SCREEN CONFORMAL EINSTEIN LIGHTLIKE HYPERSURFACES OF A LORENTZIAN SPACE FORM

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Abstract. In this paper, we study the geometry of lightlike hypersurfaces of a semi-Riemannian manifold. We prove a classification theorem for Einstein lightlike hypersurfaces $M$ of a Lorentzian space form subject such that the second fundamental forms of $M$ and its screen distribution $S(TM)$ are conformally related by some non-vanishing smooth function.

1. Introduction

It is well known that the normal bundle $TM^\perp$ of the lightlike hypersurfaces $(M, g)$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a vector subbundle of $TM$, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in $TM$ is non-degenerate distribution on $M$, called a screen distribution on $M$, such that

$$TM = TM^\perp \oplus \text{orth} S(TM),$$

where $\oplus \text{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. We use the same notation for any other vector bundle. For any null section $\xi$ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a null section $N$ of a vector bundle $\text{tr}(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)|_U).$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$TM = TM^\perp \oplus \text{tr}(TM) = \{TM^\perp \oplus \text{tr}(TM)\} \oplus \text{orth} S(TM).$$

We call $\text{tr}(TM)$ and $N$ the transversal vector bundle and the null transversal vector field of $M$ with respect to the screen distribution $S(TM)$ respectively.

Recently, Atindogbe-Ezin-Tossa have proved the following theorem for Einstein lightlike hypersurfaces of a Lorentzian space form in their paper [2]:

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Theorem A ([2]). Let \((M, g, S(TM))\) be a screen homothetic lightlike hypersurface of a Lorentzian space form \((\bar{M}^{m+2}(c), \bar{g})\), \(c \geq 0\). If \(M\) is Einstein, that is, \(\text{Ric} = \gamma g\) (\(\gamma\) constant), then \(\gamma \geq mc\) and

1. If \(\gamma = mc\), then \(M\) is locally a product manifold \(L \times M^*\), where the integral submanifold \(M^*\) of \(S(TM)\) is a Riemannian \(m\)-space form with the same curvature \(c\) as \(M\) and \(L\) is an open subset of a lightlike geodesic ray in \(\bar{M}\).

2. If \(\gamma > mc\), then \(M\) is locally a product \(L \times M^*\), where \(M^*\) is a Riemannian \(m\)-space form of positive constant curvature \(c + 2(\gamma - mc)\) which is isometric to a sphere.

The purpose of this paper is to prove a characterization theorem for screen conformal Einstein lightlike hypersurfaces \(M\) of a Lorentzian space form \((\bar{M}(c), \bar{g})\).

Theorem 1.1. Let \((M, g, S(TM))\) be a screen conformal Einstein lightlike hypersurface of a Lorentzian space form \((\bar{M}^{m+2}(c), \bar{g})\); \(m > 2\). Then \(c = 0\) and \(M\) is locally a product manifold \(L \times M^* \times M^\alpha \times M^\beta\), where \(L\) is an open subset of a lightlike geodesic ray in \(M\) and \(M^\alpha\) and \(M^\beta\) are leaves of some integrable distributions of \(M\) such that

1. If \(\gamma \neq 0\), either \(M^\alpha\) or \(M^\beta\) is an \(m\)-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of \(\gamma\) and the other is a point.

2. If \(\gamma = 0\), \(M^\alpha\) is an \((m-1)\)-dimensional Euclidean space and \(M^\beta\) is a non-null curve or a point.

Comparing our Theorem 1.1 with above result Theorem A, we observe that Theorem 1.1 has the following new features of geometric significance:

1. Since the key player of lightlike hypersurfaces is the integral submanifold \(M^* = M^\alpha \times M^\beta\) of the screen distribution \(S(TM)\), Theorem 1.1 provides more deeper geometry of \(M^*\) than Theorem A.

2. We prove \(c = 0\) if \(M\) is screen conformal and \(m > 2\). This is a significant result. The screen conformal is more weak condition than the screen homothetic. We can also find \(c = 0\) for arbitrary \(m\) (without the condition \(m > 2\) due to Note 2) if \(M\) is screen homothetic (as Theorem A). Contrary to this, there is no discussion on such a relationship in Atindogbe-Ezin-Tossa’s above result. Recall the following structure equations:

Let \(\nabla\) be the Levi-Civita connection of \(M\) and \(P\) the projection morphism of \(\Gamma(TM)\) on \(\Gamma(S(TM))\) with respect to the decomposition \((1.1)\). Then the local Gauss and Weingarten formulas are given by

\[
\begin{align*}
\nabla_X Y &= \nabla_X Y + B(X, Y)N, \\
\nabla_X N &= -A_N X + \tau(X)N, \\
\nabla_X PY &= \nabla_X^P Y + C(X, PY) \xi, \\
\nabla_X \xi &= -A^*_\xi X - \tau(X)\xi
\end{align*}
\]
for any $X, Y \in \Gamma(TM)$, where the symbols $\nabla$ and $\nabla^*$ are the induced linear connections on $TM$ and $S(TM)$ respectively, $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$ respectively, $A_N$ and $A^*_\xi$ are the shape operators on $TM$ and $S(TM)$ respectively and $\tau$ is a 1-form on $TM$.

