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A HYBRID PROXIMAL POINT ALGORITHM AND STABILITY FOR SET-VALUED MIXED VARIATIONAL INCLUSIONS INVOLVING (A, η)-ACCRETIVE MAPPINGS

Jong Kyu Kim and Hong Gang Li

Abstract. A new class of nonlinear set-valued mixed variational inclusions involving (A, η)-accretive mappings in Banach spaces is introduced and studied, which includes many kind of variational inclusion (inequality) and complementarity problems as special cases. By using the resolvent operator associated with (A, η)-accretive operator due to Lan-Cho-Verma, the existence of solution for this kind of variational inclusion is proved, and a new hybrid proximal point algorithm is established and suggested, the convergence and stability theorems of iterative sequences generated by new iterative algorithms are also given in q-uniformly smooth Banach spaces.

1. Introduction

The variational inclusion, which was introduced and studied by Hassouni and Moudafi [7], is a useful and important generalization of the variational inequality. Various variational inclusions have been intensively studied in recent years. Many authors (see, [1], [3], [4], [5], [6], [10], [11], [12], [14], [15], [19]) introduced the concepts of η-subdifferential operators, maximal η-monotone operators, H-monotone operators, A-monotone operators, (H, η)-monotone operators, (A, η)-accretive mappings, (G, η)-monotone operators, and defined resolvent operators associated with them, respectively.

Moreover, by using the resolvent operator technique, many authors constructed some approximation algorithms for some nonlinear variational inclusions in Hilbert spaces or Banach spaces. Recently, Verma [16] has developed a hybrid version of the Eckstein-Bertsekas [2] proximal point algorithm, introduced the algorithm based on the (A, η)-maximal monotonicity framework and studied convergence of the algorithm.

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The corresponding author: jongkyuk@kyungnam.ac.kr(Jong Kyu Kim).
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On the other hand, in 2008, Li [13] studied the existence of solutions and the stability of perturbed Ishikawa iterative algorithm for nonlinear mixed quasi-variational inclusions involving (A, η)-accretive mappings in Banach spaces by using the resolvent operator technique (see, [6]).

In this paper, A new class of nonlinear set-valued mixed variational inclusions involving (A, η)-accretive mappings in Banach spaces is introduced and studied, which includes many kind of variational inclusions (inequalities) and complementarity problems as special cases. By using the resolvent operator associated with (A, η)-accretive operator, an existence of solution for this kind of variational inclusion is proved, and a new hybrid proximal point algorithm is established and suggested, the convergence and stability theorems of iterative sequences generated by new iterative algorithms are also given in q-uniformly smooth Banach spaces.

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ is the dual pair between X and X^*, 2^X denotes the family of all the nonempty subsets of X, and $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. The generalized duality mapping $J_q : X \to 2^X$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that,

$$J_q(x) = \|x\|^{q-2} J_2(x)$$

for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. If $X = H$, the Hilbert space, then J_2 becomes the identity mapping on H. J_q is single-valued if X^* is strictly convex [18], or X is uniformly smooth (Hilbert space and $L_p (2 \leq p < \infty)$ space are 2-uniformly Banach space).

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, (q > 1).$$

Remark 2.1. *It is well known that Hilbert spaces, $L_p(l_p)$ spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$ are all q-uniformly smooth.*

In this paper, we consider the following variational inclusion problem:

Let $A, g : X \to X$; $\eta, N : X \times X \to X$ be single-valued mappings. Let $M : X \times X \to 2^X$ be a set-valued (A, η)-accretive mapping. For any $u \in X$, finding $x \in X$, such that

$$u \in N(x, g(x)) + M(x) \tag{2.1}$$

Above problem is called a nonlinear set-valued mixed variational inclusion problem with (A, η)-accretive mappings.

Remark 2.2. *A special case of problem (2.1) is the following:*
If $X = X^*$ is a Hilbert space, $N = 0$ is the zero operator in X, and $u = 0$, then problem (2.1) becomes the parametric usual variational inclusion

$$0 \in M(x)$$

with an (A, η)-maximal monotone mapping M, which was studied by Verma [16].

We know that a number of known special classes of variational inclusions and variational inequalities in the problem (2.1) are have studied(see [1, 8, 9, 18]).

