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A NOTE ON SEMI-SELFDECOMPOSABILITY AND OPERATOR SEMI-STABILITY IN SUBORDINATION

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Abstract. Some results on inheritance of operator semi-selfdecomposability and its decreasing subclass property from subordinator to subordinated in subordination of a Lévy process are given. A main result is an extension of results of [5] to semi-selfdecomposable subordinator. Its consequence is discussed.

1. Introduction

We use the terminology in Sato's book [17]. Let m be a nonnegative integer and let $b > 1$. A sequence of strictly decreasing subclasses $L_m(b)$ of the class of infinitely divisible distributions was introduced and characterized in Maejima and Naito [11]. A description of the classes $L_m(b)$ is given in Section 2. A distribution in the class $L_0(b)$ is called semi-selfdecomposable by Maejima and Naito [11]. This notion is a natural extension of selfdecomposability on the one hand and semi-stability on the other. Its importance comes from mathematical physics. For a relation of semi-selfdecomsability with diffusions on Sierpinski gaskets, see [14] and references therein. Later, Maejima, Sato, and Watanabe [12, 13] extended this notion to that of operator semi-selfdecomposability. We note that operator semi-selfdecomposability offers higher flexibility in stochastic modeling then semi-selfdecomposability.

It is an interesting question to see whether operator semi-selfdecomposability or the operator version of the class $L_m(b)$ property is inherited under time change. In this paper, this question is considered in relation to subordination of a Lévy process.

Subordination of a Lévy process is defined as follows.

Let $\{T(t)\}$ be an increasing Lévy process on R and $\{X(t)\}$ be a Lévy process on R^d, independent of $\{T(t)\}$. Here R^d is the d-dimensional Euclidean space with the inner product $\langle x, y\rangle$ for x, y. Subordination is a transformation of $\{X(t)\}$ to a new process $\{Y(t)\}$ defined by composition as $Y(t) = X(T(t))$

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through random time change by $\{T(t)\}$. Processes $\{X(t)\}$, $\{T(t)\}$ and $\{Y(t)\}$ are respectively called subordinand, subordinator (subordinating) and subordinated.

The importance of subordination is increasing in mathematical finance. See [1] and references given there. Particulary, interesting models including some financial models are given in [2]. Recently, this notion was extended to the general case in [15, 16].

From Theorem 6.1 of [2], it is well known that in subordination of strict stability, the subclasses L_m of the class L_0 of selfdecomposable distributions are inherited from subordinator to subordinated. This is extended to the general case in Theorem 3.1 of [19] on the one hand, and in the case where the subordinator $\{T(t)\}$ is a new selfdecomposability of stochastic processes in Theorems 7.6 and 7.7 of [1] on the other. But the problem how much we can weaken the assumption of strict stability of the subordinand is open in many important cases including the α-stable $(0 < \alpha \leq 1)$ subordinand with drift. Let $\alpha \in (0, 2]$. Under the condition of strictly α-semi-stabile subordinand with a span $b^{\frac{1}{\alpha}}$, inheritance of the class $L_m(b)$ property from subordinating to subordinated is known from Theorem 3.1 in [19]. This is given in Theorem 1.1 below. But it is not known whether the same statement of Theorem 1.1 is true if strictly α-semistable is replaced by α-semistable with drift for $\{X(t)\}$. Some related discussions are given in Section 4.

Theorem 1.1. *If the subordinand $\{X(t)\}$ is strictly α-semistable with a span $b^{\frac{1}{\alpha}}$ and the subordinator $\{T(t)\}$ belongs to the class $L_m(b)$ on R, then the subordinated process $\{Y(t)\}$ belongs to $L_m(b^{\frac{1}{\alpha}})$.*

This theorem is generalized to the operator version in Corollary 2.2 of this paper, which is a special case of Theorem 2.1 in Section 2. Theorem 2.1 is an extension of known results from [5] to the case where the subordinator $\{T(t)\}$ is in $L_m(b)$ on R. The proof of Theorem 2.1 is given in Section 3 after some preparatory. Corollary 2.2 is proved as a consequence of Theorem 2.1.

