follows that

$$
\delta(SAT) = \delta(SA)T + S\delta(T) + \alpha(SA, T).
$$

By Lemma 4.2, we have that

$$
\delta(SAT) = \delta(SA)T + S\delta(AT) - S\delta(A)T + \alpha(SA, T).
$$

Thus

$$
(12) \quad S\delta(AT) = S\delta(T) + S\delta(A)T + S\alpha(A, T).
$$

Since $I$ is a separating set of $M$, by (12), it follows that

$$
(13) \quad \delta(AT) = A\delta(T) + \delta(A)T + \alpha(A, T).
$$

For any $A, B \in A, T \in I$, by (13),

$$
\delta(BAT) = \delta(BAT) + \delta(BA)T + \alpha(BA, T),
$$

$$
\delta(BAT) = B\delta(AT) + \delta(B)AT + \alpha(B, AT)
$$

$$
= B\delta(AT) + \alpha(BA, AT) + \alpha(B, AT).
$$

Therefore, we have that

$$
\delta(BA)T = B\delta(A)T + \delta(B)AT + \alpha(BA, T) - \alpha(BA, T) + \alpha(B, AT)
$$

$$
= B\delta(A)T + \delta(B)AT + \alpha(B, A)T.
$$

Since $I$ is a separating set of $M$, it follows that $\delta(BA) = B\delta(A) + \delta(B)A + \alpha(B, A)$.

**Corollary 4.4.** Let $I$ be a separating set of $M$. Suppose that $I$ is contained in the algebra generated by idempotents in $A$. If $(\delta, \alpha)$ is a local generalized derivation from $A$ into $M$, then $(\delta, \alpha)$ is a generalized derivation.
Proof. Since \((\delta, \alpha)\) is a local generalized derivation, we have that
\[
P \perp \delta(PAQ)Q = P \perp \delta(PAQ)(PAQ)Q = P \perp (\delta(PAQ)(PA)Q + PA\delta(PAQ)(Q) + \alpha(PA,Q))Q = P \perp \alpha(PA,Q)Q^\perp
\]
for each \(A \in \mathcal{A}\) and any idempotents \(P, Q\) in \(\mathcal{A}\). And
\[
\delta(I) = \delta(I)I + I\delta(I) + \alpha(I, I) = 2\delta(I) + \alpha(I, I).
\]
Thus \(\delta(I) = -\alpha(I, I)\). By Lemma 4.1, \(\delta\) satisfies the condition \((\ast)\). By Theorem 4.3, \((\delta, \alpha)\) is a generalized derivation.

Let \(\mathcal{A}\) be an ultraweakly closed subalgebra of \(B(H)\). The Banach space \(\mathcal{M}\) is said to be a dual normal Banach \(\mathcal{A}\)-bimodule if \(\mathcal{M}\) is a Banach \(\mathcal{A}\)-bimodule, \(\mathcal{M}\) is a dual space, and for any \(m \in \mathcal{M}\), the maps \(\mathcal{A} \ni a \rightarrow am\) and \(\mathcal{A} \ni a \rightarrow ma\) are ultraweak to weak* continuous.

Corollary 4.5. Let \(\mathcal{L}\) be a CDCSL on a complex separable Hilbert space \(H\). If \(\delta\) is a linear mapping from \(\text{alg}\mathcal{L}\) into a dual normal unital Banach \(\text{alg}\mathcal{L}\)-bimodule \(\mathcal{M}\) and \(\alpha : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{M}\) is a Hochschild 2-cocycle bilinear mapping satisfying condition \((\ast)\), then \((\delta, \alpha)\) is a generalized derivation.

Proof. Let \(I = \text{span}\{T \in \text{alg}\mathcal{L}, \text{rank}T = 1\}\). Then \(I\) is an ideal of \(\text{alg}\mathcal{L}\). By [3, Lemma 2.3], \(I\) is contained in the linear span of the idempotents in \(\text{alg}\mathcal{L}\). By [5, Theorem 3], \(I\) is a separating set of \(\mathcal{M}\). By Theorem 4.3, \((\delta, \alpha)\) is a generalized derivation.

Corollary 4.6. Let \(\mathcal{L}\) be a CDCSL on a complex separable Hilbert space \(H\). If \((\delta, \alpha)\) is a local generalized derivation from \(\text{alg}\mathcal{L}\) into a dual normal unital Banach \(\text{alg}\mathcal{L}\)-bimodule \(\mathcal{M}\), then \((\delta, \alpha)\) is a generalized derivation.

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References


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