DOUBLY NONLINEAR PARABOLIC EQUATIONS RELATED TO THE LERAY-LIONS OPERATORS: TIME-DISCRETIZATION

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ABSTRACT. In this paper, we consider a doubly nonlinear parabolic equation related to the Leray-Lions operator with Dirichlet boundary condition and initial data given. By exploiting a suitable implicit time-discretization technique, we obtain the existence of global strong solution.

1. Introduction

Let Ω be a regular open bounded subset of finite dimensional $\mathbb{R}^d$ ($d \geq 3$) and $\partial\Omega$ its boundary. In this paper, we study a doubly nonlinear parabolic partial differential equation related to Leray-Lions operators. More precisely, we are interested in the existence and uniqueness of the solution of the following problem ;

$$\begin{cases} \frac{\partial\beta(u)}{\partial t} - \operatorname{div} a(x, u, \nabla u) + f(x, t, u) = 0 & \text{in} \quad \Omega \times [0, T], \\ u = 0 & \text{on} \quad \partial\Omega \times [0, T], \\ \beta(u(\cdot, 0)) = \beta(u^0) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where β is a nonlinearity of porous medium type, $-\operatorname{div} a(x, u, \nabla u) = Au$ is the Leray-Lions operator and f is a nonlinearity of reaction diffusion type. As a prime example of $a(x, u, \nabla u)$, we may choose the p-Laplacian operator. In other words, the following equation is a special case of (1).

$$\frac{\partial\beta(u)}{\partial t} - \Delta_p u + f(x, t, u) = 0, \tag{2}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Problem (2) can be found in many applications in the fields of mechanics, physics and biology (non-Newtonian fluids, gas flow in porous media, spread of biological populations, etc.). In particular, for $p = 2$, (2) has been motivated by reading of two papers. The one due

to M. Gurtin [7] gives a non phenomenological derivation of the generalized Allen-Cahn equation,

$$a(u, \nabla u, u_t)u_t - \Delta u + f(t, x, u) = 0 \tag{3}$$

with $a \geq 0$, which can be degenerate and hence we may rewrite (3) as the form of equation (2) provided a depends only on u.

On the other hand, in [9], a non-isothermal phase transition problem is modeled by A. Miranvile and G. Schimperna with Gurtin's approach to the following system of equations ;

$$(u^2)_t - \Delta u = f + u\chi\chi_t + (\chi_t)^2, \quad \chi_t - \Delta\chi + g(\chi) = -u\chi,$$

where the unknowns are the absolute temperature u and the phase field χ. If χ is given then the first equation becomes the equation of the form (2) provided $\beta(u) = u^2$ for positive u.

For the other cases with $p = 2$ in (2), we may cite the works of M. Schatzman, A. Eden, B. Michaux, J.M. Rakotoson, A. Rougirel, J.I. Diaz and J.F. Padial (cf. [4, 5, 10, 12]). A lot of works dedicated to the existence and the large time behavior of solutions has been made for the equation. When $\beta(u) = u$ and $p = 2$ in (2), M. Schatzman[12] considers the way to reduce the equation to the reaction-diffusion equation. A. Eden, B. Michaux and J.M. Rakotoson study the existence of solutions using the method of semi-discretization [4] and using Galerkin's approximation method [5], respectively. Also, A. Rougirel [10] studies the asymptotic behavior of solutions with $|\frac{\partial f}{\partial t}(t, x, u)| \leq C_M$, where $C_M > 0$.

In case of $p > 1$ in (2), A. Bensoussan, L. Boccardo and F. Murat [3] study the existence of solution for $\beta = 0$ and A. El Hachimi and H. El Ouardi [6] study the existence and regularity of this equation by Galerkin's approximation method under the assumption that f is differentiable. Thus we shall show existence of solution of (1), which is more generalized than (2), under the conditions that f is increasing and sign condition.

This is the plan of paper ; We recall our assumptions and state main results in section 2. In section 3, we shall show the existence of the corresponding discrete scheme of the equation. And, some a priori estimates on the discrete solutions is given in section 4. Next, section 5 devotes to show the passage to the limit in approximations and concludes the proof of the existence.

