DISCRETE PRESENTATIONS OF THE HOLOMONY GROUP OF A ONE-HOLED TORUS

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Abstract. A one-holed torus $\Sigma(1,1)$ is a building block of oriented surfaces. In this paper we formulate the matrix presentations of the holonomy group of a one-holed torus $\Sigma(1,1)$ by the gluing method. And we present an algorithm for deciding the discreteness of the holonomy group of $\Sigma(1,1)$.

1. Introduction

A hyperbolic structure on a smooth surface $M$ is a representation of $M$ as a quotient $\Omega/\Gamma$ of a convex domain $\Omega \subset \mathbb{H}^2$ by a discrete group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ acting properly and freely. If the Euler characteristic $\chi(M)$ of $M$ is negative, then the equivalence classes of hyperbolic structures on $M$ form a deformation space $\mathcal{T}(M)$ called the Teichmüller space.

Let $M$ be a compact connected smooth surface with $\chi(M) < 0$. Denote $\pi$ by the fundamental group $\pi_1(M)$ of $M$. For a given hyperbolic structure on $M$, the action of $\pi$ on the universal covering space $\tilde{M}$ of $M$ produces a homomorphism $h : \pi \to \text{PSL}(2, \mathbb{R})$ called the holonomy homomorphism and it is well-defined up to conjugation. Hence the Teichmüller space $\mathcal{T}(M)$ has a natural topology which identified with the open dense subset of the orbit space $\text{Hom}(\pi, \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$ corresponding to irreducible representations. Since the holonomy homomorphism $h : \pi \to \text{PSL}(2, \mathbb{R})$ is an isomorphism onto its image $\Gamma = h(\pi)$ called the holonomy group, the generators of $\pi$ can be presented by the matrices in $\text{PSL}(2, \mathbb{R})$ up to conjugation. (Goldman [2], Johnson and Millson [4]) Therefore giving a hyperbolic structure on $M$ is equivalent to finding a discrete subgroup $\Gamma$ of $\text{PSL}(2, \mathbb{R})$ up to conjugation. (Matsuzaki and Taniguchi [8])

Let $M = \Sigma(g,n)$ be a compact connected oriented surface with $g$-genus and $n$-boundary components. If $\chi(M) = 2 - 2g - n < 0$, then the Teichmüller space $\mathcal{T}(M)$ is diffeomorphic to $\mathbb{R}^{6g-6+3n}$. And $M$ can be decomposed as a disjoint union


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of $g$ one-holed tori $\Sigma(1,1)$ and $g-2+n$ pairs of pants $\Sigma(0,3)$. Thus a one-holed torus $\Sigma(1,1)$ and a pair of pants $\Sigma(0,3)$ are building blocks of an oriented surface $M$. (Wolpert [11])

The purposes of this paper are the followings: First we formulate the matrix presentations of the holonomy group of a one-holed torus $\Sigma(1,1)$ by the gluing method. The matrix presentations of the holonomy group of a pair of pants $\Sigma(0,3)$ in [6] will be used for the gluing method. Second we give an algorithm for deciding the discreteness of the holonomy group of $\Sigma(1,1)$.

2. PRELIMINARIES

Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half plane. The Lie group $\text{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$A \cdot z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$  \hfill (2.1)

An element $A$ of $\text{SL}(2,\mathbb{R})$ is said to be hyperbolic if $A$ has two distinct real eigenvalues. Since $f(x) = x^2 - tx + 1$ is the characteristic polynomial of $A \in \text{SL}(2,\mathbb{R})$ where $t = \text{tr}(A)$, $A$ is hyperbolic if and only if $\text{tr}(A)^2 > 4$. An element $A$ of $\text{PSL}(2,\mathbb{R})$ is said to be hyperbolic if $A$ has two distinct fixed points on $\partial \mathbb{H}^2$. Since the absolute value of trace is still defined, $A$ is hyperbolic if and only if $|\text{tr}(A)| > 2$. A hyperbolic element $A$ of $\text{PSL}(2,\mathbb{R})$ can be expressed by the diagonal matrix

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \text{let } \pm \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$ \hfill (2.2)

via an $\text{SL}(2,\mathbb{R})$-conjugation where $\alpha > 1$.

The following theorem is due to Kuiper [7].

**Theorem 2.1.** Suppose $M$ is a compact oriented hyperbolic surface. Then every nontrivial element of the holonomy group $\Gamma$ is hyperbolic.

A hyperbolic manifold $M$ can be developed into $\mathbb{H}^2$ as follows. (Thurston [10])

Since the universal covering space $\tilde{M}$ is simply connected, the coordinate charts on $\tilde{M}$ can globalize to define a hyperbolic map $\text{dev} : \tilde{M} \to \mathbb{H}^2$, called the developing map. Let $\Omega = \text{dev}(\tilde{M})$ be the developing image in $\mathbb{H}^2$. For a non-trivial element $A$ of the holonomy group $\Gamma \subset \text{PSL}(2,\mathbb{R})$, the translation length $\ell(A)$ is defined by $\ell(A) := \inf_{z \in \Omega} d_P(z, A(z))$ where $d_P$ is the Poincaré metric on $\Omega$. From Beardon's book [1], we get the relation

$$\left| \frac{\text{tr}(A)}{2} \right| = \cosh \left( \frac{\ell(A)}{2} \right).$$ \hfill (2.3)

Suppose that $|\text{tr}(A)| = \alpha + \alpha^{-1}$ with $\alpha > 1$. Since $\cosh^{-1}(t) = \log(t + \sqrt{t^2 - 1})$, Equation (2.3) becomes

$$\ell(A) = \log(\alpha^2)$$ \hfill (2.4)
for a hyperbolic element $A \in \text{PSL}(2, \mathbb{R})$.