2. Results

We start with the following notation we are going to use in this paper.

$\mathcal{P}(R^d)$ and $I(R^d)$ are, respectively, the collection of probability measures (distributions) defined on R^d and collection of infinitely divisible distributions defined on R^d; $\hat{\mu}(z)$, $z \in R^d$ is the characteristic function of $\mu \in \mathcal{P}(R^d)$; $\mathcal{L}(X)$ is the distribution of a random variable or vector X; $M_J^+(R^d)$ is the class of all $d \times d$ matrices all of whose eigenvalues have positive real parts in J; $\mathcal{B}(S)$ is the collection of Borel sets in S for a set $S \subset R^d$; $Q^\top$ is the transposed matrix of $Q \in M^+_{(0,\infty)}(R^d)$; I is the identity matrix; $b^Q = \sum_{n=0}^\infty (n!)^{-1}(\log b)^n Q^n$ for $b > 0$ and $Q \in M^+_{(0,\infty)}(R^d)$.

Fix b and $Q \in M^+_{(0,\infty)}(R^d)$. A distribution $\mu \in \mathcal{P}(R^d)$ is called *(b, Q)-semi-selfdecomposable* or *operator semi-selfdecomposable,* if there are some b and μ_b

$\in I(R^d)$ satisfying

$$\hat{\mu}(z) = \hat{\mu}(b^{-Q^\top} z)\hat{\mu_b}(z), \qquad z \in R^d. \tag{2.1}$$

In the case when $Q = I$, this is the usual semi-selfdecomposability. The class of all (b, Q)-semi-selfdecomposable distributions satisfying (2.1) is denoted by $L_0(b, Q)$. For $m \geq 1$, the class $L_m(b, Q)$ is defined to be the class of $\mu \in L_0(b, Q)$ such that, for some b, there exists $\mu_b \in L_{m-1}(b, Q)$ satisfying (2.1). These classes were introduced and characterized by Maejima, Sato, and Watanabe [12, 13]. For any m, the class $L_m(Q)$ is defined to be the class of distributions on R^d such that, for every $b > 0$, $\mu \in L_m(b, Q)$. μ in $L_0(Q)$ is called *Q-selfdecomposable*. In the case when $Q = I$, this is the usual selfdecomposability. Let $L_m(I) = L_m$ and $L_m(b, I) = L_m(b)$ for some b.

A Lévy process $\{Z(t)\}$ on R^d belongs to $L_m(b, Q)$ or $L_m(Q)$ if $\mathcal{L}$ $(Z(1))$ belongs to $L_m(b, Q)$ or $L_m(Q)$, respectively.

Fix b and $Q \in M^+_{(0,\infty)}(R^d)$. A distribution $\mu \in \mathcal{P}(R^d)$ is called (b, Q)-*semi-stable* (*operator semi-stable with exponent* Q) if $\mu \in I(R^d)$ and there are b and $c \in R^d$ such that

$$\hat{\mu}^b(z) = \hat{\mu}(b^{Q^\top} z)e^{i\langle c,z\rangle}, \qquad z \in R^d. \tag{2.2}$$

Moreover, a distribution $\mu \in \mathcal{P}(R^d)$ is called *strictly* (b, Q)-*semi-stable* (*strictly operator semi-stable with exponent* Q) if $\mu \in I(R^d)$ and there is b such that