2. Assumptions and main results

We let $|| \cdot ||_p, || \cdot ||_{1,p}$ and $|| \cdot ||_{-1,p}$ denote the norm in $L^p(\Omega)$, $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$, respectively, where $1 < p < \infty$. And we let $< \cdot, \cdot >$ denote the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ or the inner product in $L^2(\Omega)$. For $p \geq 1$, we define its conjugate p' by $1/p + 1/p' = 1$. Also, we let C_i and C be

positive constants and λ_i $(i = 1, \ldots, 4)$ imbedding constants [1] such that

$$|| \cdot ||_p \leq \lambda_1 || \cdot ||_{1,p}, \quad || \cdot ||_2 \leq \lambda_2 || \cdot ||_{1,p}, \quad (\frac{2d}{2+d} \leq p < 2),$$
$$\text{and } || \cdot ||_1 \leq \lambda_3 || \cdot ||_p \leq \lambda_4 || \cdot ||_{1,p}, \quad (p \geq 2).$$

Suppose β is a continuous function with $\beta(0) = 0$. By defining $\psi(t) = \int_0^t \beta(s)ds$ for $t \in \mathbb{R}$, the Legendre transform is defined as $\psi^*(\tau) = \sup_{s \in \mathbb{R}}\{\tau s - \psi(s)\}$.

Now, we are preparing our assumptions and the following well known definitions, lemmas which are used throughout this paper.

We assume that

(H1) β is an increasing and continuous from $\mathbb{R}$ to $\mathbb{R}$ with $\beta(0) = 0$ and $u^0 \in L^\infty(\Omega)$.

(H2) $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function such that

$$|a(x, s, \xi)| \leq \gamma[|s|^{p-1} + |\xi|^{p-1} + k(x)],$$
$$(a(x, s, \xi) - a(x, s, \eta))(\xi - \eta) > 0, \text{ for all } \xi \neq \eta,$$
$$a(x, s, \xi)\xi \geq \alpha|\xi|^p,$$

where $k(x) \in L^{p'}(\Omega)$, $k \geq 0$, $\gamma > 0$ and $\alpha > 0$.

(H3) For $\xi \in \mathbb{R}$, the map $(x, t) \mapsto f(x, t, \xi)$ is measurable and $\xi \mapsto f(x, t, \xi)$ is continuous and increasing $a.e.$ in $\Omega \times [0, T]$. Furthermore, we assume that there exists $C_1 > 0$ such that $\text{sign}\, \xi f(x, t, \xi) \geq -C_1$ for $a.e.$ in $\Omega \times [0, T]$.

(H4) For all $M > 0$, there exists $C_M > 0$ such that, if $|\xi| + |\xi'| \leq M$ then

$$|f(x, t, \xi) - f(x, t, \xi')|^\alpha \leq C_M(\beta(\xi) - \beta(\xi'))(\xi - \xi'),$$

where $\alpha = \begin{cases} 2, & 1 < p < 2, \\ p', & p \geq 2. \end{cases}$

(H5) For $a.e.$ $x \in \Omega$ and for all $M > 0$, there exists $\widetilde{C}_M > 0$ such that if $t + t' + |\xi| \leq M$ then

$$|f(x, t, \xi) - f(x, t', \xi)| \leq \widetilde{C}_M|t - t'|^{1/\alpha},$$

where α is as in (H4).

Definition 2.1. ([2]) Let X be a reflexive Banach space and $A : X \to X'$. We say that A is *monotone* if $\langle Ay - Az, y - z \rangle \geq 0$ for all $y, z \in X$, and *hemicontinuous* if for each $y, z, w \in X$ the real-valued function $t \to \langle A(y + tz), w \rangle$ is continuous.

