DISCRETE MULTIPLE HILBERT TYPE INEQUALITY WITH NON-HOMOGENEOUS KERNEL

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Abstract. Multiple discrete Hilbert type inequalities are established in the case of non-homogeneous kernel function by means of Laplace integral representation of associated Dirichlet series. Using newly derived integral expressions for the Mordell-Tornheim Zeta function a set of subsequent special cases, interesting by themselves, are obtained as corollaries of the main inequality.

1. Introduction

Let $\ell_p$ be the space of all complex sequences $x = (x_n)_{n=1}^{\infty}$ with the finite norm $\|x\|_p := \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$ endowed. Let $a = (a_n)_{n=1}^{\infty} \in \ell_p$, $b = (b_n)_{n=1}^{\infty} \in \ell_q$ be nonnegative sequences and $1/p + 1/q = 1$, $p > 1$. Then

$$\sum_{m,n \in \mathbb{N}} a_m b_n \frac{m+n}{\sin(\pi/p)} \leq \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q,$$

where constant $\pi/\sin(\pi/p)$ is the best possible [3, p. 253].

This is the famous discrete Hilbert double series theorem or Hilbert inequality, a topic of interest of many mathematicians now-a-days too. The accustomary approach to deriving Hilbert’s inequality is by applying the Hölder inequality to suitably transformed Hilbert type double sum expression, i.e., to the bilinear form

$$S_K^{a,b} := \sum_{m,n \in \mathbb{N}} K(m, n) a_m b_n,$$

where $a, b$ are nonnegative and $K(\cdot, \cdot)$ is the kernel function (of the double series (2)).

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Our main goal is to derive a sharp upper bound to (7).

Let us define more general Hilbert type series. Let

\[
\lambda, \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ be monotone increasing positive functions such that }
\]

\[
\lambda_1, \ldots, \lambda_m : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ be monotone increasing positive functions such that }
\]

\[
\text{Let us denote } n := (n_1, \ldots, n_m) \text{ an } m\text{-dimensional positive integer index (running on } \mathbb{N}^m). \text{ Then the multiple Hilbert-type series reads}
\]

\[
\delta_m(\rho, a, \lambda) := \sum_{n \in \mathbb{N}^m} \left( \frac{\prod_{j=1}^{m} a_{n_j}}{\lambda_1(n_1) + \cdots + \lambda_m(n_m) + \rho} \right)^\mu \quad (\rho, \mu > 0)
\]

Our main goal is to derive a sharp upper bound to (7). Here, and in what follows, \(\mathcal{I}(x) = x\) denotes the identity function, \(f^{-1}(x)\) stands for the inverse of some \(f(x)\) and \([X], \{X\}\) are the integer part and the fractional part of some \(X\). Furthermore, \(\mathcal{D}(x)\) will stand for the Laplace integral expression of the Dirichlet series [8, §5]

\[
\mathcal{D}(x) = \sum_{n \in \mathbb{N}} a_n e^{-\lambda(n)x} = x \int_0^\infty e^{-xt} \left( \sum_{n=1}^{[\lambda^{-1}(t)]} a_n \right) dt
\]
for positive monotone increasing \((\lambda(n))_{n=1}^{\infty}\) satisfying (6). The internal sum we find using the Euler-Maclaurin summation formula given in the form \([8, \S 6]\)

\[
\sum_{j=k+1}^{\ell} a_j = \int_{k}^{\ell} \vartheta_x a(x) \, dx \quad (k, \ell \in \mathbb{Z}),
\]

where the differential operator \(\vartheta_x := 1 + [x] \frac{d}{dx}\).

2. The main result

**Theorem 1.** Assume \(m \in \mathbb{N}_2, p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1, \mu > 0, \Phi = (\Phi_1, \ldots, \Phi_m), \Phi_j, j = 1, m\) positive monotone increasing functions and let \(a_j, j = 1, m\) be nonnegative sequences such that

\[
(\Phi_j^{1/q}(n_j)a_{n_j})_{n_j=1}^{\infty} \in \ell_{p_j} \quad (j = 1, m)
\]

while \(\lambda = (\lambda_1, \ldots, \lambda_m)\) and all \(\lambda_j\) satisfies (5). Then we have

\[
\mathcal{S}_m(\rho, a, \lambda) \leq C_{\mu, q}(a, \lambda) \|a_1 \Phi_1^{1/q}\|_{p_1} \cdots \|a_m \Phi_m^{1/q}\|_{p_m},
\]

where the constant factor equals

\[
C_{\mu, q}(a, \lambda) = \left( \frac{\mu q + 1}{2} \right) \int_{\Lambda(1)}^{\infty} \frac{\prod_{j=1}^{m} \vartheta_{u_j}(1/\lambda_j(\Phi_j(u_j)))}{(t_1 + \cdots + t_m + \rho)^{\mu q + 2}} \, dtdu \right)^{1/q}.
\]

Here \(\int_{\Lambda(1)}^{\infty} := \int_{\lambda_1^{-1}(1)}^{\infty} \cdots \int_{\lambda_m^{-1}(1)}^{\infty}\) stand as the suitable abbreviations for \(m\)-tuple integrals, while \(dx := dx_1 \cdots dx_m\).