The principal line of a hyperbolic element $A \in \text{PSL}(2, \mathbb{R})$ is the $A$-invariant unique geodesic in $\mathbb{H}^2$. It is the line joining the repelling and attracting fixed points of $A$. For easy understanding, see Figure 1, 2, and 3 or Beardon’s book [1]. We now consider the location of the principal line of $A$ and the relations of entries of $A$. The following Theorem 2.2 is some results in [6].

**Theorem 2.2.** Let $z_a, z_r$ be the attracting and repelling fixed points of a hyperbolic element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\text{let}}{=} \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{PSL}(2, \mathbb{R})$. Suppose both fixed points are finite (not infinite). Then we have the following relations.

1. $0 < z_a < z_r \iff a^2 < d^2, b c < 0, b d > 0$
2. $0 < z_r < z_a \iff a^2 > d^2, b c < 0, a c > 0$
3. $z_a < z_r < 0 \iff a^2 > d^2, b c < 0, a c < 0$
4. $z_r < z_a < 0 \iff a^2 < d^2, b c < 0, b d < 0$
5. $z_a < 0 < z_r \iff b c > 0, a c < 0, b d < 0$
6. $z_r < 0 < z_a \iff b c > 0, a c > 0, b d > 0$

![Figure 1](image1.png)

**Figure 1.** Fixed points with $0 < z_a < z_r$ and $0 < z_r < z_a$

![Figure 2](image2.png)

**Figure 2.** Fixed points with $z_a < z_r < 0$ and $z_r < z_a < 0$

**Proposition 2.3.** Let $\{x, y\}$ and $\{-b, b\}$ be the fixed points of hyperbolic elements $A, B \in \text{PSL}(2, \mathbb{R})$ respectively. Then the principal lines of $A$ and $B$ are perpendicular if and only if $b^2 = xy$.

**Proof.** (Case 1) In the case $x < y$. Consider the linear fractional transformation $f(z) = \frac{z-x}{z+y}$. This transformation $f : \mathbb{H}^2 \to \mathbb{H}^2$ is well-defined since $\det(f) =$
Figure 3. Fixed points with $z_a < 0 < z_r$ and $z_r < 0 < z_a$.

$y - x > 0$. Since $f$ is a conformal map which send the fixed points $\{x, y\}$ of $A$ to $\{0, \infty\}$, the principal lines of $A$ and $B$ are perpendicular if and only if those of $f(A)$ and $f(B)$ are perpendicular if and only if $f(-b) = -f(b)$; i.e. \( \left( \frac{-b-x}{b+y} \right) = -\left( \frac{b-x}{b+y} \right) \).

After some calculation we get the result $b^2 = xy$.

(Case II) In the case $y < x$. Consider $f(z) = \frac{z-x}{z-y}$. Then $f : \mathbb{H}^2 \to \mathbb{H}^2$ is well-defined since $\det(f) = -y + x > 0$. Similarly we can get the result $b^2 = xy$. \(\square\)

From the condition $xy = b^2 > 0$, we know both fixed points $\{x, y\}$ of $A$ should be positive or negative.

3. Holonomy Group of a One-holed Torus $\Sigma(1, 1)$

Suppose a one-holed torus $\Sigma(1, 1)$ is equipped with a hyperbolic structure. Since the holonomy homomorphism $h : \pi \to \text{PSL}(2, \mathbb{R})$ is an isomorphism onto its image $\Gamma = h(\pi)$, we will identified the fundamental group $\pi$ of $\Sigma(1, 1)$ with the holonomy group $\Gamma$; i.e. $\pi = \Gamma = \{ A, B, C \in \text{PSL}(2, \mathbb{R}) \mid R = CB^{-1}A^{-1}BA = I \}$.

Above argument is true for the hyperbolic structures. (See Goldman [2], Johnson and Millson [4].) But for a general geometric structure theory, the holonomy homomorphism $h$ may not be an isomorphism onto its image. We can find examples in Sullivan and Thurston’s paper [9].

Figure 4. A one-holed torus $M = \Sigma(1, 1)$
Let $A, B, C \in \text{PSL}(2, \mathbb{R})$ represent elements of the fundamental group of $M$ as in Figure 4. We will find the expression of the generators $A, B$ and $C$ of $\pi$ in terms of $\text{SL}(2, \mathbb{R})$ instead of $\text{PSL}(2, \mathbb{R})$ because $\text{SL}(2, \mathbb{R})$ is more convenient to compute and understand than $\text{PSL}(2, \mathbb{R})$.