Since $\nabla$ is torsion-free, $\nabla$ is also torsion-free and $B$ is symmetric. From the fact that $B(X, Y) = \bar{g}(\nabla_X Y, \xi)$ for all $X, Y \in \Gamma(TM)$, we know that $B$ is independent of the choice of a screen distribution and satisfies

$$B(X, \xi) = 0, \ \forall \ X \in \Gamma(TM).$$

(1.8)

The induced connection $\nabla$ of $M$ is not metric and satisfies

$$\eta(X, Y) = \bar{g}(X, N), \ \forall \ X \in \Gamma(TM).$$

(1.9)

But $\nabla^*$ is a metric connection. The above local second fundamental forms $B$ and $C$ of $M$ and on $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \quad B(A^*_\xi X, N) = 0,$$

(1.10)

$$C(X, PY) = g(A_N X, PY), \quad C(A_N X, N) = 0.$$

(1.11)

From (1.11), $A^*_\xi$ is $S(TM)$-valued and self-adjoint on $TM$ such that

$$A^*_\xi \xi = 0,$$

(1.12)

that is, $\xi$ is an eigenvector field of $A^*_\xi$ corresponding to the eigenvalue 0.

We denote by $\bar{R}$, $R$ and $R^*$ the curvature tensors of $\nabla$, $\nabla$ and $\nabla^*$ respectively. Using the Gauss-Weingarten equations for $M$ and $S(TM)$, we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$ such that, for any $X, Y, Z, W \in \Gamma(TM)$,

$$\bar{g}(\bar{R}(X, Y)Z, PW) = g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW),$$

(1.14)

$$\bar{g}(\bar{R}(X, Y)Z, \xi) = g(R(X, Y)Z, \xi)
= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)
+ B(Y, Z)\tau(X) - B(X, Z)\tau(Y),$$

(1.15)

$$\bar{g}(\bar{R}(X, Y)Z, N) = g(R(X, Y)Z, N),$$

(1.16)

$$\bar{g}(\bar{R}(X, Y)PZ, PW) = g(R^*(X, Y)PZ, PW) + C(X, PZ)B(Y, PW)
- C(Y, PZ)B(X, PW),$$

(1.17)

$$\bar{g}(\bar{R}(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)
+ C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X).$$

(1.18)
2. Screen conformal hypersurfaces

A lightlike hypersurface \((M, g, S(TM))\) of a semi-Riemannian manifold \((\bar{M}, \bar{g})\) is screen conformal [1] if the shape operators \(A_N\) and \(A^*_N\) of \(M\) and \(S(TM)\) respectively are related by \(A_N = \varphi A^*_N\), or equivalently,

\[
C(X, PY) = \varphi B(X, Y), \ \forall X, Y \in \Gamma(TM),
\]

where \(\varphi\) is a non-vanishing smooth function on a neighborhood \(U\) in \(M\). In particular, if \(\varphi\) is a non-zero constant, \(M\) is called screen homothetic.

**Note 1.** For a screen conformal \(M\), \(C\) is symmetric on \(S(TM)\). Thus, by [3], \(S(TM)\) is integrable and \(M\) is locally a product manifold \(L \times M^*\), where \(L\) is an open subset of a lightlike geodesic ray in \(\bar{M}\) and \(M^*\) is a leaf of \(S(TM)\).

Let \(M\) be a semi-Riemannian space form \(\bar{M}(c)\), by (1.15), we have

\[
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, Z)\tau(Y) - B(Y, Z)\tau(X)
\]

for all \(X, Y, Z \in \Gamma(TM)\). Using this, (1.16), (1.18) and (2.1), we obtain

\[
\{X[\varphi - 2\varphi\tau(X)]B(Y, PZ) - \{Y[\varphi - 2\varphi\tau(Y)]B(X, PZ)
\} = c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}.
\]

Replacing \(Y\) by \(\xi\) in (2.3), we obtain

\[
\{\xi[\varphi - 2\varphi\tau(\xi)]B(X, PZ) = cg(X, PZ).
\]

Using this equation, we have the following result.