**Proof.** Assume \(q > 1\) and rewrite the \(m\)-tuple sum into the form:

\[
\mathcal{S}_m(\rho, a, \lambda) = \sum_{n_1, \ldots, n_m \in \mathbb{N}} \Phi_1^{1/q}(n_1) \cdots \Phi_m^{1/q}(n_m)a_{n_1} \cdots a_{n_m}.
\]

Then, making use of the generalized Hölder inequality with \(p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1\) ([4], [6]), we get

\[
\mathcal{S}_m(\rho, a, \lambda) \leq \left( \sum_{n_1 \in \mathbb{N}} a_{n_1}^{p_1} \Phi_1^{p_1/(p_1)}(n_1) \right)^{1/p_1} \cdots \left( \sum_{n_m \in \mathbb{N}} a_{n_m}^{p_m} \Phi_m^{p_m/(p_m)}(n_m) \right)^{1/p_m}
\]

\[
\times \left( \sum_{n_1, \ldots, n_m \in \mathbb{N}} \Phi_1(n_1) \cdots \Phi_m(n_m)(\lambda_1(n_1) + \cdots + \lambda_m(n_m) + \rho)^{\mu q} \right)^{1/q} := R.
\]
Transforming the general term of the multiple series by the Gamma function, we conclude

\[ R = \prod_{j=1}^{m} \|a_j \Phi_j^{1/q}\|_{p_j} \left( \sum_{n \in \mathbb{N}^m} \Phi_1(n_1) \cdots \Phi_m(n_m) \left( \rho + \sum_{j=1}^{m} \lambda_j(n_j) \right)^{n_q} \right)^{1/q} \]

Now, we apply the Laplace integral result (8) to all Dirichlet series

\[ D_{\lambda_j}(x) = \sum_{n_j \in \mathbb{N}} e^{-\lambda_j(n_j)x} / \Phi_j(n_j) = x \int_{\lambda_j(1)}^{\infty} e^{-xt_j} \left( \sum_{n_j=1}^{\infty} \Phi_j(n_j) \right) \]
3. Integral representation of Mordell-Tornheim Zeta-function

In 1950 Tornheim [10] introduced the double series

\[ T(p, q, r) = \sum_{n \in \mathbb{N}^2} \frac{1}{n_1^{p} n_2^{q} (n_1 + n_2)^r} \quad (p, q, r > 0, r + \min\{p, q\} > 1). \]

In honor to the author it is called Tornheim double series, also known in literature as Witten Zeta function or Mordell series. Later, a various continuations of the domain for the function \( T(p, q, r) \) are given, but here we are interested in the case \( p > 1, q > 1, r > 0, \) see, for example, [1], [2], [9], [11], [12].

Matsumoto [5] introduced the related, so-called Mordell-Tornheim m-tuple Zeta-function in the form

\[ \zeta_{MT,m}(r; s) = \sum_{n \in \mathbb{N}^m} \frac{1}{n_1^{r_1} \cdots n_m^{r_m} (n_1 + \cdots + n_m)^s}, \]

where \( r := (r_1, \ldots, r_m); r_j, s \in \mathbb{C}. \) The series (16) is absolutely convergent for \( \Re \{r_j\} > 1, j = 1, m \) and \( \Re \{s\} > 0. \) It is obvious that taking \( m = 2 \) and real parameters (16) one reduces to (15).

Our goal is here to obtain an integral representation for the real parameter Mordell-Tornheim Zeta \( \zeta_{MT,m}. \) We give two different kind integral expressions for this special function.