$$\hat{\mu}^b(z) = \hat{\mu}(b^{Q^\top} z), \qquad z \in R^d. \tag{2.3}$$

The class of all (b, Q)-semi-stable distributions satisfying (2.2) and the class of all strictly (b, Q)-semi-stable distributions satisfying (2.3) are denoted by $OSS(b, Q)$ and $SOSS(b, Q)$, respectively. We note that $OSS(b, Q) = OSS(b^{-1}, Q)$ and $SOSS(b, Q) = SOSS(b^{-1}, Q)$. See [8] and [10] for a review on operator semi-stable distributions and see [4] for a review on strictly operator semi-stable distributions. In this case when $\mu \in OSS(b, \frac{1}{\alpha}I)$ or $\mu \in SOSS(b, \frac{1}{\alpha}I)$, this μ is the usual α-semi-stable distribution having a span $b^{\frac{1}{\alpha}}$ or strictly α-semi-stable distribution having a span $b^{\frac{1}{\alpha}}$, respectively. See Section 14 in [17] for a review on α-semi-stable distributions. A distribution μ is called *Q-stable* or *strictly Q-stable* if, for every $b > 0$, $\mu \in OSS(b, Q)$ or $\mu \in SOSS(b, Q)$, respectively. The class of all Q-stable distribution is denoted by $S(Q)$.

We define $L_\infty(b, Q) = \bigcap_{m<\infty} L_m(b, Q)$. Then

$$L_0(b, Q) \supset L_1(b, Q) \supset \cdots \supset L_\infty(b, Q) \supset SOSS(b^{-1}, Q). \tag{2.4}$$

This was shown by Maejima, Sato, and Watanabe [12, 13]. A distribution $\mu \in \mathcal{P}(R^d)$ is called *completely* (b, Q)-*semi-selfdecomposable* or *completely operator semi-selfdecomposable* if $\mu \in L_\infty(b, Q)$.

A Lévy process $\{Z(t)\}$ is called *selfdecomposable,* (b, Q)-*semi-selfdecomposable*, *strictly* (b, Q)-*semi-stable*, (b, Q)-*semi-stable*, *strictly Q-stable* or *Q-stable* if $\mathcal{L}(Z(1))$ is selfdecomposable, (b, Q)-semi-selfdecomposable, strictly (b, Q)-semi-stable, (b, Q)-semi-stable, strictly Q-stable or Q-stable, respectively.

Throughout this paper, let $Q \in M^+_{(0,\infty)}(R^d)$ and $b_j > 1$ for $j = 1, 2$. Our results are as follows.

Theorem 2.1. *Let $\{X(t)\}$ be strictly (b_1, Q)-semi-stable on R^d and let $\{T(t)\}$ be in $L_m(b_2)$ on R. If $\log b_1/\log b_2$ is a rational number, then the subordinated process $\{Y(t)\}$ belongs to $L_m(b, Q)$ for some b.*

Corollary 2.2. *Let $\{X(t)\}$ be strictly (b, Q)-semi-stable on R^d and let $\{T(t)\}$ be in $L_m(b)$ on R. Then the subordinated process $\{Y(t)\}$ belongs to $L_m(b, Q)$.*

Remark 2.3. Let $\{X(t)\}$ be the usual α-semi-stable process having a span $b^{\frac{1}{\alpha}}$ in Corollary 2.2. In this case, by noticing that $L_m(b, \frac{1}{\alpha}I) = L_m(b^{\frac{1}{\alpha}})$, Corollary 2.2 shows that $\{Y(t)\}$ is in $L_m(b^{\frac{1}{\alpha}})$, which is given in Theorem 1.1. This fact is extended to the general subordination in Theorem 3.1 of [19].

Remark 2.4. Let replace "$L_m(b_2)$" by "L_m", $m \geq 0$ for $\{T(t)\}$ in Theorem 2.1. Then, the statement is exactly Theorem 1.1 of [5]. In this case, we do not need the additional condition that $\log b_1/\log b_2$ is a rational number. Under this additional condition in Theorem 2.1, strict (operator) semi-stability is inherited from the subordinator to the subordinated in subordination of a strictly (operator) semi-stabile process. See Proposition 3.1 of [6] (Corollary 1.3 of [5]). Without assuming this additional condition, it is not true. See Example in [5].