**Theorem 2.1** ([6]). Let \((M, g, S(TM))\) be a screen conformal lightlike hypersurface of a semi-Riemannian space form \((\bar{M}^{m+2}(c), \bar{g})\); \(m > 2\). Then \(c = 0\).

**Proof.** Assume that \(c \neq 0\). Then \(\xi[\varphi - 2\varphi\tau(\xi)] \neq 0\) and \(B \neq 0\), that is, \(M\) is not a totally geodesic. From (2.1) and (2.4), we have

\[
B(X, Y) = \rho g(X, Y), \ \ C(X, Y) = \varphi \rho g(X, Y), \ \forall X, Y \in \Gamma(TM),
\]

where \(\rho = c(\xi[\varphi - 2\varphi\tau(\xi)])^{-1} \neq 0\). From (2.1) and (2.5), we get \(\varphi \rho \neq 0\). Thus \(M\) and \(S(TM)\) are not totally geodesic but totally umbilical. Since \(M\) is screen conformal, by Note 1, \(M\) is locally a product manifold \(L \times M^*\), where \(L\) is an open subset of a lightlike geodesic ray in \(\bar{M}\) and \(M^*\) is a leaf of \(S(TM)\). Since \(M\) is a space of constant curvature, from (1.14), (1.17) and (2.5), we have

\[
R^*(X, Y)Z = (c + 2\varphi \rho^2\{g(Y, Z)X - g(X, Z)Y\}
\]

for all \(X, Y, Z \in \Gamma(S(TM))\). Thus the leaf \(M^*\) of \(S(TM)\) is a semi-Riemannian manifold of curvature \((c + 2\varphi \rho^2)\). Let \(Ric^*\) be the induced symmetric Ricci tensor of \(M^*\). From (2.6), we have

\[
Ric^*(X, Y) = (c + 2\varphi \rho^2)(m - 1)g(X, Y), \ \forall X, Y \in \Gamma(S(TM)).
\]
Thus $M^*$ is an Einstein manifold. Since $M^*$ is a semi-Riemannian manifold and $m > 2$, we show that $(c + 2\varphi \rho^2)$ is a constant and $M^*$ has constant curvature $(c + 2\varphi \rho^2)$. Using (1.9), (2.2) and (2.5), we have

$$
(2.8) \quad \{X[\rho] + \rho \tau(X) - \rho^2 \eta(X)\}PY = \{Y[\rho] + \rho \tau(Y) - \rho^2 \eta(Y)\}PX.
$$

Suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $X_o[\rho] + \rho \tau(X_o) - \rho^2 \eta(X_o) \neq 0$ at each point $x \in M$. Then $PY = fPX_o$ for any $Y \in \Gamma(TM)$, where $f$ is a smooth function. It follows that all vectors from the fiber $S(TM)_x$ are co-linear with $(PX_o)_x$. It is a contradiction as $\dim (S(TM)_x) > 2$. Thus

$$
X[\rho] + \rho \tau(X) - \rho^2 \eta(X) = 0, \ \forall X \in \Gamma(TM).
$$

This implies $\xi[\rho] = \rho^2 - \rho \tau(\xi)$. Therefore, $0 = \xi[\varphi \rho^2] = \rho(c + 2\varphi \rho^2)$. Since $(c + 2\varphi \rho^2)$ is a constant and $\rho \neq 0$, we have $c + 2\varphi \rho^2 = 0$. Thus $M^*$ is a semi-Euclidean space and $C = 0$. Thus, from (2.4), we have $\varphi \rho = 0$. This means $c = 0$. It is contradiction to $c \neq 0$. Thus we have $c = 0$. 

3. Einstein lightlike hypersurfaces

The Ricci tensor $\bar{Ric}$ of $\bar{M}$ and the induced Ricci type tensor $R^{(0,2)}$ of $M$ are defined by

$$
(3.1) \quad \bar{Ric}(X, Y) = \text{trace}\{Z \rightarrow \bar{R}(X, Z)Y\}, \ \forall X, Y \in \Gamma(TM),
$$

$$
(3.2) \quad R^{(0,2)}(X, Y) = \text{trace}\{Z \rightarrow R(Z, X)Y\}, \ \forall X, Y \in \Gamma(TM).
$$

Substituting the Gauss–Codazzi equations (1.14) and (1.16) in (3.1) and using the relations (1.11) and (1.12), for all $X, Y \in \Gamma(TM)$, we obtain

$$
R^{(0,2)}(X, Y) = \bar{Ric}(X, Y) + B(X, Y)\text{tr}A_N - g(A_N X, A^*_N Y) - \bar{g}(\bar{R}(\xi, Y)X, N).
$$