Let us recall the following results and concepts.

Definition 2.3. *Let S be a selfmap of X, $x_0 \in X$, and let*

$$x_{n+1} = h(S, x_n)$$

define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and

$$\varepsilon_n = \|u_{n+1} - h(S, u_n)\|.$$

If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure $\{x_n\}$ is said to be S-stable or stable with respect to S.

Definition 2.4. *A single-valued mapping $A : X \to X$ is said to be*

(i) *accretive if*

$$\langle A(x_1) - A(x_2), J_q(x_1 - x_2)\rangle \geq 0, \quad \forall x_1, x_2 \in X;$$

(ii) *strictly accretive if A is accretive and $\langle A(x_1) - A(x_2), J_q(x_1 - x_2)\rangle = 0$ if and only if $x_1 = x_2 \quad \forall x_1, x_2 \in X$;*

(iii) *$r-$strongly η-accretive if there exists a constant $r > 0$ such that*

$$\langle A(x_1) - A(x_2), J_q(\eta(x_1, x_2))\rangle \geq r\|x_1 - x_2\|^q, \quad \forall x_1, x_2 \in X;$$

(iv) *α-Lipschitz continuous if there exists a constant $\alpha > 0$ such that*

$$\|A(x_1) - A(x_2)\| \leq \alpha\|x_1 - x_2\|, \forall x_1, x_2 \in X.$$

Definition 2.5. *A single-valued mapping $\eta : X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that*

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \forall x, y \in X.$$

Definition 2.6. *A single-valued mapping $N : X \times X \to X$ is said to be*

(i) *(μ, ν)-Lipschitz continuous if there exist tow constants $\mu, \nu > 0$ such that*

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leq \mu\|x_1 - x_2\| + \nu\|y_1 - y_2\| \quad \forall x_i, y_i \in X, i = 1, 2;$$

(ii) *(ψ, κ)-relaxed cocoercive with respect to A in the first argument, if there exist constants $\psi, \kappa > 0$ such that*

$$\langle N(x_1, \cdot) - N(x_2, \cdot), J_q(A(x_1) - A(x_2))\rangle \geq -\psi\|N(x_1, \cdot) - N(x_2, \cdot)\|^q + \kappa\|x_1 - x_2\|^q,$$

for all $x_1, x_2 \in X$

Definition 2.7. *Let $A : X \to X$ and $\eta : X \times X \to X$ be single-valued mappings. A set-valued mapping $M : X \to 2^X$ is said to be*

(i) *accretive if*

$$\langle u_1 - u_2, J_q(x_1 - x_2)\rangle \geq 0, \quad \forall x_1, x_2 \in X, u_1 \in M(x_1), u_2 \in M(x_2);$$

(ii) *η-accretive if*

$$\langle u_1 - u_2, J_q(\eta(x_1, x_2))\rangle \geq 0, \quad \forall x_1, x_2 \in X, u_1 \in M(x_1), u_2 \in M(x_2);$$

(iii) *m-relaxed η-accretive, if there exists a constant $m > 0$ such that*

$$\langle u_1 - u_2, J_q(\eta(x_1, x_2))\rangle \geq -m\|x_1 - x_2\|^q, \forall x_1, x_2 \in X, u_1 \in M(x_1), u_2 \in M(x_2).$$

(iv) *A-accretive if M is accretive and $(A + \rho M)(X) = X$ for all $\rho > 0$;*

(v) *(A, η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(X) = X$ for every $\rho > 0$.*

We can define the generalized resolvent operator $R^{A,\eta}_{\rho,M}$ as follows [12].

Definition 2.8. *(*[12]*) Let $\eta : X \times X \to X$ be a single-valued mapping, $A : X \to X$ be a strictly η-accretive single-valued mapping and $M : X \times X \to 2^X$ be an (A, η)-accretive mapping. The generalized resolvent operator $R^{A,\eta}_{\rho,M} : X \to X$ is defined by*

$$R^{A,\eta}_{\rho,M}(x) = (A + \rho M)^{-1}(x)$$

for all $x \in X$, where $\rho > 0$ is a constant.

