STABILITY OF FUNCTIONAL EQUATIONS RELATED TO THE
EXPONENTIAL AND BETA FUNCTIONS

YOUNG WHAN LEE

ABSTRACT. In this paper we obtain the Hyers-Ulam stability of functional equations
\[ f(x + y) = f(x) + f(y) + \ln a^{2xy-1} \]
and
\[ f(x + y) = f(x) + f(y) + \ln \beta(x, y)^{-1} \]
which is related to the exponential and beta functions.

1. INTRODUCTION

In 1940, S. M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems [19]. One of those was the question concerning the stability of homomorphisms;

Let \( G_1 \) be a group and \( G_2 \) a metric group with a metric \( d(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( h : G_1 \rightarrow G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \rightarrow G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)?

In the next year, D. H. Hyers [7] answered the Ulam’s question for the case of the additive mapping on the Banach spaces \( G_1, G_2 \) as follows;

Let \( G_1 \) and \( G_2 \) are Banach spaces. Assume that a mapping \( f : G_1 \rightarrow G_2 \) satisfies the inequality
\[ ||f(x + y) - f(x) - f(y)|| \leq \epsilon \]

Received by the editors August 10, 2010. Revised October 28, 2010. Accepted November 22, 2010.

2000 Mathematics Subject Classification. 39B72, 39B22.

Key words and phrases. Cauchy functional equation, exponential functional equation, beta functional equation, stability of functional equation, solution of functional equation.

© 2010 The Korean Society of Mathematical Education
for all \( x, y \in G_1 \). Then the limit \( g(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) exists for all \( x \in G_1 \) and \( g \) is the unique additive mapping satisfying

\[
\|f(x) - g(x)\| \leq \varepsilon
\]

for all \( x \in G_1 \).

In 1978, Th. M. Rassias [16] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded;

Let \( G_1 \) be a vector space and \( G_2 \) a Banach space. Assume that a mapping \( f : G_1 \to G_2 \) satisfies

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)
\]

for all \( x, y \in G_1 \), \( \varepsilon > 0 \) and \( p < 1 \). Then the limit \( g(x) := \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) exists for all \( x \in G_1 \) and \( g \) is the unique additive mapping satisfying

\[
\|f(x) - g(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p
\]

for all \( x \in G_1 \).

In 1991, Z. Gajda [3] gave an affirmative solution to this question by the same approach as in Th. M. Rassias [16]. It was also shown by Z. Gajda [3], as well as by Th. M. Rassias and P. Šemrl [17] that one cannot prove the Th. M. Rassias’ type theorem when \( p = 1 \). These results provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept.

P. Găvruţa [4] provided a further generalization of Th. M. Rassias’ Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [1]-[19]).

Gilányi [5] and Râtz [18] showed that if satisfies the functional inequality

\[
\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|,
\]

then \( f \) satisfies the Jordan-von Neumann functional equation

\[
2f(x) + 2f(y) = f(xy) + f(xy^{-1}).
\]


\[
\|af(x) + bf(y) + cf(z)\| \leq \|f(ax + by + cz)\| + \phi(x, y, z).
\]
Now, we consider the following functional equations (1) and (2)

\[(1) \quad f(x + y) = f(x) + f(y) + \ln a^{2xy-1}\]

and

\[(2) \quad f(x + y) = f(x) + f(y) + \ln \beta(x, y)^{-1} .\]

Then functional inequalities with a perturbing term \(\delta\) are represented by

\[|f(x + y) - (f(x) + f(y) + \ln a^{2xy-1})| \leq \delta\]

and

\[|f(x + y) - (f(x) + f(y) + \ln \beta(x, y)^{-1})| \leq \delta.\]

The purpose of this paper is to prove that if \(f\) satisfies the above inequalities, then we can find the stable mappings satisfying the equation (1) and (2) near an approximate mapping \(f\), and thus we prove the Hyers-Ulam stability of the above functional inequalities.

2. Solutions of the Equation (1) and (2)

Let \(f(x) = \ln a^{x^2+x+1}\). Then \(f(x + y) = f(x) + f(y) + \ln a^{2xy-1}\). Thus this function is a solution of the functional equation (1). Now consider the gamma and beta functions

\[\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}dt\]

and

\[\beta(x, y) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1}dt.\]

It is well known that these gamma and beta functions satisfy the equation

\[\beta(x, y)\Gamma(x + y) = \Gamma(x)\Gamma(y) .\]

Thus if we let \(f(x) = \ln \Gamma(x)\), then

\[f(x + y) = f(x) + f(y) + \ln \beta(x, y)^{-1},\]

and so this logarithm of the gamma function is a solution of the functional equation (2).