3. Proofs of results

We use the Lévy representation (A, ν, γ) of $\mu \in I(R^d)$ in the sense that

$$\hat{\mu}(z) = \exp\left[i\langle\gamma, z\rangle - \frac{1}{2}\langle Az, z\rangle + \int_{R^d} G(z, x)\nu(dx)\right]$$

and

$$G(z, x) = e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle I_{\{|x|\leq 1\}}$$

for $z \in R^d$, where A is a symmetric nonnegative-definite operator on R^d, ν is a measure on R^d satisfying $\nu(\{0\}) = 0$ and $\int_{R^d-\{0\}}(|x|^2 \wedge 1)\nu(dx) < \infty$, and $\gamma \in R^d$. Here $I_D(x)$ is the indicator function of D. A Lévy process $\{Z(t)\}$ is said to have the Lévy representation (A, ν, γ) if $\mathcal{L}(Z(1))$ has the Lévy representation (A, ν, γ). We note that these A, ν, and γ are uniquely determined by μ, A is called the Gaussian variance and ν is called the Lévy measure of μ.

Let $\{X(t)\}$ be strictly (b, Q)-semi-stable with Lévy representation (A, ν, γ). It is a necessary and sufficient condition for $\mu \in \mathcal{P}(R^d)$ with the Lévy representation (A, ν, γ) to be in $SSOS(b, Q)$ for some b that A and ν are expressed as

$$\langle Ab^{Q^\top} z, b^{Q^\top} z\rangle = b\langle Az, z\rangle, \qquad z \in R^d \tag{3.1}$$

and

$$\nu(b^Q E) = b^{-1}\nu(E) \tag{3.2}$$

for every $E \in \mathcal{B}(R^d - \{0\})$.

For any signed measure σ, b and $E \in \mathcal{B}(R^d - \{0\})$, define $\phi_{b,Q,\sigma}(E) = \sigma(E) - \sigma(b^Q E)$, where $b^Q E = \{b^Q x : x \in E\}$. Let $\phi^j_{b,Q,\sigma}$ be the j-th iteration of $\phi_{b,Q,\sigma}$. Then, for any $E \in \mathcal{B}(R^d - \{0\})$, the equation (3.2) leads to

$$\phi^j_{b,Q,\nu}(E) = (1 - b^{-1})^j \nu(E) \qquad \text{for } j = 1, \ldots, m+1. \tag{3.3}$$

Maejima, Sato, and Watanabe showed the following in Theorem 3.1 of [12]. It is a necessary and sufficient condition for $\mu_0 \in \mathcal{P}(R^d)$ with Lévy measure ν_0 to be in $L_m(b, Q)$ for some b that ν_0 is expressed as

$$\phi^j_{b,Q,\nu_0}(E) \geq 0 \qquad \text{for } j = 1, \ldots, m+1. \tag{3.4}$$

By combining (3.1) and (3. 2), this leads to the following:

Lemma 3.1. *Let l be a positive integer. Then, for each l and m,*

$$SOSS(b, Q) \subseteq SOSS(b^l, Q) \quad \textit{and} \quad L_m(b) \subseteq L_m(b^l).$$

For a real-valued function $k(s)$ defined on $(0, \infty)$ and b, define as $\psi_b k(s) = k(s) - k(bs)$. For $\epsilon > 0$, let $\Delta_\epsilon \psi^j_b k(s) = \psi^j_b k(s) - \psi^j_b k(s + \epsilon)$, where $\psi^j_b k(s)$ is the j-th iteration of the operator $\psi_b k(s)$. In the following, we are using the words increase and decrease in the weak sense.