Remark 2.9. *The (A, η)-accretive mappings is more general than (H, η)-monotone mappings and m-accretive mappings in Banach space or Hilbert space, and the resolvent operators associated with (A, η)-accretive mappings include the corresponding resolvent operators associated with (H, η)-monotone operators, m-accretive mappings, A-monotone operators, η-subdifferential operators [5, 8, 9, 18].*

Lemma 2.10. *([12]) Let $\eta : X \times X \to X$ be a τ-Lipschtiz continuous mapping, $A : X \to X$ be a r-strongly η-accretive mapping, and $M : X \times X \to 2^X$ be an (A, η)-accretive mapping. Then the generalized resolvent operator $R^{A,\eta}_{\rho,M} : X \to X$ is $\tau^{q-1}/(r - m\rho)$-Lipschitz continuous, that is,*

$$\|R^{A,\eta}_{\rho,M}(x) - R^{A,\eta}_{\rho,M}(y)\| \leq \frac{\tau^{q-1}}{r - m\rho}\|x - y\|$$

for all $x, y \in X$. where $\rho \in (0, r/m)$.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu[18] proved the following result .

Lemma 2.11. *([18]) Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Lemma 2.12. *([17]) Let $\{\xi_n\}_{n=0}^{\infty}$ be a nonnegative real sequence and $\{\varphi_n\}_{n=0}^{\infty}$ be a real sequence in $[0, 1]$ such that $\sum\limits_{n=0}^{\infty} \varphi_n = \infty$. If there exists a positive integer n_1 such that*

$$\xi_{n+1} \leq (1 - \varphi_n)\xi_n + \varphi_n\chi_n, \quad \forall n \geq n_1,$$

where $\chi_n \geq 0$ for all $n \geq 0$ and $\chi_n \to 0 (n \to \infty)$, then $\lim\limits_{n\to\infty} \xi_n = 0$.

3. Main Results

We can get the following result from the definition of $R^{A,\eta}_{\rho,M(x)}$.

Lemma 3.1. *Let X be a Banach space. Let $\eta : X \times X \to X$ be a τ-Lipschtiz continuous mapping, $A : X \to X$ be a r-strongly η-accretive mapping, and $M : X \to 2^X$ be an (A, η)-accretive mapping. Then the following statements are mutually equivalent:*

(i) *An element $x \in X$ is a solution of problem (2.1).*

(ii) *For a $x \in X$, we have*

$$x = R^{A,\eta}_{\rho,M}(A(x) - \rho N(x, g(x)) + u), \tag{3.1}$$

where $\rho > 0$ is a constant.

Now, We give the existence theorem of the problem (2.1).

Theorem 3.2. *Let X be a q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a τ-Lipschtiz continuous mapping, and $A : X \to X$ be a r-strongly η-accretive mapping and α-Lipschitz continuous. Let $g : X \to X$ be Lipschitz continuous with constants β. Let $N : X \times X \to X$ be (μ, ν)-Lipschitz continuous, and (ψ, κ)-relaxed cocoercive with respect to A in the first argument. Let $M : X \times$*

$X \to 2^X$ be a set-valued (A,η)-accretive mapping. If the following condition holds

$$\tau^q(\sqrt[q]{\alpha^q + q\rho(c_q\psi\mu^q - q\kappa) + \mu^q\rho^q} + \rho\nu\beta) < \tau(r - m\rho), \tag{3.2}$$

where $c_q > 0$ is the same as in Lemma 2.11, and $\rho \in (0, \frac{r}{m})$. Then the problem (2.1) has a solution $x^* \in X$.

Proof. Define a mapping $\tilde{G} : X \to 2^X$ as follows:

$$\tilde{G}(x) = R^{A,\eta}_{\rho,M}(A(x) - \rho N(x, g(x)) - u), \quad \forall x \in X. \tag{3.3}$$

For any $\varepsilon > 0$ and any elements $x_1, x_2 \in X$, if $F(x_i) \in \tilde{G}(x_i)$ and

$$s_i = A(x_i) - \rho N(x_i, g(x_i)) - u \quad (i = 1, 2),$$

then by (3.1), (3.3) and Lemma 2.10, we have

$$\begin{aligned}
\|F(x_1) - F(x_2)\| &= \|R^{A,\eta}_{\rho,M}(s_1) - R^{A,\eta}_{\rho,M}(s_2)\| \\
&\le \frac{\tau^{q-1}}{r - m\rho}(\rho\|N(x_2, g(x_1)) - N(x_2, g(x_2))\| \qquad (3.4)\\
&\quad + \|A(x_1) - A(x_2) - \rho(N(x_1, g(x_1)) - N(x_2, g(x_1)))\|).
\end{aligned}$$