**Theorem 2.** Let \( r_1 > 1, \ldots, r_m > 1, s > 0. \) Then the series \( \zeta_{MT,m}(r; s) \) has the following integral representation:

\[ \zeta_{MT,m}(r; s) = \frac{1}{2^m \Gamma(s) \prod_{j=1}^{m} \Gamma(r_j)} \int_{\mathbb{R}^m_+} \frac{x^{s-1}}{\prod_{j=1}^{m} \sinh \frac{x + r_j}{2}} \prod_{j=1}^{m} \frac{t_j^{r_j-1}}{t_j^{r_j-1} e^{x + t_j} - 1} \, dt. \]

**Proof.** First, let us rewrite \( \zeta_{MT,m} \) into

\[ \zeta_{MT,m}(r_1, \ldots, r_m; s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \prod_{j=1}^{m} \left( \sum_{n_j \in \mathbb{N}} e^{-n_j x} \right) \, dx. \]

Then the inner sums become

\[ \sum_{n_j \in \mathbb{N}} e^{-n_j x} = \frac{1}{\Gamma(r_j)} \int_0^{\infty} t_j^{r_j-1} \sum_{n_j \in \mathbb{N}} e^{-(t_j + x)n_j} \, dt_j = \frac{1}{\Gamma(r_j)} \int_0^{\infty} t_j^{r_j-1} \frac{e^{-x} - 1}{e^{x + t_j} - 1} \, dt_j = \frac{1}{2 \Gamma(r_j)} \int_0^{\infty} t_j^{r_j-1} e^{-\frac{x + t_j}{2}} \sinh \frac{t_j}{2} \, dt_j. \]

Since all above series are convergent, their integral expressions converge simultaneously. The integration order interchanges are legitimate. Thus, straightforward steps result in the desired relation (17). \( \square \)
Now, taking $m = 2$ we get the integral representation for the Tornheim-Witten Zeta function.

**Corollary 1.** Let $p > 1, q > 1, r > 0$. Then the series $T(p, q, r)$ possesses the integral representation:

$$T(p, q, r) = \frac{1}{4\Gamma(p)\Gamma(q)\Gamma(r)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{-1}t_1^{-p-1}t_2^{-q-1}e^{-x-\frac{x+t_1}{2}}}{\sinh \frac{x+t_1}{2} \sinh \frac{x+t_2}{2}} \, dx \, dt_1 \, dt_2.$$  

Another kind integral expression can be given specifying $\Phi_j(n_j) = n_j^{r_j}, \lambda_j(n_j) = n_j, j = 1, m; \mu q = s$ in (12). Then, by means of (13)-(14) we deduce the following result.

**Theorem 3.** Let the situation be the same as then in the previous theorem. Then we have

$$\zeta_{MT,m}(r_1, \ldots, r_m; s) = \frac{s(s+1)}{2} \int_1^\infty \int_0^{[t]} \int_0^{[t]} \frac{\prod_{j=1}^m u_j^{-r_j}}{(t_1 + \ldots + t_m)^{s+2}} \, dt_1 \, dt_2 \, du_1 \, du_2$$

where the notations coincide with the ones in (10).

If we recall the proof of Theorem 3 and specify $m = 2$, we get a new integral representation of the Tornheim series $T(p, q, r)$.

**Corollary 2.** Let $p, q, r > 0, r + \min\{p, q\} > 1$. Then the series (15) possesses the following integral expression

$$T(p, q, r) = \left( \frac{r+1}{2} \right) \int_1^\infty \int_1^{[t]} \frac{\prod_{j=1}^m u_j^{r_j}}{(t_1 + t_2)^{s+2}} \, dt_1 \, dt_2 \, du_1 \, du_2.$$  

4. **Special cases**

In this chapter we return to our multiple Hilbert-type series in which specifying step-by-step the functions $\Phi_j, \lambda_j, j = 1, m$ we get various special cases for $\mathcal{H}_m(\rho, a, \lambda)$ and its upper bounds.

**4.1.** $\Phi_j(n_j) = n_j^{r_j}, r_j > 1$

Following the proving procedure of Theorem 1, we conclude

$$\mathcal{H}_m(\rho, a, \lambda) \leq \left( \sum_{n_1 \in \mathbb{N}} a_{n_1} n_1^{r_1}/q \right)^{1/p_1} \cdots \left( \sum_{n_m \in \mathbb{N}} a_{n_m} n_m^{r_m}/q \right)^{1/p_m} \times \left( \sum_{n \in \mathbb{N}} n \lambda_1(n_1) + \cdots + \lambda_m(n_m) + \rho \right)^{1/q}.$$
\[ D_{\lambda_j}(x) = \sum_{n_j=1}^{\infty} e^{-\lambda_j(n_j)x} n_j^{-r_j} = x \int_{\lambda_j(1)}^{\infty} e^{-xt_j} \left( \sum_{n_j=1}^{\infty} n_j^{-r_j} \right) dt_j, \]

by (9) we have

\[ D_{\lambda_j}(x) = x \int_{\lambda_j(1)}^{\infty} \int_0^{[\lambda_j^{-1}(t_j)]} e^{-xt_j} \partial_{\lambda_j}(n_j^{-r_j}) dt_j du_j. \]

So, we easily deduce the subsequent result.