A tensor field $R^{(0,2)}$ of $M$ is called its induced Ricci tensor, denoted by $\bar{Ric}$, if it is symmetric. If $M$ is a semi-Riemannian space form $(\bar{M}(c), \bar{g})$, then we have $\bar{R}(\xi, Y)X = c g(X, Y)\xi$ and $\bar{Ric}(X, Y) = (m + 1)c g(X, Y)$. Thus

$$
(3.3) \quad R^{(0,2)}(X, Y) = mc g(X, Y) + B(X, Y)\text{tr}A_N - g(A_N X, A^*_N Y).
$$

For the rest of this section, by $(M, g, S(TM))$ we shall mean a screen conformal lightlike hypersurfaces of a Lorentzian space form $(M^{m+2}(c), \bar{g})$; $m > 2$ unless otherwise specified. In this case, $S(TM)$ is Riemannian and integrable distribution and the sectional curvature $c$ of $\bar{M}(c)$ satisfies $c = 0$. For this class of lightlike hypersurfaces, $R^{(0,2)}$ is a symmetric Ricci tensor $\bar{Ric}$.

Note 2. It is well known that $R^{(0,2)}$ is symmetric if and only if each 1-form $\tau$ is closed, i.e., $d\tau = 0$, on any $U \subset M$ [5]. Therefore, suppose $R^{(0,2)}$ is symmetric, there exists a smooth function $f$ on $U$ such that $\tau = df$. Consequently we get $\tau(X) = X(f)$. If we take $\xi = \alpha \xi$, it follows that $\tau(X) = \tau(X) + X(\ln \alpha)$. Setting $\alpha = \exp(f)$ in this equation, we get $\tau(X) = 0$ for any $X \in \Gamma(TM|U)$. We call the pair $\{\xi, N\}$ on $U$ such that the corresponding 1-form $\tau$ vanishes
the distinguished null pair of $M$. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM^2 = TM/\text{Rad}(TM)$ considered by Kupeli [7]. Thus all $S(TM)$ are mutually isomorphic. For this reason, let $(M, g, S(TM))$ be a screen conformal Einstein lightlike hypersurface equipped with the distinguished null pair $(\xi, N)$ of a Lorentzian space form $(\bar{M}^{m+2}(c), \bar{g})$; $m > 2$. Under this hypothesis, we show that $\xi[\bar{g}]B(X, Y) = cg(X, Y)$ due to (2.4). Thus if $M$ is screen homothetic, then we have $c = 0$.

Let $M$ be an Einstein manifold, that is, $R^{(0,2)} = Ric = \gamma g$, where $\gamma$ is a constant if $m > 2$. Since $\xi$ is an eigenvector field of $A^*_\xi$ corresponding to the eigenvalue 0 due to (1.13) and $A^*_\xi$ is $\Gamma(S(TM))$-valued real symmetric, $A^*_\xi$ have $m$ real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of $A^*_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then

$$A^*_\xi E_i = \lambda_i E_i, \quad 1 \leq i \leq m.$$  

Since $M$ is screen conformal and $Ric = \gamma g$, the equation (3.3) reduces to

$$g(A^*_\xi X, A^*_\xi Y) - sg(A^*_\xi X, Y) + \varphi^{-1}\gamma g(X, Y) = 0,$$

where $s = \text{tr} A^*_\xi$. Put $X = Y = E_i$ in (3.4), $\lambda_i$ is a solution of equation

$$x^2 - sx + \varphi^{-1}\gamma = 0.$$  

The equation (3.5) has at most two distinct solutions which are smooth real valued function on $M$. Assume that there exists $p \in \{0, 1, \ldots, m\}$ such that $\lambda_1 = \cdots = \lambda_p = \alpha$ and $\lambda_{p+1} = \cdots = \lambda_m = \beta$, by renumbering if necessary. From (3.5), we have

$$s = \alpha + \beta = pa + (m-p)\beta; \quad \alpha \beta = \varphi^{-1}\gamma.$$  

Theorem 3.1. Let $(M, g, S(TM))$ be a screen conformal Einstein lightlike hypersurface of a Lorentzian space form $(M^{m+2}(c), \bar{g})$; $m > 2$. Then $M$ is locally a product manifold $L \times M_{\alpha} \times M_{\beta}$, where $L$ is an open subset of a lightlike geodesic ray in $\bar{M}$ and $M_{\alpha}$ and $M_{\beta}$ are totally umbilical leaves of some integrable distributions of $M$.

Proof. If the equation (3.5) has only one solution $\alpha$, then, by Note 1, we have $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$, where $M^* = M_{\alpha}$. Since $B(X, Y) = g(A^*_\xi X, Y) = \alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$, $M$ is totally umbilical. By (2.1), we get $C(X, Y) = \varphi \alpha g(X, Y)$ for all $X, Y \in \Gamma(TM)$. Thus $M^*$ is also totally umbilical. In this case, our assertion is true.