3. Stability of the Functional Equation (1) and (2)

The following theorem is the Hyers-Ulam stability of the functional equation

\[f(x + y) = f(x) + f(y) + \ln a^{2xy-1} \quad (a > 0)\]

which is related to the exponential function.
Theorem 1. Let $\delta > 0$ and $a > 0$ be given. Assume that a mapping $f : R \to R$ satisfies the functional inequality

$$|f(x + y) - f(x) - f(y) - \ln a^{2xy - 1}| < \delta$$

for all $x, y \in R$. Then there exists a unique mapping $g : R \to R$ such that

$$g(x + y) = g(x) + g(y) + \ln a^{2xy - 1}$$

for all $x, y \in R$ and

$$|f(x) - g(x)| \leq \delta$$

for all $x \in R$. In particular, $g$ is defined by

$$g(x) := \lim_{n \to \infty} P_n(x)$$

where

$$P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=1}^{n} a^{2^{i-1} \cdot 2^{i-1}}$$

for all $x \in R$.

Proof. If we replace $y$ by $x$ and dividing 2 in (3), we get

$$\left| \frac{f(2x)}{2} - \ln a^{x^2 - 1} - f(x) \right| \leq \frac{\delta}{2}$$

for all $x \in R$. We use induction on $n$ to prove

$$\left| \frac{f(2^n x)}{2^n} - \ln a^{\sum_{i=1}^{n} \frac{2^{i-1} \cdot 2^{i-1}}{2^i}} - f(x) \right| \leq \delta \sum_{i=1}^{n} \frac{1}{2^i}$$

for all $x \in R$. On account of (4), the inequality holds for $n = 1$. Suppose that the inequality (5) holds true for some integer $n > 1$. Then (4) and (5) imply

$$\left| \frac{f(2^{n+1} x)}{2^{n+1}} - \ln a^{\sum_{i=0}^{n} \frac{2^{i+1} \cdot 2^{i-1}}{2^{i+1}}} - f(x) \right|$$

$$\leq \left| \frac{f(2^n \cdot 2x)}{2^n} - \frac{1}{2} \ln a^{\sum_{i=1}^{n} \frac{2^{i-1} \cdot (2x)^2 - 1}{2^i} - f(2x)} - \frac{f(2x)}{2} \right| + \left| \frac{f(2x)}{2} - \ln a^{x^2 - 1} - f(x) \right|$$

$$\leq \delta \sum_{i=1}^{n+1} \frac{1}{2^i}$$

which ends the proof of (5). For any $x \in R$ and for every positive integer $n$ we define that

$$P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=1}^{n} a^{2^{i-1} \cdot 2^{i-1}}.$$
Let \( m, n > 0 \) be integers with \( n > m \). Then it follows from (3)

\[
|P_n(x) - P_m(x)|
\leq \frac{1}{2^n} \left| f\left(2^{n-m} (2^m x)\right) - \ln \left( \prod_{i=1}^{n-m} a \frac{2^{i-1} (2^m x)^{2^{-i}} - 1}{2^{i}} \right) - f(2^m x) \right|
\leq \delta \sum_{i=1}^{n-m} \frac{1}{2^i} \to 0
\]
as \( m \to \infty \). Therefore, the sequence \( \{P_n(x)\} \) is a Cauchy sequence, and we may define a function \( g : R \to R \) by

\[
g(x) := \lim_{n \to \infty} P_n(x)
\]
for all \( x \in R \). Now we prove that

\[
g(x + y) = g(x) + g(y) + \ln a^{2xy-1}
\]
for all \( x, y \in R \). For this, we consider the following property.

\[
\ln \left( \prod_{i=1}^{n} a \frac{2^{(2i-1)x_{2^{-i}} - 1}}{2^{i}} \prod_{i=1}^{n} a \frac{2^{(2i-1)y_{2^{-i}} - 1}}{2^{i}} \right) = \ln \frac{a^{-(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n})}}{a^{2xy-1} a^{2xy(1+2+\ldots+2^{n-1})}}
\leq \frac{1}{2^n} \ln \left[ \frac{1}{a^{2(2^nx)(2^ny)-1}} \right]
\]
for all \( x, y \in R \). Thus we have

\[
|g(x + y) - g(x) - g(y) - \ln a^{2xy-1}| = \lim_{n \to \infty} \left| f\left(2^n x + 2^n y\right) - f\left(2^n x\right) - f\left(2^n y\right) - \frac{1}{2^n} \ln a^{2(2^nx)(2^ny)-1}\right|
\]