Lemma 2.2. (Minty theorem [13]) *Let X be a reflexive Banach space. If $A : X \to X'$ is monotone and hemicontinuous, then*

$$Ay = f \quad \text{if and only if} \quad \langle f - Az, y - z \rangle \geq 0 \text{ for all } z \in X.$$

Lemma 2.3. ([3]) *Let Ω be a bounded subset of $\mathbb{R}^d$. Let $1 < p < \infty$ be fixed and A a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ defined by*

$$A(u) = -\operatorname{div} a(x, u, Du)$$

where a is a Caratheodory function satisfying (H2). *Let g also be a Caratheodory function such that*

$$g(x, s, \xi)s \geq 0 \quad \text{and} \quad |g(x, s, \xi)| \leq b(|s|)(|\xi|^p + c(x)),$$

where b is a continuous and increasing function with (finite) values on $\mathbb{R}^+$ and $c \in L^1(\Omega)$, $(c \geq 0)$. Then, for $h \in W^{-1,p'}(\Omega)$, the problem

$$Au + g(x, u, \nabla u) = h, \quad u \in W_0^{1,p}(\Omega)$$

has at least one solution.

Lemma 2.4. ([11]) *If $u \in W_0^{1,p}(\Omega)$ is a solution of the equation*

$$-\Delta_p u + F(x, u) = T,$$

where $T \in W^{-1,r}$, $r > d/(p-1)$ and F satisfies $\xi F(x, \xi) \geq 0$ in $\Omega \times \mathbb{R}$, then $u \in L^\infty(\Omega)$.

Finally, we state our main result as followings ;

Theorem 2.5.   (A) *Under assumptions* (H1)–(H5), *there exists*

$$u \in L^p(0, T : W_0^{1,p}(\Omega)) \cap L^\infty(0, T : L^\infty(\Omega))$$

fulfilling (1) *if $p \geq 2$.*
      (B) *Under assumptions* (H1)–(H5), *there exists*

$$u \in L^2(0, T : L^2(\Omega))$$

fulfilling (1) *if $\frac{2d}{2+d} \leq p < 2$.*

3. Existence of schemes

We consider the corresponding discrete scheme related to (1), which is represented by

$$\begin{cases} \frac{\beta(u_i) - \beta(u_{i-1})}{\tau} - \operatorname{div} a(x, u_i, \nabla u_i) + f(x, i\tau, u_i) = 0, & \text{in } \Omega, \\ v_i = \beta(u_i), & \\ u_i = 0, & \text{in } \partial\Omega, \\ \beta(u_0) = \beta(u^0), & \text{in } \Omega, \end{cases} \tag{4}$$

for $i = 1, 2, \ldots, N$, where $N\tau = T$ with a fixed positive number T.

We shall show that (4) has a solution u_i, $i = 1, 2, \ldots, N$.

Theorem 3.1. *Assuming* (H1), (H2), (H3), *there exist unique solutions u_i, $i = 1, 2, \ldots, N$, of (4) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for sufficiently small τ.*

Proof. We may rewrite (4) as the equation

$$-\tau \operatorname{div} a(x, u_i, \nabla u_i) + \beta(u_i) + \tau f(x, i\tau, u_i) + \tau C_1 \operatorname{sign}(u_i)$$
$$= \beta(u_{i-1}) + \tau C_1 \operatorname{sign}(u_i), \quad u_{i-1} \in W_0^{1,p}(\Omega).$$

By setting

$$F(x, u_i) = \beta(u_i) + \tau f(x, i\tau, u_i) + \tau C_1 \operatorname{sign}(u_i) \quad \text{and}$$
$$\varphi_i = \beta(u_{i-1}) + \tau C_1 \operatorname{sign}(u_i),$$

F is a Caratheodory function such that $u_i F(x, u_i) \geq 0$ and $|F(x, u_i)| \leq \beta(|u_i|) + 2\tau C_1$ by (H1) and (H3). Thus, by (H1), the second condition of Lemma 2.3 is satisfied. Since β is a continuous function and $u^0 \in L^\infty(\Omega)$, there exists a solution u_i of (4) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for $i = 1, 2, \ldots, N$ by Lemma 2.3 and 2.4.