We now explain about the gluing method. Let $C_1, C_2, C_3 \in \text{SL}(2, \mathbb{R})$ represent the boundary components of a pair of pants $\Sigma(0,3)$ as in Figure 5. Then the fundamental group $\pi$ of $\Sigma(0,3)$ is identified with $\pi = \langle C_1, C_2, C_3 \in \text{SL}(2, \mathbb{R}) \mid R = C_3C_2C_1 = I \rangle$.

![Figure 5. A pair of pants $M = \Sigma(0,3)$](image)

Suppose two boundary components $C_1, C_2$ of a pair of pants $\Sigma(0,3)$ have the same translation lengths; i.e. $\ell(C_1) = \ell(C_2)$. Then a one-holed torus $\Sigma(1,1)$ can be obtained by gluing two boundaries $C_1, C_2$ of a pair of pants $\Sigma(0,3)$. By the orientations of boundary components $C_1$ and $C_2$, the boundary $C_1$ is identified with $C_2^{-1}$ up to conjugation. For an easy understanding, see the Figure 6. Thus there exists a matrix $Q \in \text{SL}(2, \mathbb{R})$ such that $C_1 = Q^{-1}C_2^{-1}Q$.

![Figure 6. Gluing boundary components $C_1$ with $C_2^{-1}$](image)

Without loss of generality, we may assume that $\text{tr}(C_1) > 2$ and $\text{tr}(C_2) > 2$. In the cases $\text{tr}(C_1) < -2$ or $\text{tr}(C_2) < -2$, we replace $C_1$ to $-C_1$ or $C_2$ to $-C_2$. Suppose $\lambda, \mu$ are the eigenvalues of $C_1, C_2$ respectively with $\lambda > 1$ and $\mu > 1$. Since
\( \ell(C_1) = \log(\lambda^2) \) and \( \ell(C_2) = \log(\mu^2) \) in Equation (2.4), the condition \( \ell(C_1) = \ell(C_2) \) induces \( \lambda = \mu \).

**Theorem 3.1.** Suppose \( C_1, C_2, C_3 \in \text{SL}(2, \mathbb{R}) \) are the generators of the fundamental group of a pair of pants \( \Sigma(0, 3) \) with \( \ell(C_1) = \ell(C_2) \). If \( Q \in \text{SL}(2, \mathbb{R}) \) is a hyperbolic matrix such that \( C_1 = Q^{-1}C_2^{-1}Q \), then \( A := Q, B := C_2^{-1}, C := C_3 \) are the generators of the fundamental group of a one-holed torus \( \Sigma(1, 1) \).

**Proof.** By assumption, we have \( C_1 = Q^{-1}C_2^{-1}Q \). If we define \( A = Q, B = C_2^{-1}, C = C_3 \), then they are hyperbolic matrices and satisfy

\[
CB^{-1}A^{-1}BA = C_3C_2Q^{-1}C_2^{-1}Q = C_3C_2C_1 = I.
\]

Therefore \( A, B, C \) form the generators of the fundamental group of \( \Sigma(1, 1) \). \( \square \)

Now we find the matrix presentations of the holonomy group of a one-holed torus \( \Sigma(1, 1) \) by Theorem 3.1. The following Theorem 3.2 is one of the main results in [6]. Since the matrices \( C_1, C_2, C_3 \in \text{SL}(2, \mathbb{R}) \) are represented up to conjugation, without loss of generality, we may assume \( C_2 \) is diagonal.

**Theorem 3.2.** The following matrices \( C_1, C_2, C_3 \in \text{SL}(2, \mathbb{R}) \) with

\[
(3.1) \quad \lambda > 1, \; \mu > 1, \; a < \lambda^{-1}, \; c \neq 0
\]

form the generators of the holonomy group of a pair of pants \( \Sigma(0, 3) \).

\[
(3.2) \quad C_1 = \begin{pmatrix} a & -(\lambda - a)(\lambda^{-1} - a)c^{-1} \\ c & \lambda + \lambda^{-1} - a \end{pmatrix}, \quad C_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},
\]

and

\[
(3.3) \quad C_3 = \begin{pmatrix} \mu^{-1}(\lambda + \lambda^{-1} - a) & \mu(\lambda - a)(\lambda^{-1} - a)c^{-1} \\ -\mu^{-1}c & \mu a \end{pmatrix}.
\]

**Proof.** See Kim’s paper [6]. \( \square \)

The conditions in (3.1) are from the locations of principal lines of \( C_1, C_2, \) and \( C_3 \). See the Figure 7. In the case \( c < 0 \) (\( c > 0 \)), the fixed points of \( C_1 \) and \( C_3 \) are positive (negative) respectively. (Compare with the results (1) and (4) in Theorem 2.2.)

We now remind some relations between two hyperbolic elements \( A \) and \( \tilde{A} \) in \( \text{SL}(2, \mathbb{R}) \). See [6] for detail. For a hyperbolic element \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \), we denote \( \tilde{A} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \). Then \( z \) is a fixed point of \( A \) if and only if \( -z \) is a fixed point of \( \tilde{A} \) since \( \tilde{A}(-z) = -A(z) \). And the principal lines of \( A \) and \( \tilde{A} \) are symmetric with respect to the imaginary axis. See the Figure 8.