By (ψ,κ)-relaxed cocoercive with respect to A in the first argument and Lemma 2.10, we obtain

$$\begin{aligned}
&\|A(x_1) - A(x_2) - \rho(N(x_1, g(x_1)) - N(x_2, g(x_1)))\|^q \\
&\le \|A(x_1) - A(x_2)\|^q + c_q\rho^q\|N(x_1, g(x_1)) - N(x_2, g(x_1))\|^q \\
&\quad - q\rho\langle N(x_1, \cdot) - N(x_2, \cdot), J_q(A(x_1) - A(x_2))\rangle \\
&\le (\alpha^q + qc_q\psi\mu^q\rho - q\kappa\rho + \mu^q\rho^q)\|x_1 - x_2\|^q \qquad (3.5)
\end{aligned}$$

and

$$\|N(x_2, g(x_1)) - N(x_2, g(x_2))\| \le \nu\beta\|x_1 - x_2\|. \tag{3.6}$$

Combing (3.4), (3.5) and (3.6), we have

$$\|F(x_1) - F(x_2)\| \le \theta\|x_1 - x_2\|. \tag{3.7}$$

where

$$\theta = \frac{\tau^{q-1}}{r - m\rho}(\sqrt[q]{\alpha^q + q\rho(c_q\psi\mu^q - \kappa) + \mu^q\rho^q} + \rho\nu\beta)$$

From (3.7), we know that

$$\sup_{F(x_1)\in\tilde{G}(x_1)} d(F(x_1), \tilde{G}(x_2)) \le \theta\|x_1 - x_2\|, \forall x_1, x_2 \in X. \tag{3.8}$$

Similarly, we have

$$\sup_{F(x_2)\in\tilde{G}(x_2)} d(F(x_2), \tilde{G}(x_1)) \le \theta\|x_1 - x_2\|, \forall x_1, x_2 \in X. \tag{3.9}$$

It follows from (3.8), (3.9) and the definition of Hausdorff metric that

$$D(\tilde{G}(x_1), \tilde{G}(x_2)) \le \theta\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X,$$

where

$$\theta = \frac{\tau^{q-1}}{r - m\rho}(\sqrt[q]{\alpha^q + q(c_q\psi\mu^q - q\kappa)\rho + \mu^q\rho^q} + \rho\nu\beta).$$

It follows from (3.2), (3.3), and (3.9) that $\tilde{G}$ has a fixed point in X, i.e., there exists a point $x^* \in X$ such that $x^* \in \tilde{G}(x^*)$, and

$$x^* = R^{A,\eta}_{\rho,M}(A(x^*) - \rho N(x^*, g(x^*)) + u).$$

This completes the proof.

Remark 3.3. *If X is 2-uniformly smooth and there exist $\rho > 0$ such that*

$$\begin{cases} |\rho - \dfrac{\tau^2(\kappa - \psi\mu^2) - rt}{c_2\tau^2\mu^2 - t^2}| < \\ \dfrac{\sqrt{[\tau^2(\kappa-\psi\mu^2) - rt]^2 - (c_2\tau^2\mu^2 - t^2)(\tau^2\alpha^2 - r^2)}}{c_2\tau^2\mu^2 - t^2}, \\ \kappa\tau^2 > rt + \tau^2\psi\mu^2 + \sqrt{(c_2\tau^2\mu^2 - t^2)(\tau^2\alpha^2 - r^2)}, \\ \tau^2\mu^2 > t^2, \\ t = \tau\nu\beta + m \end{cases} \tag{3.10}$$

then (3.2) holds, and the problem (2.1) has a solution.