Next, we shall show the uniqueness of the above u_i. If u_i and u_i^* are two solution of (4), then we obtain that

$$-\tau \operatorname{div} a(x, u_i, \nabla u_i) + \tau \operatorname{div} a(x, u_i^*, \nabla u_i^*)$$
$$+\beta(u_i) - \beta(u_i^*) + \tau f(x, i\tau, u_i) - \tau f(x, i\tau, u_i^*) = 0. \tag{5}$$

Multiplying (5) by $u_i - u_i^*$ and integrating over Ω, we have

$$\langle -\tau \operatorname{div} a(x, u_i, \nabla u_i) + \tau \operatorname{div} a(x, u_i^*, \nabla u_i^*), u_i - u_i^* \rangle \tag{6}$$
$$+ \int_\Omega (\tau(f(x, i\tau, u_i) - f(x, i\tau, u_i^*)) + (\beta(u_i) - \beta(u_i^*)))(u_i - u_i^*)dx = 0.$$

Then, we get by (H1) and (H3) that

$$\langle -\tau \operatorname{div} a(x, u_i, \nabla u_i) + \tau \operatorname{div} a(x, u_i^*, \nabla u_i^*), u_i - u_i^* \rangle \tag{7}$$
$$+\tau \int_\Omega -(\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx + \int_\Omega (\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx \leq 0.$$

By (H2), (7) is reduced to

$$(1 - \tau) \int_\Omega (\beta(u_i) - \beta(u_i^*))(u_i - u_i^*)dx \leq 0.$$

Hence, by (H1), we get $u_i = u_i^*$ for sufficiently small τ. $\qquad\square$

In the forthcoming discussion the following notation will be used expensively. Letting $\{u_i\}_{i=0}^N$ be vectors, we denote by u_τ and $\bar{u}_\tau$ two functions of the time interval $[0, T]$ which interpolate the values of the vector $\{u_i\}_{i=0}^N$ piecewise linearly and backward constantly on partition of diameter $\tau := T/N$, respectively. Namely, for $t \in ((i-1)\tau, i\tau]$, $i = 1, 2, \ldots, N$,

$$u_\tau(0) := u_0, \quad u_\tau(t) := u_i + \frac{u_i - u_{i-1}}{\tau}(t - i\tau),$$
$$\bar{u}_\tau(0) := u_0, \quad \bar{u}_\tau(t) := u_i = u(\cdot, i\tau).$$

Also, we let $\bar{f}_\tau(t) := f_i = f(\cdot, i\tau, u_i)$ for $t \in ((i-1)\tau, i\tau]$, $i = 1, 2, \ldots, N$. Hence we are entitled to rewrite (4) in a more compact form as

$$
\begin{cases}
v_\tau' - \operatorname{div} a(\bar{u}_\tau, \nabla\bar{u}_\tau) + \bar{f}_\tau = 0 & \text{a.e. in } [0, T], \\
\bar{v}_\tau = \beta(\bar{u}_\tau), & \text{a.e. in } [0, T], \\
\beta(\bar{u}_\tau(0)) = \beta(u_0).
\end{cases}
\tag{8}
$$

4. Estimates

Assuming the hypotheses (H1)–(H5) and $p > 1$, we shall show the bound of the discrete scheme which satisfies (8) in this section. We begin the process with the following theorem.

Theorem 4.1. *Suppose* (H1)–(H3). *Then there exist* $C(C_1, T, u_0) > 0$ *and* $C > 0$ *which are independent of* τ *such that for all* $i = 1, 2, \ldots, N$,

(a) $\|u_i\|_\infty \leq C(C_1, T, u_0)$,

(b) $\tau \sum_{i=1}^m \|u_i\|_{1,p}^p \leq C$,

(c) $\|\beta(u_m)\|_2^2 + \sum_{i=1}^m \|\beta(u_i) - \beta(u_{i-1})\|_2^2 \leq C$ *for all* $m = 1, 2, \ldots, N$.

Here, u_i *satisfies* (4) *for* $i = 1, 2, \ldots, N$.