**Corollary 3.** Assume \( m \in \mathbb{N}_2, q > 1, p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1, \mu > 0 \) and let \( a_j \) be nonnegative sequences such that

\[ (n_j^{r_j} a_j)_{n_j \in \ell_{p_j}} \quad (j = 1, m) \]

with a conveniently chosen \( r := (r_1, \ldots, r_m) \). When \( \lambda_1, \ldots, \lambda_m : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) are positive monotone increasing functions such that satisfy (5), we have

\[ \sum_{n \in \mathbb{N}_m} \frac{a_1 \cdots a_m}{(\rho + \sum_{j=1}^{m} \lambda_j(n_j))^\mu} \leq C_{\mu,q}(a,\lambda) \|a_1 n_1^{r_1/q}\|_{p_1} \cdots \|a_m n_m^{r_m/q}\|_{p_m}, \]

where the constant factor equals

\[ C_{\mu,q}(a,\lambda) = \left( \frac{\mu q + 1}{2} \right) \int_{\lambda(1)}^{\infty} \int_0^{[\lambda^{-1}(s)]} \prod_{j=1}^{m} \partial_{\lambda_j}(n_j^{-p_j}) \frac{dt du}{(\rho + \sum_{j=1}^{m} t_j)^{\mu q + 2}} \right)^{1/q}, \]

making use of abbreviations from Theorem 1.

**4.2.** \( \Phi_j(n_j) = n_j^{r_j}, \; r_j > 0, \lambda_j(x) = I(x) = x, \; \rho = 0 \)

In this case we get

\[ S_m(0, a, I) \leq \left( \sum_{n \in \mathbb{N}_m} n_1^{r_1} \cdots n_m^{r_m} (n_1 + \cdots + n_m)^{\mu q} \right)^{1/q} \prod_{j=1}^{m} \|a_j n_j^{r_j/q}\|_{p_j}, \]

Now, there are two possible approaches to estimate the series \( S_m(0, a, I) \). First, we apply the integral expression (17) for the Mordell-Tornheim Zeta function \( \zeta_{MT,r} \) achieved by Theorem 2.
Corollary 4. Let $m \in \mathbb{N}_2$, $q > 1$, $p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1$, $\mu > 0$, $a_j$ be nonnegative sequences such that $(n_j^{r_j/q} a_n)_{n_j \in \mathbb{N}} \in \ell_{p_j}, j = 1, m$ where $\mu + \min_{1 \leq j \leq m} \{r_j\} > 1$. Then we have

$$
\delta_m(0, a, \mathcal{I}) = \sum_{n \in \mathbb{N}^m} \frac{a_{n_1} \cdots a_{n_m}}{(n_1 + \cdots + n_m)\mu} \leq \zeta_{\mu, q}(a, \mathcal{I}) \prod_{j=1}^{m} \|a_j n_j^{r_j/q}\|_{p_j},
$$

where

$$
\zeta_{\mu, q}(a, \mathcal{I}) = \left( \frac{2^{-m}}{\Gamma(\mu q)} \prod_{j=1}^{m} \Gamma(r_j) \int_{\mathbb{R}^{m+1}} e^{x^{\mu q - 1}} \prod_{i=1}^{m} \cosh \frac{x^{p_i}}{2} \ dx \ du \right)^{1/q}
$$

with the use of notations from Theorem 1.

Second, we can apply the calculated integral representation (21) for $\zeta_{MT, m}$ exposed in Theorem 3.

Corollary 5. Suppose $m \in \mathbb{N}_2$, $q > 1$, $p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1$, $\mu > 0$, $a_j$ are nonnegative sequences such that $\|a_j n_j^{r_j/q}\|_{p_j} \in \ell_{p_j}, j = 1, m$. Then we have

$$
\delta_m(0, a, \mathcal{I}) = \sum_{n \in \mathbb{N}^m} \frac{a_{n_1} \cdots a_{n_m}}{(n_1 + \cdots + n_m)\mu} \leq \xi_{\mu, q}(a, \mathcal{I}) \prod_{j=1}^{m} \|a_j n_j^{r_j/q}\|_{p_j},
$$

where the constant factor equals

$$
\xi_{\mu, q}(a, \mathcal{I}) = \left( \left( \frac{\mu q + 1}{2} \right) \int_{1}^{\infty} \int_{0}^{\infty} \frac{\prod_{j=1}^{m} u_{j}^{-p_j}}{(t_1 + \cdots + t_m)^{\mu q + 2}} \ du \ dt \right)^{1/q}
$$

with the use of abbreviations analogous to ones in Theorem 1.