and so

\[
g(x + y) = g(x) + g(y) + \ln a^{2xy-1}
\]
for all \( x, y \in R \). From the inequality (5), we have

\[
|f(x) - g(x)| \leq \delta
\]
for all \( x \in R \). Now suppose that \( h \) satisfies the equation

\[
h(x + y) = h(x) + h(y) + \ln a^{2xy-1}
\]
for all \( x, y \in R \) and

\[
|f(x) - h(x)| \leq \delta
\]
for all $x \in R$. Then
\[
|g(x) - h(x)| 
\leq \lim_{n \to \infty} \frac{1}{2^n} \left| f(2^n x) - h(2^n x) \right| + \lim_{n \to \infty} \left| \frac{h(2^n x)}{2^n} - \ln \prod_{i=1}^{n} \frac{2^{2i-1}x^2 - 1}{x^2} - h(x) \right|
\]
\[
= \lim_{n \to \infty} \frac{\delta}{2^n} + 0 = 0
\]
as $n \to \infty$ and for all $x \in R$, and thus $g$ is unique. \qed

**Corollary 2.** Let $A$ be a Banach algebra and $\delta > 0$ be given. Assume that a mapping $f : A \to A$ satisfies the functional inequality
\[
||f(x + y) - f(x) - f(y) - (2xy - 1)|| < \delta
\]
for all $x, y \in A$. Then there exists a unique mapping $g : A \to A$ such that
\[
g(x + y) = g(x) + g(y) + 2xy - 1
\]
for all $x, y \in A$ and
\[
|f(x) - g(x)| \leq \delta
\]
for all $x \in A$. In particular, $g$ is defined by
\[
g(x) := \lim_{n \to \infty} P_n(x)
\]
where
\[
P_n(x) = \frac{f(2^n x)}{2^n} - \sum_{i=1}^{n} \left( \frac{2^{2i-1}x^2 - 1}{2^i} \right)
\]
for all $x, y \in A$.

**Proof.** From Theorem 1 with $a = e$, we complete the proof. \qed

The following theorem is the Hyers-Ulam stability of the functional equation
\[
f(x + y) = f(x) + f(y) + \ln \beta(x, y)^{-1}
\]
which is related to the beta function.

**Theorem 3.** Let $\delta > 0$ and $a > 0$ be given. Assume that a mapping $f : (0, \infty) \to (0, \infty)$ satisfies the functional inequality
\[
|f(x + y) - f(x) - f(y) - \ln \beta(x, y)^{-1}| < \delta
\]
for all $x, y \in (0, \infty)$. Then there exists a unique mapping $g : (0, \infty) \to (0, \infty)$ such that
\[
g(x + y) = g(x) + g(y) + \ln \beta(x, y)^{-1}
\]
for all \( x, y \in (0, \infty) \) and
\[
|f(x) - g(x)| \leq \delta
\]
for all \( x \in (0, \infty) \). In particular, \( g \) is defined by
\[
g(x) := \lim_{n \to \infty} P_n(x)
\]
where
\[
P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i x)^{\frac{1}{2i+1}}
\]
for all \( x, y \in (0, \infty) \).

**Proof.** If we replace \( y \) by \( x \) and dividing 2 in (6), we get
\[
\left| \frac{f(2x)}{2} + \ln \beta(x, x)^{\frac{1}{2}} - f(x) \right| \leq \frac{\delta}{2}
\]
for all \( x \in R \). We use induction on \( n \) to prove
\[
\left| \frac{f(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i x)^{\frac{1}{2i+1}} - f(x) \right| \leq \delta \sum_{i=1}^{n-1} \frac{1}{2i+1}
\]
for all \( x > 0 \). On account of (7), the inequality holds for \( n = 1 \). Suppose that inequality (8) holds true for some integer \( n > 1 \). Then (7) and (8) imply
\[
\left| \frac{f(2^{n+1} x)}{2^{n+1}} + \ln \prod_{i=0}^{n} \beta(2^i x, 2^i x)^{\frac{1}{2i+1}} - f(x) \right|
\]
\[
\leq \left| \frac{f(2^n \cdot 2x)}{2 \cdot 2^n} + \frac{1}{2} \ln \prod_{i=0}^{n-1} \beta(2^i \cdot 2x, 2^i \cdot 2x)^{\frac{1}{2i+1}} - \frac{f(2x)}{2} \right|
\]
\[
+ \left| \frac{f(2x)}{2} + \ln \beta(x, x)^{\frac{1}{2}} - f(x) \right|
\]
\[
\leq \delta \sum_{i=0}^{n} \frac{1}{2i+1}
\]
for any \( x > 0 \), which ends the proof of (8). For any \( x > 0 \) and for every positive integer \( n \) we define that
\[
P_n(x) = \frac{f(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i x)^{\frac{1}{2i+1}}
\]
for all \(x, y > 0\). Let \(m, n > 0\) be integers with \(n > m\). Then it follows from (6) that for all \(x > 0\)