Proof. (a) Multiplying the equation (4) by $|\beta(u_i)|^k \beta(u_i)$ and integrating over Ω,

$$
\int_\Omega |\beta(u_i)|^k \beta(u_i)\beta(u_i)dx - \int_\Omega \tau \operatorname{div} a(x, u_i, \nabla u_i)|\beta(u_i)|^k \beta(u_i)dx
$$
$$
\leq \|\beta(u_i)\|_{k+2}^{k+1}\|\beta(u_{i-1})\|_{k+2} + m(\Omega)^{1/k+2}\tau C_1\|\beta(u_i)\|_{k+2}^{k+1}
$$

by Hölder's inequality. Thus, we have

$$
\|\beta(u_i)\|_{k+2} \leq m(\Omega)^{1/k+2}\tau C_1 + \|\beta(u_{i-1})\|_{k+2}.
$$

By induction, we get $\|\beta(u_i)\|_{k+2} \leq m(\Omega)^{1/k+2}C_1 T + \|\beta(u_0)\|_{k+2}$. As $k \to \infty$, $\|u_i\|_\infty \leq C(C_1, T, u_0)$ by (H1).

(b) Let $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be fixed. Multiplying the equation (4) by $u_i - z$ and integrating over Ω,

$$
\langle \beta(u_i) - \beta(u_{i-1}), u_i \rangle - \langle \beta(u_i) - \beta(u_{i-1}), z \rangle + \tau\alpha\|u_i\|_{1,p}^p
$$
$$
\leq \tau \int_\Omega \gamma[|u_i|^{p-1} + |\nabla u_i|^{p-1} + k(x)]|\nabla z|dx + \tau C_1 \int_\Omega |u_i - z|dx.
$$

by (H2) and (H3). We apply the Young's inequality to get

$$\langle\beta(u_i)-\beta(u_{i-1}),u_i\rangle-\langle\beta(u_i)-\beta(u_{i-1}),z\rangle+\frac{\tau\alpha}{2}\|u_i\|_{1,p}^p$$
$$\leq(\tau+\tau^2)C(\alpha,\gamma,\lambda_1,p)\|z\|_{1,p}^p+\tau C_1\|u_i\|_\infty m(\Omega)$$
$$+\tau C_1\|z\|_\infty m(\Omega)+\frac{\tau^2}{p'}C_1^{p'}m(\Omega)+2^p\lambda_1\frac{\tau}{p}\sum_{j=1}^i\tau\|u_j\|_{1,p}^p+\frac{\tau}{p'}\|k\|_{L^{p'}(\Omega)}^{p'},$$

for $i=1,2,\ldots,m$ and $m=1,2,\ldots,N$. Applying

$$\int_\Omega\psi^*(\beta(u_i))-\psi^*(\beta(u_{i-1}))dx\leq\int_\Omega(\beta(u_i)-\beta(u_{i-1}))u_idx \qquad(9)$$

and then summing up from $i=1$ to $i=m$ for $m=1,2,\ldots,N$, we obtain

$$\int_\Omega\psi^*(\beta(u_m))-\psi^*(\beta(u_0))dx-\langle\beta(u_m)-\beta(u_0),z\rangle+\frac{\tau\alpha}{2}\sum_{i=1}^m\|u_i\|_{1,p}^p$$
$$\leq C_2+C_3\tau\sum_{i=1}^m\sum_{j=1}^i\tau\|u_j\|_{1,p}^p,$$

where $C_2=C(T,\alpha,\gamma,p,\lambda_1,C_1,C(C_1,T,u_0),m(\Omega),\|z\|_\infty,\|z\|_{1,p},\|k\|_{L^{p'}(\Omega)})$ and $C_3=C(\lambda_1,p)$ by Theorem 4.1(a). For sufficiently small $\tau<\bar\tau=\alpha/4C_3$,

$$\int_\Omega\psi^*(\beta(u_m))dx-\langle\beta(u_m),z\rangle+\frac{\tau\alpha}{4}\sum_{i=1}^m\|u_i\|_{1,p}^p$$
$$\leq\int_\Omega\psi^*(\beta(u_0))dx-\langle\beta(u_0),z\rangle+C_2+C_3\tau\sum_{i=1}^{m-1}\sum_{j=1}^i\tau\|u_j\|_{1,p}^p.$$