Based on Lemma 3.1, we can develop a new hybrid proximal point algorithm for finding an iterative sequence solving problem (2.1) as follows:

Algorithm 3.4. *Let x^* be a solution of problem (2.1). Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be three nonnegative sequences such that*

$$\lim_{n\to\infty} b_n = 0, \quad a = \limsup_{n\to\infty} a_n < 1, \quad \rho_n \uparrow \rho \le \infty, \qquad (n = 0, 1, 2, \cdots).$$

***Step 1**: For an arbitrarily chosen initial point $x_0 \in X$, set*

$$x_1 = (1 - a_0)x_0 + a_0 y_0,$$

where the y_0 satisfies

$$\|y_0 - R^{A,\eta}_{\rho_n,M}(A(x_0) - \rho_n N(x_0, g(x_0)) + u)\| \le b_0\|y - x_0\|;$$

***Step 2**: The sequence $\{x_n\}$ is generated by an iterative procedure*

$$x_{n+1} = (1 - a_n)x_n + a_n y_n, \tag{3.11}$$

and y_n satisfies

$$\|y_n - R^{A,\eta}_{\rho_n,M}(A(x_n) - \rho_n N(x_n, g(x_n)) + u)\| \le b_n\|y_n - x_n\|,$$

where $n = 1, 2, \cdots$.

We consider the following sequences for the stability problem for the convergence.

Let $\{w_n\}_{n=0}^{\infty}$ *be any sequence in* X *and define* $\{\varepsilon_n\}_{n=0}^{\infty}$ *by*

$$\begin{cases} \varepsilon_n = \|w_{n+1} - [(1-a_n)w_n + a_n y_n]\|, \\ \|y_n - R^{A,\eta}_{\rho_n,M}(A(w_n) - \rho_n N(x_n, g(w_n)) + u)\| \le b_n \|y_n - w_n\|, \end{cases} \tag{3.12}$$

where $n = 0, 1, \cdots$.

Remark 3.5. *For a suitable choice of the mappings* $A, \eta, N, M, g,$ *and space* X*, the Algorithm 3.4 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities and variational inclusions[1, 8, 9, 16, 18].*

We are in a position to construct the convergence theorem of the iterative algorithm.

Theorem 3.6. *Let* X, A, N, M, g *be the same as in Theorem 3.2, and the condition (3.2) holds. Let* $\{a_n\}_{n=0}^{\infty}$*,* $\{b_n\}_{n=0}^{\infty}$ *and* $\{\rho_n\}_{n=0}^{\infty}$ *be the same as in Algorithm 3.4. Then the sequence* $\{x_n\}$ *in Algorithm 3.4 converges to a solution* x^* *of problem (2.1) with the convergence rate*

$$\begin{aligned} \vartheta &= (1-a) + a\frac{\tau^{q-1}}{r - m\rho}(\sqrt[q]{\alpha^q + q(c_q\psi\mu^q - \kappa)\rho + \mu^q\rho^q} + \rho\nu\beta) \\ &< 1. \end{aligned} \tag{3.13}$$

Proof. Suppose that $\{x_n\}$ is the sequence generated by the hybrid proximal point Algorithm 3.4, and that x^* is a solution of (2.1). From Lemma 3.1, we have

$$x^* = (1-a_n)x^* + a_n R^{A,\eta}_{\rho_n,M}(A(x^*) - \rho_n N(x^*, g)(x^*) + u).$$

For all $n \ge 0$, set

$$z_{n+1} = (1-a_n)x_n + a_n R^{A,\eta}_{\rho_n,M}(A(x_n) - \rho_n N(x_n, g(x_n)) + u).$$

Next, we find the estimate

$$\begin{aligned} &\|z_{n+1} - x^*\| \\ &\le (1-a_n)\|x_n - x^*\| + a_n\|R^{A,\eta}_{\rho_n,M)}(A(x_n) - \rho_n N(x_n, g(x_n)) + u) \\ &\quad - R^{A,\eta}_{\rho_n,M}(A(x^*) - \rho_n N(x^*, g(x^*)) + u)\| \\ &\le (1-a_n)\|x_n - x^*\| \\ &\quad + a_n\frac{\tau^{q-1}}{r - m\rho_n}[\|A(x_n) - A(x^*) - \rho_n(N(x_n, g(x_n)) - N(x^*, g(x_n)))\| \\ &\quad + \rho_n\|N(x^*, g(x_n)) - N(x^*, g(x^*))\|]. \end{aligned} \tag{3.14}$$