Without loss of generality, we assume the fixed points of hyperbolic matrices \( C_1, C_3 \) in (3.2) and (3.3) are positive; i.e. we shall assume \( c < 0 \) from now on. In the case the fixed points are negative, we just consider \( \tilde{C}_1, \tilde{C}_3 \) instead of \( C_1, C_3 \). Then we can reformulate the matrix presentations of \( C_1, C_2, C_3 \) as follows.
Figure 7. The locations of the principal lines of a pair of pants Σ(0, 3)

Figure 8. The fixed points of the matrices A and \( \tilde{A} \)

**Theorem 3.3.** The following matrices \( C_1, C_2, C_3 \in \text{SL}(2, \mathbb{R}) \) with

\begin{equation}
\lambda > 1, \quad \mu > 1, \quad a < \lambda^{-1}
\end{equation}

form the generators of the holonomy group of a pair of pants \( \Sigma(0, 3) \).

\begin{equation}
C_1 = \begin{pmatrix}
\alpha & (\lambda - a)(\lambda^{-1} - a) \\
-1 & \lambda + \lambda^{-1} - a
\end{pmatrix}, \quad C_2 = \begin{pmatrix}
\mu & 0 \\
0 & \mu^{-1}
\end{pmatrix},
\end{equation}

and

\begin{equation}
C_3 = \begin{pmatrix}
\mu^{-1}(\lambda + \lambda^{-1} - a) & -\mu(\lambda - a)(\lambda^{-1} - a) \\
\mu^{-1} & \mu a
\end{pmatrix}.
\end{equation}
Proof. We rename the matrices in (3.2) and (3.3) as $B_1, B_2,$ and $B_3$. Let $P = \begin{pmatrix} \sqrt{-c} & 0 \\ 0 & \sqrt{-c^{-1}} \end{pmatrix}$. It is well-defined since we assume $c < 0$. Then we can calculate
\[
PAP^{-1} = \begin{pmatrix} \sqrt{-c} & 0 \\ 0 & \sqrt{-c^{-1}} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \sqrt{-c^{-1}} & 0 \\ 0 & \sqrt{-c} \end{pmatrix} = \begin{pmatrix} \alpha & -c\beta \\ -c^{-1}\gamma & \delta \end{pmatrix}.
\]
Thus we get the results $PB_1P^{-1} = C_1, PB_2P^{-1} = C_2,$ and $PB_3P^{-1} = C_3$ which are the matrices in (3.5) and (3.6). Since the generators of the holonomy group of surfaces are up to conjugation, we can think the matrices $C_1, C_2$ and $C_3$ form the generators of the holonomy group of a pair of pants $\Sigma(0, 3)$. 

From now on, we denote $C_1, C_2, C_3$ as the matrices (3.5) and (3.6) in Theorem 3.3 instead of the matrices (3.2) and (3.3) in Theorem 3.2.

Consider the hyperbolic matrices $C_1, C_2 \in \text{SL}(2, \mathbb{R})$ in (3.5). Suppose that $C_1$ and $C_2$ have the same translation lengths; i.e. $\ell(C_1) = \ell(C_2)$. Since $\lambda > 1$ and $\mu > 1$, it is equivalent to $\lambda = \mu$ by Equation (2.4). Now we shall find a hyperbolic matrix $Q \in \text{SL}(2, \mathbb{R})$ such that $C_1 = Q^{-1}C_2^{-1}Q$.

Let $Q = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. After some calculations, the relation $QC_1 = C_2^{-1}Q$ induces $y = -(\lambda^{-1} - a)x$ and $w = -(\lambda - a)z$; i.e. $Q$ becomes
\[
Q = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & -(\lambda^{-1} - a)x \\ z & -(\lambda - a)z \end{pmatrix}.
\]
Then the condition $\det(Q) = 1$ implies that $1 = xw - yz = -xz(\lambda - \lambda^{-1})$. Plug in $x = 1/\sqrt{\lambda - \lambda^{-1}}$ and $z = -x = -1/\sqrt{\lambda - \lambda^{-1}}$. Then the following matrix $Q \in \text{SL}(2, \mathbb{R})$ satisfies the condition $C_1 = Q^{-1}C_2^{-1}Q$;
\[
(3.7) \quad Q = \frac{1}{\sqrt{\lambda - \lambda^{-1}}} \begin{pmatrix} 1 & -(\lambda^{-1} - a) \\ -1 & (\lambda - a) \end{pmatrix}.
\]

Proposition 3.4. Suppose we have another $\tilde{Q} \in \text{SL}(2, \mathbb{R})$ such that $C_1 = \tilde{Q}^{-1}C_2^{-1}\tilde{Q}$. Then there exists a diagonal matrix $D \in \text{SL}(2, \mathbb{R})$ such that $\tilde{Q} = DQ$.