By the discrete Gronwall's lemma, we have

$$\int_\Omega\psi^*(\beta(u_m))dx-\langle\beta(u_m),z\rangle+\frac{\tau\alpha}{4}\sum_{i=1}^m\|u_i\|_{1,p}^p$$
$$\leq C(\beta(u_0),T,\alpha,\gamma,p,\lambda_1,C_1,C(C_1,T,u_0),m(\Omega),\|z\|_\infty,\|z\|_{1,p},\|k\|_{L^{p'}(\Omega)}).$$

Since $\int_\Omega\psi^*(\beta(u_i))dx-\langle\beta(u_i),z\rangle>-\infty$, we finally get $\tau\sum_{i=1}^m\|u_i\|_{1,p}^p\leq C$.

(c) Multiplying the equation (4) by $\beta(u_i)$ and integrating over Ω,

$$\int_\Omega\frac{1}{2}|\beta(u_i)|^2-\frac{1}{2}|\beta(u_{i-1})|^2+\frac{1}{2}|\beta(u_i)-\beta(u_{i-1})|^2dx\leq C_1\tau\int_\Omega|\beta(u_i)|dx.$$

by (H2). Here we have used the equality $a(a-b)=\frac{1}{2}a^2-\frac{1}{2}b^2+\frac{1}{2}(a-b)^2$.
Then we get

$$\|\beta(u_i)\|_2^2-\|\beta(u_{i-1})\|_2^2+\|\beta(u_i)-\beta(u_{i-1})\|_2^2\leq 2C_1\tau\|\beta(u_i)\|_1.$$

Summing the above inequality with respect to $i = 1, 2, \ldots, m$ for arbitrary $m = 1, 2, \ldots, N$,

$$||\beta(u_m)||_2^2 + \sum_{i=1}^{m}||\beta(u_i) - \beta(u_{i-1})||_2^2 \leq 2C_1 Tm(\Omega)C(C_1, T, u_0) + ||\beta(u_0)||_2^2.$$

Thus we have our result such as

$$||\beta(u_m)||_2^2 + \sum_{i=1}^{m}||\beta(u_i) - \beta(u_{i-1})||_2^2 \leq C$$

. $\square$

From the above results we may conclude that

$$\bar{u}_\tau \text{ is bounded in } L^p(0, T : W_0^{1,p}(\Omega)) \cap L^\infty(0, T : L^\infty(\Omega)), \quad (10)$$

$$v_\tau \text{ is bounded in } C(0, T : L^2(\Omega)), \quad (11)$$

$$- \operatorname{div} a(\bar{u}_\tau, \nabla \bar{u}_\tau) \text{ is bounded in } L^{p'}(0, T : W^{-1,p'}(\Omega)), \quad (12)$$

where all of bounds are independent of τ.

From now on, we devote to show the existence of a bound for v_τ'. At fist, we consider for the case of $p \geq 2$. By (H4) and (H5),

$$\sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} ||f(x, t, u) - f(x, t, \bar{u}_\tau)||_{-1,p'}^{p'} dt$$

$$\leq \lambda_1^{p'} \sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} \int_\Omega C_M(\beta(u) - \beta(\bar{u}_\tau))(u - \bar{u}_\tau) dx dt$$

$$\leq \lambda_1^{p'} C_M \tau (\int_0^T ||u||_2||\frac{\partial\beta}{\partial s}||_2 + ||\bar{u}_\tau||_2||\frac{\partial\beta}{\partial s}||_2 dt)$$

$$\leq \lambda_1^{p'} C_M \tau (||u||_{L^2(0,T:L^2(\Omega))} + ||\bar{u}_\tau||_{L^2(0,T:L^2(\Omega))})||\frac{\partial\beta}{\partial s}||_{L^2(0,T:L^2(\Omega))}$$