Following Theorem 30.1 of [17], it is well-known that $\{T(t)\}$ is a subordinator with the Lévy representation $(A^\natural, \rho, \beta)$ and $\kappa = \mathcal{L}(T(1))$ if and only if

$$\hat{\kappa}(z) = \exp\left[\int_{(0,\infty)} (e^{izs} - 1)\rho(ds) + i\beta_0 z\right]$$

with $\beta_0 = \beta - \int_{(0,1]} s\rho(ds) \geq 0$. Combining this, Theorem 4.1 and Theorem 4.2 in [11], we have the following:

Lemma 3.2. *Let $\{T(t)\}$ be a subordinator in $L_m(b)$. Then the Lévy measure ρ has the expression*

$$\rho(E) = C \int_0^\infty I_E(s) d\{-k(s)\}, \quad E \in \mathcal{B}((0, \infty)),$$

where C is a constant and $k(s)$ is right continuous, decreasing in s, nonnegative, $\lim_{s\to\infty} k(s) = 0$ and $\Delta_\epsilon \psi^j_b k(s) \geq 0$ for every $\epsilon > 0$, $s > 0$ and $j = 1, \ldots, m+1$

Lemma 3.3. *Let $\{Z(s)\}$ be a Lévy process on R^d. Let C and $k(s)$ be as in Lemma 3.2. Then, for every $E \in \mathcal{B}(R^d - \{0\})$ and $j = 1, \ldots, m+1$,*

$$\int_{(0,\infty)} \mu^s(E) d\{\psi^j_b k(s) - \psi^j_b k(bs)\} \leq 0, \qquad \textit{where} \quad \mu^s = \mathcal{L}(Z(s)).$$

Proof. Using the fact that $\Delta_\epsilon \psi^j_b k(s) \geq 0$ for every $\epsilon > 0$, $s > 0$ and $j = 1, \ldots, m+1$, we see that

$$-\int_{(0,\infty)} \mu^s(E) d\{\psi^j_b k(s) - \psi^j_b k(bs)\} \geq 0.$$

□

Let $\{X(t)\}$ be a Lévy process with the Lévy presentation (A, ν, γ) on R^d and $\{T(t)\}$ be a subordinator with the Lévy representation $(A^\natural, \rho, \beta)$ on R. Then, from Theorem 30.1 of [17], the subordinated process $\{Y(t)\} = \{X(T(t))\}$ is a Lévy process with the Lévy representation $(A^\sharp, \nu^\sharp, \gamma^\sharp)$ on R^d such that

$$A^\sharp = \beta_0 A,$$

$$\nu^\sharp(E) = \beta_0 \nu(E) + \int_{(0,\infty)} \mu^s(E)\rho(ds), \quad E \in \mathcal{B}(R^d - \{0\}), \tag{3.5}$$

$$\gamma^\sharp = \beta_0 + \int_{(0,\infty)} \rho(ds) \int_{|x|\le 1} x\mu^s(dx),$$

where $\mu^s = \mathcal{L}(X(s))$.

Proof of Theorem 2.1. Let $\{X(t)\}$, $\{T(t)\}$ and $\{Y(t)\}$ be as above.

Let $\log b_1 / \log b_2$ be a rational number. Then there exist some positive integers M and N such that $b_1^M = b_2^N$. Let $b_2^N = b$, then by (3.5), we see that

$$\nu^\sharp(b^Q E) = \beta_0 \nu(b^Q E) + \int_{(0,\infty)} \mu^s(b^Q E)\rho(ds), \quad E \in \mathcal{B}(R^d - \{0\}).$$

Let $\{X(t)\}$ be strictly (b_1, Q)-semi-stable. Then by using (2.3) and Lemma 3.1, we have that, for every $E \in \mathcal{B}(R^d - \{0\})$ and $s > 0$, $\mu^s(b^Q E) = \mu^{sb^{-1}}(E)$.

For $m = 0$, we suppose that $\{T(t)\}$ is in $L_0(b_2)$. Then, by Lemma 3.1 and Lemma 3.2, we see that

$$\begin{aligned} \int_{(0,\infty)} \mu^s(b^Q E)\rho(ds) &= C\int_{(0,\infty)} \mu^{sb^{-1}}(E)d\{-k(s)\} \\ &= C\int_{(0,\infty)} \mu^s(E)d\{-k(sb)\}. \end{aligned}$$

From the facts that $\nu(b^Q E) = b^{-1}\nu(E)$ and Lemma 3.3, this shows that, for some b, $\phi_{b,Q,\nu^\sharp}(E) \ge 0$, which means that the Lévy measure of $\{Y(t)\}$, $\nu^\sharp$ satisfies the condition to be in $L_0(b, Q)$ by (3.4).