Assume the equation (3.5) has exactly two distinct solutions $\alpha$ and $\beta$. If $p = 0$ or $p = m$, then we also show that $M = L \times M^* \cong L \times M^* \times \{x\}$ for any $x \in M$ and $M^* = M_{\alpha}$ or $M_{\beta}$. In these cases, $M$ and $M^*$ are also totally umbilical. Let $0 < p < m$. Consider the following four distributions
\(D_\alpha, D_\beta, D_\alpha^p\) and \(D_\beta^p\) on \(M\):

\[
\Gamma(D_\alpha) = \{X \in \Gamma(TM) \mid A_\alpha^p X = \alpha PX\}, \quad D_\alpha^p = PD_\alpha;
\]

\[
\Gamma(D_\beta) = \{U \in \Gamma(TM) \mid A_\beta^p U = \beta PU\}, \quad D_\beta^p = PD_\beta.
\]

Then \(D_\alpha \cap D_\beta = TM^{\perp}\) and \(D_\alpha^p \cap D_\beta^p = \{0\}\). As \(A_\alpha^p PX = A_\alpha^p X = \alpha PX\) for all \(X \in \Gamma(D_\alpha)\) and \(A_\beta^p PU = A_\beta^p U = \beta PU\) for all \(U \in \Gamma(D_\beta)\), \(PX\) and \(PU\) are eigenvector fields of the real symmetric operator \(A_\alpha^p\) corresponding to the different eigenvalues \(\alpha\) and \(\beta\) respectively. Thus \(PX \perp PU\) and \(g(X, U) = g(PX, PU) = 0\), that is, \(D_\alpha \perp D_\beta\). Also, since \(B(X, U) = g(A_\alpha^p X, U) = \alpha g(PX, PU) = 0\), we show that \(D_\alpha \perp D_\beta\).

Since \(\{E_1\}_{1 \leq i \leq p}\) and \(\{E_a\}_{p+1 \leq a \leq m}\) are vector fields of \(D_\alpha^p\) and \(D_\beta^p\) respectively and \(D_\alpha^p\) and \(D_\beta^p\) are mutually orthogonal vector subbundle of \(\Gamma(TM)\), \(D_\alpha^p\) and \(D_\beta^p\) are non-degenerate distributions of rank \(p\) and rank \((m-p)\) respectively. Thus we have \(\Gamma(TM) = D_\alpha^p \oplus_{\text{orth}} D_\beta^p\).

From (3.4), we show that \((A_\alpha^p)^2 - (\alpha + \beta)A_\alpha^p + \alpha \beta P = 0\). Let \(Y \in \text{Im}(A_\alpha^p - \alpha P)\), then there exists \(X \in \Gamma(TM)\) such that \(Y = (A_\alpha^p - \alpha P)X\). Then \((A_\alpha^p - \beta P)Y = 0\) and \(Y \in \Gamma(D_\beta)\). Thus \(\text{Im}(A_\alpha^p - \alpha P) \subseteq \Gamma(D_\beta)\). Since the morphism \(A_\alpha^p - \alpha P\) maps \(\Gamma(TM)\) onto \(\Gamma(S(TM))\), we have \(\text{Im}(A_\alpha^p - \alpha P) \subseteq \Gamma(D_\beta^p)\). By duality, we also have \(\text{Im}(A_\alpha^p - \beta P) \subseteq \Gamma(D_\beta^p)\).

For \(X, Y \in \Gamma(D_\alpha^p)\) and \(U \in \Gamma(D_\beta^p)\), we have

\[
(\nabla_X B)(Y, U) = -g((A_\alpha^p - \alpha P)\nabla_X Y, U) + \alpha B(X, Y)\eta(U)
\]

and \((\nabla_Y B)(Y, U) = (\nabla_Y B)(X, U)\) due to (1.15). Thus \(g((A_\alpha^p - \alpha P)[X, Y], U) = 0\). Since the distribution \(D_\beta^p\) is non-degenerate and \(\text{Im}(A_\alpha^p - \alpha P) \subseteq \Gamma(D_\beta^p)\), we have \((A_\alpha^p - \alpha P)[X, Y] = 0\). Thus \([X, Y] \in \Gamma(D_\alpha^p)\) and \(D_\alpha^p\) is integrable. By duality, \(D_\beta^p\) is also integrable. Since \(\Gamma(TM)\) is integrable, for any \(X, Y \in \Gamma(D_\alpha^p)\), we have \([X, Y] \in \Gamma(D_\alpha^p)\) and \([X, Y] \in \Gamma(S(TM))\). Thus \([X, Y] \in \Gamma(D_\alpha^p)\) and \(D_\alpha^p\) is integrable. So is \(D_\beta^p\).