Proof. From the condition $Q^{-1}C_2^{-1}Q = C_1 = \tilde{Q}^{-1}C_2^{-1}\tilde{Q}$, we have $(\tilde{Q}Q^{-1})C_2^{-1} = C_2^{-1}(\tilde{Q}Q^{-1})$. Since $C_2^{-1}$ is a diagonal matrix, the commutativity of $(\tilde{Q}Q^{-1})$ with $C_2^{-1}$ implies $(\tilde{Q}Q^{-1})$ should be diagonal. Therefore there exists a diagonal matrix $D \in \text{SL}(2, \mathbb{R})$ such that $\tilde{Q} = DQ$. \qed

Let $D \in \text{SL}(2, \mathbb{R})$ be a diagonal matrix with entries $D_{11} = t$ and $D_{22} = t^{-1}$. Then we have
\[
(3.8) \quad \tilde{Q} = DQ = \frac{1}{\sqrt{\lambda - \lambda^{-1}}} \begin{pmatrix} t & -t(\lambda^{-1} - a) \\ -t^{-1} & t^{-1}(\lambda - a) \end{pmatrix}.
\]
Now we shall show that the matrix $\tilde{Q}$ in (3.8) is hyperbolic.

Proposition 3.5. Let $\tilde{Q}$ be the matrix in (3.8). Then
(1) \( \text{tr}(\bar{Q}) > 2 \) if and only if \( t > 0 \).
(2) \( \text{tr}(\bar{Q}) < -2 \) if and only if \( t < 0 \).

Proof. From Theorem 3.3, we have the conditions \( a < \lambda^{-1} < 1 < \lambda \). Suppose \( t > 0 \). Then we also have \( t^{-1}(\lambda - a) > 0 \). Thus

\[
\text{tr}(\bar{Q}) = \frac{t + t^{-1}(\lambda - a)}{\sqrt{\lambda - \lambda^{-1}}} \geq \frac{2\sqrt{\lambda - a}}{\sqrt{\lambda - \lambda^{-1}}} > 2
\]

since \( (\lambda - a) > (\lambda - \lambda^{-1}) \). Conversely, suppose \( \text{tr}(\bar{Q}) > 2 \). Since \( (\lambda - a) > 0 \), the sign of \( t \) should be positive. Similarly we can show \( t < 0 \) if and only if \( \text{tr}(\bar{Q}) < 2 \). \( \square \)

Thus the matrix \( \bar{Q} \) in (3.8) is hyperbolic and satisfies \( C_1 = Q^{-1}C_2^{-1}Q \). Consider the fixed points and the principal line of \( \bar{Q} \). Let \( w_r \) and \( w_a \) be the repelling and attracting fixed points of \( \bar{Q} \). We denote \( \sqrt{\lambda - \lambda^{-1}} \bar{Q} \) by \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \). Then we have \( \beta \gamma = (\lambda^{-1} - a), \alpha \gamma = -1, \) and \( \beta \delta = -(\lambda^{-1} - a)(\lambda - a) \). From the results in Theorem 2.2, the conditions \( \beta \gamma > 0, \alpha \gamma < 0, \beta \delta < 0 \) induces \( w_a < 0 < w_r \).

Proposition 3.6. Suppose \( \bar{Q} \) is the hyperbolic matrix in (3.8). Then the attracting and repelling fixed points \( w_a \) and \( w_r \) of \( \bar{Q} \) are

\[
w_a = \frac{-D - \sqrt{E}}{2}, \quad w_r = \frac{-D + \sqrt{E}}{2}
\]

where \( D = [t^2 - (\lambda - a)] \) and \( E = D^2 + 4t^2(\lambda^{-1} - a) \).

Proof. Since \( w_a, w_r \) are the fixed points of the transformation \( \bar{Q}(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \), they are the roots of the equation \( \gamma z^2 + (\delta - \alpha) z - \beta = 0 \). Thus the fixed points \( w_a, w_r \) of \( \bar{Q} \) are

\[
\frac{(\alpha - \delta) \pm \sqrt{(\alpha - \delta)^2 + 4\beta \gamma}}{2\gamma} = \frac{[t-t^{-1}(\lambda - a)] \pm \sqrt{[t-t^{-1}(\lambda - a)]^2 + 4(\lambda^{-1} - a)}}{-2t^{-1}}
\]

\[
= \frac{-[\sqrt{2(\lambda - a)}]^2 + \sqrt{[t^2 - (\lambda - a)]^2 + 4t^2(\lambda^{-1} - a)}}{-2t^{-1}}
\]

From the above argument, we know the condition \( w_a < 0 < w_r \). Therefore we get the results \( w_a = \frac{-D - \sqrt{E}}{2} \) and \( w_r = \frac{-D + \sqrt{E}}{2} \). \( \square \)

Proposition 3.7. The principal line of \( \bar{Q} \) intersects those of \( C_1 \) and \( C_2 \).

Proof. Since the principal line of \( C_2 \) is the upper half imaginary axis, the condition \( w_a < 0 < w_r \) implies that the principal line of \( \bar{Q} \) intersects that of \( C_2 \).