Next, for $m \ge 1$, we suppose that $\{T(t)\}$ is in $L_m(b_2)$. This means that

$$\Delta_\epsilon \psi_b^j k(s) \ge 0, \qquad \forall \epsilon > 0, \quad j = 1, \ldots, m+1.$$

This, combined with the equation (3.3) and Lemma 3.3, leads to

$$\phi^j_{b,Q,\nu^\sharp}(E) \ge 0 \qquad \text{for } j = 1, \ldots, m+1, \tag{3.6}$$

which shows that, by (3.4), the Lévy measure of $\{Y(t)\}$, $\nu^\sharp$ satisfies the condition to be in $L_m(b, Q)$ for $m \ge 1$.

For $m = \infty$, this assertion is a consequence of that for finite m.

Using (2.4), we note that, $A^\sharp$ has the same the property of Gaussian matrix from the class $L_\infty(b, Q)$. Combining this and (3.6), we see that $\{Y(t)\}$ belongs to $L_m(b, Q)$ for $m \ge 0$. □

Proof of Corollary 2.2. Let $\{X(t)\}$ and $\{T(t)\}$ be the processes in Corollary 2.2. Let M, N and b_i be as above for $i = 1, 2$. Then we see that $M = N = 1$ and $b_1 = b_2 = b$ in the above proof. This says that $\{Y(t)\}$ belongs to $L_m(b, Q)$ for $m \geq 0$. □

4. Remark

In this paper, we treat the case where $\mathcal{L}(X(1)) \in SOSS(b, Q)$. In the simple case where $Q = \frac{1}{\alpha} I$, let $SOSS(b, Q) = SS_0(\alpha)$, $OSS(b, Q) = SS(\alpha)$ and $S(Q) = S(\alpha)$ for some b. Then $\alpha \in (0, 2]$ and $S(2) = SS(2)$. We recall that α is uniquely determined by $\mathcal{L}(X(1))$ (see [3]). Even in case where $\mathcal{L}(X(1)) \in SS(\alpha)$ and $\mathcal{L}(X(1)) \notin SS_0(\alpha)$ for $\alpha \in (0, 2)$, we do not know whether $\mathcal{L}(Y(1)) \in L_0(b)$ for $\mathcal{L}(T(1)) \in L_0$. We note that the Brownian motion is strictly 2-stable and Brownian motions with drift are stable but not strictly 2-stable. Suppose that $\{X(t)\}$ is an α-stable process on R, which is not strictly stable. Halgreen [7] asked a question whether $\mathcal{L}(Y(1)) \in L_0$ for $\mathcal{L}(X(1)) \in S(2)$ and $\mathcal{L}(T(1)) \in L_0$. After 22 years, Sato affirmatively settled this question in Theorem 1.1 of [18]. He [18] raised a question whether this remains true for $\mathcal{L}(X(1)) \in S(\alpha)$ with $0 < \alpha < 2$. For $\alpha \in (1, 2)$, Kozubowski [9] pointed out that this is not true. But, for $\alpha \in (0, 1]$, it is not known whether $\mathcal{L}(Y(1)) \in L_0$ for $\mathcal{L}(X(1)) \in S(\alpha)$ and $\mathcal{L}(T(1)) \in L_0$. Sato [19] again raised a question whether $\mathcal{L}(Y(1)) \in L_m$ for $\mathcal{L}(X(1)) \in S(\alpha)$ with $\alpha \in (0, 2]$ and $\mathcal{L}(T(1)) \in L_m$. He [19] also raised a question whether the above statement is true with "L_0" replaced by "$L_0(b)$".

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