By (ψ, κ)-relaxed cocoercive with respect to A in the first argument and Lemma 2.11, we obtain

$$\begin{aligned}&\|A(x_n) - A(x^*) - \rho_n(N(x_n, g(x_n)) - N(x^*, g(x_n)))\|^q\\ &\leq \|A(x_n) - A(x^*)\|^q + c_q\rho_n^q\|N(x_n, g(x_n)) - N(x^*, g(x_n))\|^q\\ &\quad -q\rho_n\langle N(x_n, g(x_n)) - N(x^*, g(x_n)), J_q(A(x_n) - A(x^*))\rangle\\ &\leq (\alpha^q + qc_q\psi\mu^q\rho_n - q\kappa\rho_n + \mu^q\rho_n^q)\|x_n - x^*\|^q,\end{aligned} \tag{3.15}$$

and

$$\|N(x^*, g(x_n)) - N(x^*, g(x^*))\| \leq \nu\beta\|x_n - x^*\|. \tag{3.16}$$

It follows that

$$\|z_{n+1} - x^*\| \leq \theta_n\|x_n - x^*\|, \tag{3.17}$$

where

$$\theta_n = (1 - a_n) + a_n\frac{\tau^{q-1}}{r - m\rho_n}(\sqrt[q]{\alpha^q + q(c_q\psi\mu^q - \kappa)\rho_n + \mu^q\rho_n^q} + \rho_n\nu\beta).$$

Since $x_{n+1} = (1 - a_n)x_n + a_ny_n$, we have $x_{n+1} - x_n = a_n(y_n - x_n)$. It follows that

$$\begin{aligned}&\|x_{n+1} - z_{n+1}\|\\ &\leq \|(1 - a_n)x_n + a_ny_n - [(1 - a_n)x_n + a_nR^{A,\eta}_{\rho_n,M}(A(x_n) - \rho_nN(x_n, g(x_n)) + u)]\|\\ &\leq a_n\|y_n - R^{A,\eta}_{\rho_n,M}(A(x_n) - \rho_nN(x_n, g(x_n)) + u)\|\\ &\leq a_nb_n\|y_n - x_n\|.\end{aligned}$$

Next, we can obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|z_{n+1} - x^*\| + \|x_{n+1} - z_{n+1}\|\\ &\leq \|z_{n+1} - x^*\| + a_nb_n\|y_n - x_n\|\\ &\leq \|z_{n+1} - x^*\| + b_n\|x_{n+1} - x_n\|\\ &\leq \|z_{n+1} - x^*\| + b_n\|x_{n+1} - x^*\| + b_n\|x^* - x_n\|.\end{aligned} \tag{3.18}$$

This implies that

$$\|x_{n+1} - x^*\| \leq \frac{\theta_n + b_n}{1 - b_n}\|x_n - x^*\|. \tag{3.19}$$

Let

$$\begin{aligned}\vartheta &= \limsup_{n\to\infty}\frac{\theta_n + b_n}{1 - b_n} = \limsup_{n\to\infty}\theta_n\\ &= (1 - a) + a\frac{\tau^{q-1}}{r - m\rho}(\sqrt[q]{\alpha^q + qc_q\psi\mu^q\rho - q\kappa\rho + \mu^q\rho^q} + \rho\nu\beta).\end{aligned}$$

Then, we have

$$\|x_{n+1} - x^*\| \leq \vartheta\|x_n - x^*\|.$$

By (3.2), it follows that $0 < \vartheta < 1$, and the sequence $\{x_n\}$ in Algorithm 3.4 converges to a solution x^* of problem (2.1) with convergence rate ϑ. This completes the proof.

Now, we will give the stability problem for the convergence of the sequence $\{x_n\}$.