For \(X, Y \in \Gamma(D_\alpha^p)\), we have

\[
(\nabla_X B)(Y, Z) = -g((A_\alpha^p - \alpha P)\nabla_X Y, Z) + \alpha B(X, Y)\eta(Z)
\]

\[
+ (X\alpha)g(Y, Z) + \alpha^2 \eta(Y)g(X, Z).
\]

Using this and the fact that \((\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z)\), we obtain

\[(3.7) \quad \{X\alpha - \alpha^2 \eta(X)\}g(Y, Z) = \{Y\alpha - \alpha^2 \eta(Y)\}g(X, Z),\]

due to \((A_\alpha^p - \alpha P)[X, Y] = 0\). Therefore, for \(X, Y \in \Gamma(D_\alpha^p)\) and \(Z \in \Gamma(S(TM))\), we obtain \((X\alpha)g(Y, Z) = (Y\alpha)g(X, Z)\). Since \(S(TM)\) is non-degenerate, we have \(d\alpha(X)Y = d\alpha(Y)X\). Suppose there exists a vector field \(X_o \in \Gamma(D_\alpha^p)\) such that \(d\alpha(X_o)_x \neq 0\) at each point \(x \in M\). Then \(Y = fX_o\) for any \(Y \in \Gamma(D_\alpha^p)\), where \(f\) is a smooth function. It follows that all vectors from the fiber \((D_\alpha^p)_x\) are collinear with \((X_o)_x\). It is a contradiction as \(\dim((D_\alpha^p)_x) = p > 1\). Thus we have \(d\alpha|_{D_\alpha^p} = 0\). By duality, we also have \(d\beta|_{D_\beta^p} = 0\). Thus \(\alpha\) is a constant.
along $D^*_\alpha$ and $\beta$ is a constant along $D^*_\beta$. Since $(p - 1)\alpha = -(m - p - 1)\beta$, $\alpha$ and $\beta$ are constants along $S(TM)$.

From (2.3) with $\epsilon = 0$, we have

\begin{equation}
(X\varphi)B(Y, Z) = (Y\varphi)B(X, Z), \quad \forall X, Y, Z \in \Gamma(TM).
\end{equation}

Take $X, Y, Z \in \Gamma(D^*_\alpha)$, the equation (3.8) reduces to

\begin{equation}
(X\varphi)\alpha g(Y, Z) = (Y\varphi)\alpha g(X, Z), \quad \text{i.e., } d(X\varphi)\alpha Y = (Y\varphi)\alpha X.
\end{equation}

Since $\dim(D^*_\alpha)_x > 1$, we have $(X\varphi)\alpha = 0$ for all $X \in \Gamma(D^*_\alpha)$. While, take $X \in \Gamma(D^*_\alpha)$ and $Y, Z \in \Gamma(D^*_\alpha)$ in (3.8), we have $(X\varphi)\alpha = 0$ for all $X \in \Gamma(D^*_\alpha)$. Consequently, we obtain $(X\varphi)\alpha = 0$ for all $X \in \Gamma(S(TM))$. By duality, we get $(X\varphi)\beta = 0$ for all $X \in \Gamma(S(TM))$. Since $(\alpha, \beta) \neq (0, 0)$, we have $X\varphi = 0$ for all $X \in \Gamma(S(TM))$, that is, $\varphi$ is a constant along $S(TM)$. For all $X, Y \in \Gamma(D^*_\alpha)$, we have $[\varphi]\alpha = 0$ due to (2.3). Also, for all $X, Y \in \Gamma(D^*_\alpha)$, we have $[\varphi]\beta = 0$. Thus we have $[\varphi]\alpha = 0$. Consequently we have $X\varphi = 0$ for all $X \in \Gamma(TM)$, i.e., $\varphi$ is a constant on $M$. For all $X \in \Gamma(D^*_\alpha)$ and $U \in \Gamma(D^*_\beta)$, since $(\nabla_X B)(U, Z) = (\nabla_U B)(X, Z)$, we get

\begin{equation}
\frac{1}{2}((A^*_\alpha - \beta P)\nabla_X U - (A^*_\alpha - \alpha P)\nabla_U X), \quad \forall Z \in \Gamma(S(TM)).
\end{equation}