The fixed points of \( C_1 \) in (3.5) are \( z_a = (\lambda^{-1} - a) \) and \( z_r = (\lambda - a) \) with \( 0 < z_a < z_r \). From the conditions \( 0 < z_a < z_r \) and \( w_a < 0 < w_r \), the principal line of \( \bar{Q} \) intersects that of \( C_1 \) if and only if \( z_a < w_r < z_r \)

\[
\iff 2(\lambda^{-1} - a) < -D + \sqrt{E} < 2(\lambda - a)
\]

\[
\iff 2(\lambda^{-1} - a) + D < \sqrt{E} < 2(\lambda - a) + D = [t^2 + (\lambda - a)]
\]

\[
\iff [2(\lambda^{-1} - a) + D]^2 < E < [t^2 + (\lambda - a)]^2 = D^2 + 4t^2(\lambda - a)
\]

\[
\iff 4(\lambda^{-1} - a)^2 + 4(\lambda^{-1} - a)D < 4t^2(\lambda^{-1} - a) < 4t^2(\lambda - a)
\]

\[
\iff (\lambda^{-1} - a) + D < t^2 < t^2(\lambda - a)(\lambda^{-1} - a)^{-1}.
\]
Both inequalities hold since \( a < \lambda^{-1} < \lambda \). \( \square \)

Suppose that the trace of \( \bar{Q} \) is positive; i.e. \( t > 0 \). Now we find when \( \bar{Q} \) has the smallest trace. From the proof of Proposition 3.5, we know the smallest value of the trace of \( \bar{Q} \) is \( 2 \sqrt{\lambda-a} / \sqrt{\lambda-a} \). And it takes when \( t = t^{-1}(\lambda-a) \). i.e. \( t = \sqrt{\lambda-a} \) since \( t > 0 \). Let \( Q_0 \) be the \( \bar{Q} \) plug in \( t = \sqrt{\lambda-a} \). Then

\[
Q_0 = \bar{Q} \big|_{t=\sqrt{\lambda-a}} = \frac{\sqrt{\lambda-a}}{\sqrt{\lambda-a}} \begin{pmatrix} 1 & -1 \lambda^{-1} - a \\ -(\lambda-a)^{-1} & 1 \end{pmatrix}.
\]

**Proposition 3.8.** Suppose that \( Q_0 \) is the hyperbolic matrix in (3.9). Then the attracting and repelling fixed points \( w_a \) and \( w_r \) of \( Q_0 \) are

\[
w_a = -\sqrt{(\lambda^{-1} - a)(\lambda-a)}, \quad w_r = \sqrt{(\lambda^{-1} - a)(\lambda-a)}.
\]

**Proof.** We denote \( Q_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \). Then the fixed points of \( Q_0 \) are \( (\alpha - \delta) \pm \sqrt{(\alpha - \delta)^2 + 4\beta \gamma} \)

\[
= \pm \frac{2\sqrt{\beta \gamma}}{2\gamma} = \mp \sqrt{(\lambda^{-1} - a)(\lambda-a)}.
\]

Since \( w_a < 0 < w_r \), we get the results. \( \square \)

**Theorem 3.9.** The principal line of \( Q_0 \) orthogonally intersects those of \( C_1 \) and \( C_2 \).

**Proof.** Denote \( b = \sqrt{(\lambda^{-1} - a)(\lambda-a)} > 0 \). Then the fixed points of \( Q_0 \) are \( \pm b \). Thus the principal line of \( Q_0 \) orthogonally intersect \( C_2 \). Since the fixed points \( x, y \) of \( C_1 \) are \( (\lambda^{-1} - a) \) and \( (\lambda-a) \), we have \( b^2 = xy \). From Proposition 2.3, the principal line of \( Q_0 \) orthogonally intersect \( C_1 \). \( \square \)

Since \( \bar{Q} \) is hyperbolic such that \( C_1 = \bar{Q}^{-1}C_2^{-1}\bar{Q} \), we have the following theorem.

**Theorem 3.10.** The generators of the holonomy group of a one-holed torus \( \Sigma(1,1) \) are expressed by

\[
(3.10) \quad A = \frac{1}{\sqrt{\lambda-a}} \begin{pmatrix} t & -t(\lambda^{-1} - a) \\ -t^{-1} & t^{-1}(\lambda-a) \end{pmatrix}, \quad B = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix},
\]

and

\[
(3.11) \quad C = \begin{pmatrix} \lambda^{-1}(\lambda + \lambda^{-1} - a) & -\lambda(\lambda-a)(\lambda^{-1} - a) \\ \lambda^{-1} & \lambda a \end{pmatrix}
\]

with \( \lambda > 1, t > 0, a < \lambda^{-1} \) up to conjugation.

**Proof.** Consider the matrices \( C_1, C_2, C_3 \) in (3.5) and (3.6). First let \( \mu = \lambda \) since \( \ell(C_1) = \ell(C_2) \). From Theorem 3.1, \( A = \bar{Q}, B = C_2^{-1}, C = C_3 \) are the generators of the holonomy group of a one-holed torus \( \Sigma(1,1) \) up to conjugation. \( \square \)
4. Application: Algorithm for Deciding the Discreteness

Finally we can present an algorithm for deciding the discreteness of the holonomy group of a one-holed torus $\Sigma(1,1)$. Let $A_r, B_r, A_a, B_a$ be the repelling and attracting fixed points of hyperbolic matrices $A$ and $B$ respectively. We define $\text{CR}(A, B)$ by the cross ratio of $B_a, A_r, A_a, B_r$; that is

$$\text{CR}(A, B) = [B_a, A_r, A_a, B_r] = \frac{(B_a - A_a)(A_r - B_r)}{(B_a - A_r)(A_a - B_r)}.$$

Then $\text{CR}(A, B) = \text{CR}(B, A)$ and represents the relations between the principal lines of $A$ and $B$.