Proof. Consider the starting inequality (23). Now, we have

$$
\delta_m(0, a, \mathcal{I}) \leq \frac{\|a_1 n_1^{r_1/q}\|_{p_1} \cdots \|a_m n_m^{r_m/q}\|_{p_m}}{\Gamma^{1/q}(\mu q)}
$$

$$
\times \left( \int_{0}^{\infty} e^{-u_{1} x_{1}} \left( \sum_{n_1 \in \mathbb{N}} e^{-n_1 x_{1}} \frac{x_{1}}{n_1} \right) \cdots \left( \sum_{n_m \in \mathbb{N}} e^{-n_m x_{m}} \frac{x_{m}}{n_m} \right) \ dx \right)^{1/q}.
$$

Summing the associated Dirichlet series, having in mind their Laplace integral representations, by means of (9) we get

$$
\sum_{n_j \in \mathbb{N}} e^{-n_j x_{j} / n_j^{r_j}} = x \int_{1}^{\infty} e^{-x t} \left( \sum_{n_j = 1}^{[t_j]} n_j^{r_j} \right) dt = x \int_{1}^{\infty} \int_{0}^{[t_j]} e^{-x t} \mathcal{M}_{j}(u_{j}^{r_j}) \ du_{j} \ dt_{j}.
$$

Now, obvious transformations lead to the desired inequality. \qed
4.3. Two-dimensional Theorem 1

Taking \( m = 2 \) in Theorem 1 we deduce the following result.

**Corollary 6.** Suppose \( p_1^{-1} + p_2^{-1} + q^{-1} \geq 1, \mu > 0 \) and \( a_i, i = 1, 2 \) are nonnegative sequences such that

\[
(\Phi_i^{1/q}(n_i)a_{n_i})_{n_i=1}^{\infty} \in \ell_{p_i} \quad (i = 1, 2).
\]

Then

\[
\sum_{n_1, n_2 \in \mathbb{N}} \frac{a_{n_1}a_{n_2}}{(\lambda_1(n_1) + \lambda_2(n_2))^\mu} \leq C_{\mu, q}(a, \lambda)\|a_1\Phi_1^{1/q}\|_{p_1}\|a_2\Phi_2^{1/q}\|_{p_2}.
\]

Here the constant \( C_{\mu, q}(a, \lambda) \) takes the value

\[
\left(\frac{\mu q + 1}{2}\right)^{\frac{1}{q}} \int_1^\infty \int_1^\infty \frac{dt_1 dt_2}{(t_1 + t_2)^{\mu q + 2}} \times \int_0^{[\lambda_1^{-1}(t_1)]} \int_0^{[\lambda_2^{-1}(t_2)]} d\nu_{a_1}(1/\Phi_1(u_1))d\nu_{a_2}(1/\Phi_2(u_2)) du_1 du_2 \right)^{1/q}.
\]

4.4. \( m = 2, \Phi_i(n_i) = n_i^{\gamma_i}, \lambda_i = I \)

**Corollary 7.** Suppose \( p_1^{-1} + p_2^{-1} + q^{-1} \geq 1, \mu > 0, \Phi_i, i = 1, 2 \) positive monotone increasing functions, \( a_i, i = 1, 2 \) are nonnegative sequences such that

\[
(\Phi_i^{1/q}(n_i)a_{n_i}^{\gamma_i/q})_{n_i=1}^{\infty} \in \ell_{p_i} \quad (i = 1, 2)
\]

and \( \lambda_1, \lambda_2 \) are positive monotone increasing functions satisfying (5). Then

\[
\sum_{n_1, n_2 \in \mathbb{N}} \frac{a_{n_1}a_{n_2}}{(n_1 + n_2)^\mu} \leq C_{\mu, q}^{\gamma_2}(a, I)\|a_1n_1^{\gamma_1/q}\|_{p_1}\|a_2n_2^{\gamma_2/q}\|_{p_2},
\]

where the inequality’s constant factor \( C_{\mu, q}^{\gamma_2}(a, I) \) equals

\[
\left(\frac{\mu q + 1}{2}\right)^{\frac{1}{q}} \int_1^\infty \int_1^\infty \int_0^{[\lambda_2^{-1}(t_2)]} \frac{d\nu_{a_1}(u_1, u_2) d\nu_{a_2}(u_2, p_2)}{(t_1 + t_2)^{\mu q + 2}} dt_1 dt_2 du_1 du_2 \right)^{1/q}.
\]

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