Since $S(TM)$ is non-degenerate, we have $(A^*_\alpha - \beta P)\nabla_X U = (A^*_\alpha - \alpha P)\nabla_U X$. Since the left term of the last equation is in $\Gamma(D^*_\alpha)$ and the right term is in $\Gamma(D^*_\beta)$ and $D^*_\alpha \cap D^*_\beta = \{0\}$, we have $(A^*_\alpha - \beta P)\nabla_X U = 0$ and $(A^*_\alpha - \alpha P)\nabla_U X = 0$. This implies that $\nabla_X U \in \Gamma(D^*_\beta)$ and $\nabla_U X \in \Gamma(D^*_\alpha)$. On the other hand, $\nabla_X U = \nabla^*_U U$ and $\nabla_U X = \nabla^*_U X$ due to $D^*_\alpha \perp B D^*_\beta$, we have

\begin{equation}
\nabla^*_X U \in \Gamma(D^*_\beta), \quad \nabla^*_U X \in \Gamma(D^*_\alpha), \quad \forall X \in \Gamma(D^*_\alpha); \quad \forall U \in \Gamma(D^*_\beta).
\end{equation}

For $X, Y \in \Gamma(D^*_\alpha)$ and $U, V \in \Gamma(D^*_\beta)$, since $g(X, U) = 0$, we have

\begin{equation}
g(\nabla_X U, \nabla_X U) + g(X, \nabla_X U) = 0, \quad g(\nabla_U U, X) + g(U, \nabla_U X) = 0.
\end{equation}

Using (3.9), we have $g(X, \nabla_Y U) = g(U, \nabla_Y X) = 0$. Thus we get

\begin{equation}
\nabla_X U = 0; \quad g(X, \nabla_Y U) = 0.
\end{equation}

Since the leaf $M^*$ of $S(TM)$ is a Riemannian manifold and $S(TM) = D^*_\alpha \oplus_{orth} D^*_\beta$, where $D^*_\alpha$ and $D^*_\beta$ are parallel and integrable distributions with respect to the induced connection $\nabla^*$ on $M^*$ due to (3.10), by the decomposition theorem of de Rham [8], we have $M^* = M_\alpha \times M_\beta$, where $M_\alpha$ and $M_\beta$ are some leaves of $D^*_\alpha$ and $D^*_\beta$ respectively. Thus we have our theorem.  

**Proof of Theorem 1.1.** First, we prove that $\gamma = 0$ and $\alpha \beta = 0$ for $0 < p < m$. From the facts that $(p - 1)\alpha = -(m - p - 1)\beta$ and $m \geq 2$, if $p = 1$, then $\beta = 0$ and if $p = m - 1$, then $\alpha = 0$. Thus we have $\gamma = 0$. Let $1 < p < m - 1$. Then, for $X \in \Gamma(D^*_\alpha)$ and $U \in \Gamma(D^*_\beta)$, using (3.9) and (3.10), we have

\begin{equation}
g(R(X, U)U, X) = g(\nabla_X \nabla_U U, X).
\end{equation}
From the second equation of (3.10), we know that $\nabla U$ has no component of $D_\alpha$. Since $P$ maps $\Gamma(D_\beta)$ onto $\Gamma(D_\beta)$ and $S(TM) = D_\alpha \oplus \text{orth} D_\beta$, we have
\[ \nabla U = P(\nabla U) + \eta(\nabla U)\xi; \quad P(\nabla U) \in \Gamma(D_\beta). \]
It follows that
\[ g(\nabla U, X) = g(\nabla X P(\nabla U), X) + (\nabla X \eta)(\nabla U) g(\xi, X) + \eta(\nabla U) g(\nabla X \xi, X) = -\alpha \eta(\nabla U) g(X, X). \]
Since $\eta(\nabla U) = g(U, A_N U) = \phi g(U, A^*_N U) = \phi \beta g(U, U)$, we have
\[ g(R(X, U) U, X) = -\phi \beta \alpha g(X, X) g(U, U). \]
While, from the Gauss equation (1.14), we have
\[ g(R(X, U) U, X) = \phi \alpha \beta g(X, X) g(U, U). \]
From the last two equations, we get $\gamma = \phi \alpha \beta = 0$ for $1 < p < m - 1$. Consequently we show that if $0 < p < m$, then $\gamma = 0$ and $\alpha \beta = 0$. \hfill \Box