\[
|P_n(x) - P_m(x)|
= \frac{1}{2^m} \left| \frac{f(2^{n-m}(2^m x))}{2^{n-m}} + \ln \prod_{i=0}^{n-1} \frac{\beta(2^i x, 2^i x)}{2^{n-m}} \beta(2^i x, 2^i x) \frac{1}{\frac{1}{2^{n-m}} - f(2^m x)} \right|
\leq \delta \sum_{i=m}^{n-1} \frac{1}{2^{i+1}} \to 0
\]
as \(m \to \infty\). Therefore, the sequence \(\{P_n(x)\}\) is a Cauchy sequence, and we may define a function \(g : (0, \infty) \to (0, \infty)\) by

\[
g(x) := \lim_{n \to \infty} P_n(x)
\]
for all \(x > 0\). Now we prove that

\[
g(x + y) = g(x) + g(y) + \ln \beta(x, y)^{-1}
\]
for all \(x, y > 0\). For this, we consider the following property of the beta function.

\[
\beta(x + y, x + y) = \frac{\beta(x, x + y) \beta(y, y + 2x)}{\beta(x, y)} = \frac{\beta(x, x) \beta(y, y) \beta(2x, 2y)}{\beta(x, y)^2}
\]
for all \(x, y > 0\). By this property, we have the equation

\[
\prod_{i=0}^{n-1} \left[ \frac{\beta(2^i x, 2^i y)}{\beta(2^i x, 2^i y) \beta(2^i y, 2^i y)} \right]^{\frac{1}{n+1}} = \prod_{i=0}^{n-1} \left[ \frac{\beta(2^i+1 x, 2^i+1 y)}{\beta(2^i x, 2^i y)^2} \right]^{\frac{1}{n+1}}
\]

\[
\beta(2^x, 2^y)^{\frac{1}{n+1}} \cdot \frac{\beta(2^x, 2^y)^{\frac{1}{2}}}{\beta(2x, 2y)^{\frac{1}{2}}} \cdot \frac{\beta(2^x, 2^y)^{\frac{1}{2}}}{\beta(2x, 2y)^{\frac{1}{2}}} \cdots \frac{\beta(2^x, 2^y)^{\frac{1}{2}}}{\beta(2x, 2y)^{\frac{1}{2}}} \frac{1}{\beta(x, y)^{\frac{1}{2}}}
\]

\[
\beta(2^x, 2^y)^{\frac{1}{2}}
\]

for all \(x, y > 0\). From this equation (9) we get

\[
|g(x + y) - g(x) - g(y) - \ln \beta(x, y)^{-1}|
= \lim_{n \to \infty} \left| \frac{f(2^n x + 2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} - \frac{1}{2^n} \ln \beta(2^n x, 2^n y)^{-1} \right|
\leq \lim_{n \to \infty} \frac{\delta}{2^n} = 0
\]
for all $x, y > 0$ and thus
\[ g(x + y) = g(x) + g(y) + \ln \beta(x, y)^{-1} \]
for all $x, y > 0$. From the inequality (8), we have
\[ |f(x) - g(x)| \leq \delta \]
for all $x > 0$. Now suppose that $h$ satisfies the equation
\[ h(x + y) = h(x) + h(y) + \ln \beta(x, y)^{-1} \]
for all $x, y > 0$ and
\[ |f(x) - h(x)| \leq \delta \]
for all $x > 0$. Then for all $x > 0$
\[
|g(x) - h(x)| \\
\leq \lim_{n \to \infty} \frac{1}{2^n} \left| f(2^n x) - h(2^n x) \right| + \lim_{n \to \infty} \left| \frac{h(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} \beta(2^i x, 2^i y)^{\frac{1}{2^{i+1}}} - h(x) \right| \\
\leq \lim_{n \to \infty} \frac{\delta}{2^n} + 0 = 0
\]
as $n \to \infty$ and for all $x > 0$, and thus $g$ is unique.

\[\square\]

REFERENCES


*Department of Computer Hacking and Information Security, Daejeon University, Daejeon 300-716, Korea*

*Email address: ywlee@du.ac.kr*