**Definition 4.1.** The principal lines of two hyperbolic elements $A$ and $B$ are said to be **intersect** if they intersect in $\mathbb{H}^2$, **intersect at infinity** if they intersect in $\partial \mathbb{H}^2$, **separated with the same orientation** (separated with the opposite orientation) if they do not intersect in $\mathbb{H}^2 \cup \partial \mathbb{H}^2$ and $A_r < A_a < B_r < B_a$ or $A_a < A_r < B_a < B_r$ ($A_r < A_a < B_a < B_r$ or $A_a < A_r < B_r < B_a$) up to conjugation.

Consider Figure 9. The principal lines of $A, B$ are intersect, those of $B, C$ are separated with the opposite orientation, and those of $A, C$ are separated with the same orientation.

**Theorem 4.2.** Suppose $A, B$ are hyperbolic matrices in $\text{SL}(2, \mathbb{R})$. Then the principal lines of $A, B$ are

1. intersect $\iff \text{CR}(A, B) < 0$
2. intersect at infinity $\iff \text{CR}(A, B) = 0$ or $\infty$
3. separated with the opposite orientation $\iff 0 < \text{CR}(A, B) < 1$
4. separated with the same orientation $\iff \text{CR}(A, B) > 1$

**Proof.** Suppose $f$ is a linear fractional transformation such that $f(B_r) = \infty$ and $f(B_a) = 0$. Since the cross ratio is invariant under the linear fractional transformations,

$$\text{CR}(A, B) = \frac{(0 - z_a)(z_r - \infty)}{(0 - z_r)(z_a - \infty)} = \frac{z_a}{z_r},$$

where $z_a = f(A_a)$ and $z_r = f(A_r)$. If $\text{CR}(A, B) = z_a/z_r < 0$, then the fixed points of $f(A)$ have the opposite signs. Thus the principal lines of $f(A), f(B)$ are intersect. Therefore those of $A, B$ are also intersect since they are invariant under linear fractional transformations. If $\text{CR}(A, B) = z_a/z_r = 0$, then $z_a = 0$ or $z_r = \infty$. Then the principal lines of $f(A), f(B)$ are intersect at infinity. Thus those of $A, B$ have the same result. We can prove similarly the case $\text{CR}(A, B) = z_a/z_r = \infty$. If $0 < \text{CR}(A, B) = z_a/z_r < 1$, then both fixed point have the same signs. Thus if $z_r > 0$ then $0 < z_a < z_r$, and if $z_r < 0$ then $z_r < z_a < 0$. Therefore they are separated with the opposite orientation. The cases $\text{CR}(A, B) > 1$ can be similarly proved. \hfill $\square$

**Remark 4.3.** Since $A$ is hyperbolic, $A$ has two distinct fixed points. Thus the case $\text{CR}(A, B) = z_a/z_r = 1$ can not be happen.
Suppose $A, B, C$ are hyperbolic elements. Then the holonomy group
\[ \pi = \langle A, B, C \mid R = CB^{-1}A^{-1}BA = I \rangle \]
is discrete if and only if the principal lines of $A, B, C$ are located as in Figure 9 up to conjugation. (Keen [5], Goldman [3])

![Figure 9. The locations of the principal lines of a one-holed torus $\Sigma(1,1)$](image)

**Proposition 4.4.** Suppose that the matrix $C$ in (3.11) is hyperbolic. Let $C_a, C_r$ be the fixed point of $C$. Then we have
\[
C_a, C_r = \frac{F \pm \sqrt{G}}{2}
\]
where $F = (\lambda + \lambda^{-1} - a - \lambda^2 a)$ and $G = (\lambda + \lambda^{-1} - a + \lambda^2 a)^2 - 4\lambda^2$.

**Proof.** We give the same proof in Proposition 3.6. We denote $C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then the fixed points of $C$ are
\[
\frac{(\alpha - \delta) \pm \sqrt{(\alpha - \delta)^2 + 4\beta \gamma}}{2\gamma} = \frac{(\alpha - \delta) \pm \sqrt{(\alpha + \delta)^2 - 4}}{2\gamma}.
\]
Thus $C_a, C_r$ are
\[
\frac{[\lambda^{-1}(1 + \lambda^{-1} - a) - \lambda a] \pm \sqrt{[\lambda^{-1}(1 + \lambda^{-1} - a) + \lambda a]^2 - 4}}{2\lambda^{-1}}.
\]
Therefore we get the results. \(\square\)

**Theorem 4.5 (Main Theorem).** Suppose $A, B$ are hyperbolic matrices in $\text{SL}(2, \mathbb{R})$ with $\text{tr}(A) > 2$ and $\text{tr}(B) > 2$. Let $\pi = \langle A, B, C \mid CB^{-1}A^{-1}BA = I \rangle$. Then $\pi$ is discrete if and only if $\text{CR}(A, B) < 0$ and $\text{tr}(C) < -2$.