$$\leq \widetilde{C}_M^{1/p'} \tau \sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} \lambda_4^{p'} dt = \widetilde{C}_M^{1/p'} \tau \lambda_4^{p'} T,$$

where u is the weak limit of $\bar{u}_\tau$. It implies that

$$||\bar{f}_\tau(x, t, \bar{u}_\tau) - f(x, t, u)||_{L^{p'}(0,T:W^{-1,p'}(\Omega))}^{p'} \quad (13)$$

$$\leq 2^{p'}(\lambda_1^{p'} C_M \tau (||u||_{L^2(0,T:L^2(\Omega))}$$

$$+ ||\bar{u}_\tau||_{L^2(0,T:L^2(\Omega))})||\frac{\partial\beta}{\partial s}||_{L^2(0,T:L^2(\Omega))} + \widetilde{C}_M^{1/p'} \tau \lambda_4^{p'} T).$$

Applying (12) and (13) to (8), we conclude that for $p \geq 2$

$$v_\tau' \text{ is bounded in } L^{p'}(0, T : W^{-1,p'}(\Omega)). \quad (14)$$

Now, we consider for the case of $1 < p < 2$. As before, by (H4) and (H5), we also have

$$\sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} ||f(x,t,u(x,t)) - f(x,t,\bar{u}_\tau(x,t))||_2^2 dt$$

$$\leq \sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} \int_\Omega C_M(\beta(u) - \beta(\bar{u}_\tau))(u - \bar{u}_\tau)dxdt$$

$$\leq \tau C_M(||u||_{L^2(0,T:L^2(\Omega))} + ||\bar{u}_\tau||_{L^2(0,T:L^2(\Omega))})||\frac{\partial\beta}{\partial s}||_{L^2(0,T:L^2(\Omega))}$$

$$\leq \sum_{i=1}^{N} \int_{(i-1)\tau}^{i\tau} \int_\Omega (\widetilde{C}_M|t - i\tau|^{1/2})^2 dxdt \leq T\widetilde{C}_M^2\tau m(\Omega),$$

where u is weak convergence of $\bar{u}_\tau$. Hence

$$||\bar{f}_\tau(x,t,\bar{u}_\tau) - f(x,t,u)||_{L^2(0,T:L^2(\Omega))}^2 \qquad (15)$$
$$\leq 2(\tau C_M(||u||_{L^2(0,T:L^2(\Omega))}$$
$$+ ||\bar{u}_\tau||_{L^2(0,T:L^2(\Omega))})||\frac{\partial\beta}{\partial s}||_{L^2(0,T:L^2(\Omega))} + T\widetilde{C}_M^2\tau m(\Omega)).$$

We also conclude that for $1 < p < 2$

$$v_\tau' \text{ is bounded in } L^2(0,T:L^2(\Omega)). \qquad (16)$$

5. Limits

In this section, we show the existence of limits of discrete schemes using the priori estimates which are obtained in the previous sections. Since we have the similar results for both cases, $p \geq 2$ and $1 < p < 2$, we shall give the proof for the case of $p \geq 2$ in detail, and then we accept the same result without proof for the case of $1 < p < 2$.

We assume that $p \geq 2$. From the priori estimates (10), (11), (13) and (14), we find functions u, v and f such that, for some not relabeled subsequence,

$$\bar{u}_\tau \to u \text{ weakly in } L^p(0,T:W_0^{1,p}(\Omega)) \cap L^\infty(0,T:L^\infty(\Omega)), \qquad (17)$$

$$v_\tau \to v \text{ weakly in } W^{1,p'}(0,T:W^{-1,p'}(\Omega)), \qquad (18)$$

$$v_\tau \to v \text{ strongly in } C(0,T:L^2(\Omega)),$$

$$\bar{v}_\tau \to v \text{ weakly in } L^{p'}(0,T:W^{-1,p'}(\Omega)), \qquad (19)$$

$$\bar{v}_\tau \to v \text{ strongly in } L^\infty(0,T:L^2(\Omega)),$$

$$\bar{f}_\tau \to f \text{ strongly in } L^{p'}(0,T:W^{-1,p'}(\Omega)). \qquad (20)$$