Theorem 3.7. *Let X, A, N, M, g be the same as in Theorem 3.2, and the condition (3.2) holds. If $0 < \lambda \le a_n$, then $\lim_{n\to\infty} w_n = x^*$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$, where ε_n is defined by (3.12), that is, the sequence $\{x_n\}$ in (3.11) is stable.*

Proof. Let the x^* be a unique solution of problem (2.1). Then it follows from Lemma 3.1 that

$$x^* = (1-a_n)x^* + a_n R^{A,\eta}_{\rho_n,M}(A(x^*) - \rho_n N(x^*, g(x^*)) + u) \tag{3.20}$$

From (3.19) and (3.20), it follows that

$$\|x_{n+1} - x^*\| \le \frac{\theta_n + b_n}{1 - b_n}\|x_n - x^*\|. \tag{3.21}$$

By (3.11) and the proof of inequality (3.17), we obtain

$$\begin{aligned}
&\|w_{n+1} - x^*\| \\
&\le \|w_{n+1} - [(1-a_n)w_n + a_n y_n]\| + \|[(1-a_n)w_n + a_n y_n] - x^*\| \qquad (3.22)\\
&\le \varepsilon_n + (1-a_n)\|w_n - x^*\| + a_n\|y_n - R^{A,\eta}_{\rho_n,M}(A(x^*) - \rho_n N(x^*, g(x^*)) + u)\| \\
&\le \varepsilon_n + (1-a_n)\|w_n - x^*\| + a_n\|y_n - R^{A,\eta}_{\rho_n,M}(A(w_n) - \rho_n N(w_n, g(w_n)) + u)]\| \\
&\quad + \|R^{A,\eta}_{\rho_n,M}(A(w_n) - \rho_n N(w_n, g(w_n)) + u)] - R^{A,\eta}_{\rho_n,M}(A(x^*) - \rho_n N(x^*, g(x^*)) + u)\| \\
&\le \varepsilon_n + \{(1-a_n) + a_n b_n \\
&\quad + a_n \frac{\tau^{q-1}}{r - m\rho_n}[\sqrt[q]{\alpha^q + q(c_q\psi\mu^q - \kappa)\rho_n + \mu^q\rho_n^q} + \nu\beta\rho_n]\}\|w_n - x^*\|.
\end{aligned}$$

Since $0 < \lambda \le a_n$, by (3.22), we have

$$\|z_{n+1} - x^*\| \le [1 - a_n(1-\theta)]\|z_n - x^*\| + (1-\theta)a_n\frac{\varepsilon_n}{\lambda(1-\theta)}, \tag{3.23}$$

where

$$\begin{aligned}
\theta \quad &= \limsup_{n\to\infty}[b_n + \frac{\tau^{q-1}}{r - m\rho_n}(\sqrt[q]{\alpha^q + (qc_q\psi\mu^q - q\kappa)\rho_n + \mu^q\rho_n^q} + \rho_n\nu\beta)] \\
&= \frac{\tau^{q-1}}{r - m\rho}(\sqrt[q]{\alpha^q + q(c_q\psi\mu^q - \kappa)\rho + \mu^q\rho^q} + \rho\nu\beta) \\
&< 1.
\end{aligned}$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $\xi_n = \|z_n - x^*\|$, $\varphi_n = a_n(1-\theta)$, and $\chi_n = \frac{\varepsilon_n}{\varphi_n(1-\theta)}$. From $\sum_{n=0}^{\infty} a_n = \infty$ and Lemma 2.12, we have $\lim_{n\to\infty} z_n = x^*$.

Conversely, if $\lim\limits_{n\to\infty} z_n = x^*$, then we get

$$\begin{aligned} \varepsilon_n \quad &= \|w_{n+1} - [(1-a_n)w_n + a_n y_n]\| \\ &\le \|z_{n+1} - x^*\| + \|(1-a_n)w_n + a_n y_n - x^*\| \\ &\le \|z_{n+1} - x^*\| + [1 - a_n(1-\theta)]\|z_n - x^*\| \\ &\to 0, (n \to \infty). \end{aligned} \tag{3.24}$$

The sequence $\{x_n\}$ generated by (3.11) is stable. This completes the proof.

Remark 3.8. *For a suitable choice of the mappings A, η, N, M, g, we can obtain several known results [8, 18] as special cases of Theorem 3.2, 3.6, 3.7.*

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Jong Kyu Kim
Department of Mathematics Education
Kyungnam University
Masan, Kyungnam, 631-701, Korea
E-mail address: jongkyuk@kyungnam.ac.kr

Hong Gang Li
Institute of Applied Mathematics Research
Chongqing University of Posts and Telecommunications
Chongqing 400065, China
E-mail address: lihg12@126.com