**Proof.** ($\Rightarrow$) Without loss of generality we may assume $A, B, C$ are the matrices in (3.10) and (3.11). Suppose $\pi$ is discrete. Since the principal lines of $A, B$ are intersect, we have $\text{CR}(A, B) < 0$ by Theorem 4.2. Let $C_a, C_r$ and $C_{ij}$ stand for the fixed points and the $(i, j)$-th entry of the matrix $C$. Since $0 < C_a < C_r$, we have $|C_{11}| < |C_{22}|$ and $C_{12}C_{22} > 0$ by Theorem 2.2. Then the condition
\[
C_{12}C_{22} = -\lambda^2 a(\lambda - a)(\lambda^{-1} - a) > 0
\]
implies $a < 0$ since $\lambda > 1$ and $a < \lambda^{-1}$. And the condition
\[
|C_{11}| = |\lambda^{-1}||\lambda + \lambda^{-1} - a| < |C_{22}| = |\lambda||a|
\]
induces \( \lambda^{-1}(\lambda + \lambda^{-1} - a) < \lambda(-a) \). Thus
\[
\text{tr}(C) = \lambda^{-1}(\lambda + \lambda^{-1} - a) + \lambda a < 0.
\]
Since \( C \) is hyperbolic, the trace of \( C \) should be less than \(-2\).

\((\Leftarrow)\) Since \( \text{CR}(A, B) < 0 \), the principal lines of \( A, B \) are intersect. Without loss of

generality we may assume the principal line of \( B \) is the upper half of the imaginary
axis. Thus the fixed points of \( A \) are \( w_a < 0 < w_r \) up to conjugation. To show the
discreteness, we have to show that if \( \text{tr}(C) < -2 \), then \( w_r < C_a < C_r \).

The condition \( \text{tr}(C) = \lambda^{-1}(\lambda + \lambda^{-1} - a) + \lambda a < -2 \) implies \( a < 0 \) since \( \lambda > 1 \)
and \( a < \lambda^{-1} \). Therefore we have the followings;
\[
\begin{align*}
C_{12}C_{21} & = -(\lambda - a)(\lambda^{-1} - a) < 0 \\
C_{12}C_{22} & = -\lambda^2 a(\lambda - a)(\lambda^{-1} - a) > 0 \\
|C_{11}||C_{22}| & = \lambda^{-1}(\lambda + \lambda^{-1} - a) - \lambda(-a) = \text{tr}(C) < -2.
\end{align*}
\]
By Theorem 2.2, we get \( 0 < C_a < C_r \). Hence, from Proposition 4.4, the fixed point
\( C_a \) and \( C_r \) of \( C \) should be
\[
C_a = \frac{F - \sqrt{G}}{2} \quad \text{and} \quad C_r = \frac{F + \sqrt{G}}{2}.
\]
In the proof of Proposition 3.7, we showed \( w_r < z_r = (\lambda - a) \). Thus to show \( w_r < \)
\( C_a < C_r \), it is enough to show that \( z_r < C_a \). This is equivalent to \( 2(\lambda - a) < F - \sqrt{G} \).

Since
\[
\sqrt{G} < F - 2(\lambda - a) = -\lambda + \lambda^{-1} + a - \lambda^2 a,
\]

and \( -\lambda + \lambda^{-1} + a - \lambda a > -\lambda - \lambda^{-1} + a - \lambda^2 a = -\text{tr}(C)\lambda > 0 \), it is equivalent to show that
\[
G = (\lambda + \lambda^{-1} - a + \lambda^2 a)^2 - 4\lambda^2 < (-\lambda + \lambda^{-1} + a - \lambda^2 a)^2.
\]

After some calculations, we can get the equivalent condition
\[
(\lambda^2 - 1)(\lambda - a) > 0.
\]
This is true since \( \lambda > 1 \) and \( \lambda > a \). It proves the main theorem.

We give an algorithm for deciding the discreteness of a holonomy group of a
one-holed torus \( \Sigma(1, 1) \). For given two hyperbolic matrices \( A, B \) in \( \text{SL}(2, \mathbb{R}) \),

**Step 1:** Compute \( \text{tr}(A) \) and \( \text{tr}(B) \). If \( \text{tr}(A) < -2 \), then replace \( A \) by \(-A\).
Similarly if \( \text{tr}(B) < -2 \), then replace \( B \) by \(-B\).

**Step 2:** By step 1, without loss of generality, we may assume that \( \text{tr}(A) > 2 \)
and \( \text{tr}(B) > 2 \). Compute the attracting and repelling fixed points \( A_a, A_r \) of
\( A \) and \( B_a, B_r \) of \( B \).

**Step 3:** Compute \( \text{CR}(A, B) = [B_a, A_r, A_a, B_r] \). If \( \text{CR}(A, B) < 0 \), then go to
step 4. Otherwise the hyperbolic matrices \( A, B \) can not generate a discrete
holonomy group of \( \Sigma(1, 1) \).
Step 4: Compute $C = A^{-1}B^{-1}AB$. If $\text{tr}(C) < -2$, then
\[ \pi = \langle A, B, C \in \text{SL}(2, \mathbb{R}) \mid R = CB^{-1}A^{-1}BA = I \rangle \]
is a discrete group. If $\text{tr}(C) \geq -2$, then $\pi$ is not discrete.

Using above algorithm we can make a computer program determine the discreteness of a holonomy group.

REFERENCES


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