(1) Let $\gamma \neq 0$: In case $(\text{tr} A^*_\gamma)^2 \neq 4 \phi^{-1} \gamma$. The equation (3.5) has two non-vanishing distinct solutions $\alpha$ and $\beta$. If $0 < p < m$, then we have $\gamma = 0$. Thus $p = 0$ or $p = m$. If $p = 0$, then $D_\alpha = \{0\}$ and $D_\beta = S(TM)$. If $p = m$, then $D_\alpha = S(TM)$ and $D_\beta = \{0\}$. From (1.14) and (1.18), we have
\[ R^*(X, Y) Z = 2 \phi \alpha^2 \{g(Y, Z) X - g(X, Z) Y\}, \quad \forall X, Y, Z \in \Gamma(D_\alpha); \]
\[ R^*(U, V) W = 2 \phi \beta^2 \{g(V, W) U - g(U, W) V\}, \quad \forall U, V, W \in \Gamma(D_\beta). \]
Thus either $M_\alpha$ or $M_\beta$, which are leaves of $D_\alpha$ or $D_\beta$ respectively, is a Riemannian manifold $M^*$ of constant curvature $2 \phi \alpha^2$ or $2 \phi \beta^2$ respectively and the other leaf is a point $\{x\}$. If $p = m$, that is, $M^* = M_\alpha$, since $B(X, Y) = \alpha g(X, Y)$ for all $X, Y \in \Gamma(S(TM))$, we have $C(X, Y) = \phi \alpha g(X, Y)$ for all $X, Y \in \Gamma(S(TM))$. If $p = 0$, that is, $M^* = M_\beta$, since $B(U, V) = \beta g(U, V)$ for all $U, V \in \Gamma(S(TM))$, we have $C(U, V) = \phi \beta g(U, V)$ for all $U, V \in \Gamma(S(TM))$. Thus the leaf $M^*$ is a totally umbilical which is not a totally geodesic. Consequently $M$ is locally a product manifold $L \times M^* \times \{x\}$ or $L \times \{x\} \times M^*$, where $M^*$ is an $m$-dimensional totally umbilical Riemannian manifold of constant curvature $2 \phi \beta^2$ or $2 \phi \alpha^2$ which is isometric to a sphere or a hyperbolic space, $\{x\}$ is a point.

In case $(\text{tr} A^*_\gamma)^2 = 4 \phi^{-1} \gamma$. The equation (3.5) has only one non-zero constant solution, named by $\alpha$ and $\alpha$ is only one eigenvalue of $A^*_\gamma$. In this case, the equations (3.6) reduce to $s = 2 \alpha = m \alpha; \alpha^2 = \phi^{-1} \gamma$. Thus we have $m = 2$. Thus this case is not appear.

(2) Let $\gamma = 0$. The equation (3.6) reduces to $x(x - s) = 0$. In case $\text{tr} A^*_\gamma \neq 0$. Let $\alpha = 0$ and $\beta = s$. Then we have $s = \beta = (m - p) \beta$, i.e., $(m - p - 1) \beta = 0$. So $p = m - 1$. Thus the leaf $M_\alpha$ of $D_\alpha$ is totally geodesic $(m - 1)$-dimensional Riemannian manifold and the leaf $M_\beta$ of $D_\beta$ is a spacelike curve. In the sequel,
let \( X, Y, Z \in \Gamma(D^a_\alpha) \) and \( U \in \Gamma(D^a_\beta) \). From (1.14), (1.18) and \( c = 0 \), we have \( R^*(X, Y)Z = R(X, Y)Z = \bar{R}(X, Y)Z = 0 \). Using (3.10) and the fact that the connection \( \nabla^* \) is metric, we have

\[
g(\nabla^*_X Y, U) = -g(Y, \nabla^*_X U) = -g(Y, \nabla X U) = 0.
\]

Thus \( \nabla^*_X Y \in \Gamma(D^a_\alpha) \). From this result, (1.6), (3.9) and the integrable property of \( D^a_\alpha \), we have \( g(R^*(X, Y)Z, U) = 0 \). This implies \( \pi_\alpha R^*(X, Y)Z \) and \( \pi_\alpha R^* \) is the curvature tensor of \( D^a_\alpha \). Thus \( M^\alpha \) is a Euclidean manifold.

In case \( \text{tr}A^*_\alpha = 0 \), Then we have \( \alpha = \beta = 0 \) and \( A^*_\alpha = 0 \) or equivalently \( B = 0 \) and \( D^a_\alpha = D^a_\beta = S(TM) \). Thus \( M \) is totally geodesic in \( \bar{M} \). Since \( M \) is screen conformal, we also have \( C = A_N = 0 \). Thus the leaf \( M^* \) of \( S(TM) \) is also totally geodesic. Thus we have \( \nabla_X Y = \nabla^*_X Y \) for any tangent vector fields \( X \) and \( Y \) to the leaf \( M^* \). This implies that \( M^* \) is a Euclidean \( m \)-space. Thus \( M \) is locally a product \( L \times M^\alpha \times M_\beta \), where \( M^\alpha \) is an \((m - 1)\)-dimensional Euclidean space and \( M_\beta \) is a spacelike curve in \( \bar{M} \).

References