Of course, by (14), v_τ and $\bar{v}_\tau$ have the same limit since

$$\int_0^T \langle u, \bar{v}_\tau - v_\tau \rangle dt \le \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega u(t)(\frac{v_i - v_{i-1}}{\tau}) dx dt$$

$$= \tau \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega u(t) v_\tau'(t) dx dt = \tau \int_0^T \langle u, v_\tau' \rangle dt$$

for all $u \in L^p(0, T : W_0^{1,p}(\Omega))$. And, since $\beta(\bar{u}_\tau) = \bar{v}_\tau$, by (17), (19), (H1) and Lemma 2.2, $v = \beta(u)$. By (9), we get

$$\limsup_{\tau \to 0} \int_0^T \langle - \operatorname{div} a(x, \bar{u}_\tau, \nabla \bar{u}_\tau), \bar{u}_\tau \rangle dt$$

$$\le \limsup_{\tau \to 0} \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \int_\Omega \frac{-\psi^*(\beta(u_i)) + \psi^*(\beta(u_{i-1}))}{\tau} dx dt$$

$$+ \limsup_{\tau \to 0} \int_0^T \int_\Omega - \bar{f}_\tau \bar{u}_\tau dx dt.$$

Using the fact

$$\psi^*(\beta(u(i\tau))) - \psi^*(\beta(u((i-1)\tau))) = \frac{\partial \psi^*}{\partial s}(\beta(u(\sigma)))(i\tau - (i-1)\tau)$$

for $\sigma \in ((i-1)\tau, i\tau]$, by (17) and (20),

$$\limsup_{\tau \to 0} \int_0^T \langle - \operatorname{div} a(x, \bar{u}_\tau, \nabla \bar{u}_\tau), \bar{u}_\tau \rangle dt$$

$$\le \limsup_{\tau \to 0} \sum_{i=1}^N \int_{(i-1)\tau}^{i\tau} \frac{1}{\tau} \int_\Omega - \tau \frac{\partial \psi^*(\beta(u(s)))}{\partial s} dx dt + \int_0^T \langle -f, u \rangle dt$$

$$= \int_0^T \int_\Omega - \frac{\partial \psi^*(\beta(u(s)))}{\partial s} dx dt + \int_0^T \langle -f, u \rangle dt.$$

But, since $\psi(t) = \int_0^t \beta(s) ds$, $\psi'(t) = \beta(t)$. Using $(\psi^*)' = (\psi')^{-1}$, we have

$$\limsup_{\tau \to 0} \int_0^T \langle - \operatorname{div} a(x, \bar{u}_\tau, \nabla \bar{u}_\tau), \bar{u}_\tau \rangle dt \le \int_0^T \langle -\frac{\partial \beta}{\partial t} - f, u \rangle dt.$$

By Lemma 2.2, for $p \ge 2$ there exist a solution u of (1) such that

$$u \in L^p(0, T : W^{1,p}(\Omega)) \cap L^\infty(0, T : L^\infty(\Omega)).$$

For the case of $1 < p < 2$, since we have the priori estimates (10), (11), (15) and (16), we also have functions u, v and f such that

$$\bar{u}_\tau \to u \text{ strongly in } L^2(0, T : L^2(\Omega)),$$

$$v_\tau \to v \text{ strongly in } W^{1,2}(0, T : L^2(\Omega)) \cap C(0, T : L^2(\Omega)),$$

$$\bar{v}_\tau \to v \text{ strongly in } L^2(0, T : L^2(\Omega)) \cap L^\infty(0, T : L^2(\Omega)),$$

$$\bar{f}_\tau \to f \text{ strongly in } L^2(0, T : L^2(\Omega)).$$

Therefore, by similar steps of the above process, we conclude that for $1 < p < 2$ there exist a solution u such that

$$u \in L^2(0, T : L^2(\Omega)).$$

Remark 5.1. Under the assumptions (H1), (H3), (H4) and (H5) (without using the condition that f is increasing), we can prove the existence of the solution of (2) by the above method similarly.

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