REAL HALF LIGHTLIKE SUBMANIFOLDS WITH TOTALLY
UMBILICAL PROPERTIES

DAE HO JIN

ABSTRACT. In this paper, we prove two characterization theorems for real half
lightlike submanifold $(M, g, S(TM))$ of an indefinite Kaehler manifold $\tilde{M}$ or an
indefinite complex space form $\tilde{M}(c)$ subject to the conditions: (a) $M$ is totally
umbilical in $\tilde{M}$, or (b) its screen distribution $S(TM)$ is totally umbilical in $M$.

1. INTRODUCTION

It is well known that the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$ of the
half lightlike submanifolds $M$ of a semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ of codimension
2 is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of
rank 1. Then there exists complementary non-degenerate distributions $S(TM)$ and
$S(TM^\perp)$ of $\text{Rad}(TM)$ in $TM$ and $TM^\perp$ respectively, which called the screen and
co-screen distribution on $M$, such that

\begin{equation}
TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),
\end{equation}

where the symbol $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a half
lightlike submanifold by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth
functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle
$E$ (same notation for any other vector bundle) over $M$. Choose $L \in \Gamma(S(TM^\perp))$ as
a unit vector field with $\tilde{g}(L, L) = \epsilon = \pm 1$. Consider the orthogonal complementary
distribution $S(TM)^\perp$ to $S(TM)$ in $TM$. Certainly $\xi$ and $L$ belong to $\Gamma(S(TM)^\perp)$.
Hence we have the following orthogonal decomposition

$$S(TM)^\perp = S(TM^\perp) \oplus_{\text{orth}} S(TM^\perp)^\perp,$$

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where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. For any null section $\xi$ of $\text{Rad}(TM)$ on a coordinate neighborhood $U \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(\text{ltr}(TM))$ [1] satisfying

\begin{equation}
\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).
\end{equation}

We call $N, \text{ltr}(TM)$ and $\text{tr}(TM) = S(TM^\perp) \oplus_{\text{orth}} \text{ltr}(TM)$ the *lightlike transversal vector field*, *lightlike transversal vector bundle* and *transversal vector bundle* of $M$ with respect to $S(TM)$ respectively. Therefore the tangent space $T\bar{M}$ of the ambient manifold $\bar{M}$ is decomposed as follows:

\begin{equation}
T\bar{M} = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM)
= \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).
\end{equation}

The purpose of this paper is to study the geometry of real half lightlike submanifolds $(\bar{M}, g, S(TM))$ of an indefinite Kaehler manifold $\bar{M}$ or an indefinite complex space form $\bar{M}(c)$ subject to the conditions: (a) $M$ is totally umbilical in $\bar{M}$ or (b) its screen $S(TM)$ is totally umbilical in $M$. In section 3, we prove a characterization theorem for totally umbilical real half lightlike submanifolds $M$ of an indefinite Kaehler manifold $\bar{M}$. This theorem shows that the second fundamental forms $B$ and $D$ of such a real half lightlike submanifold $M$ satisfy $B = D = 0$, i.e., $M$ is totally geodesic (Theorem 3.1). Furthermore, if $\bar{M} = \bar{M}(c)$, then we have also $c = 0$, i.e., $\bar{M}$ is a semi-Euclidean space (Theorem 3.3). In section 4, we prove a characterization theorem for real half lightlike submanifolds $M$ of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in $\bar{M}$. This theorem shows that the second fundamental form $C$ of $S(TM)$ and the constant holomorphic sectional curvature $c$ satisfy $C = c = 0$, i.e., $S(TM)$ is totally geodesic and $\bar{M}$ is a semi-Euclidean space (Theorem 4.2). Using these theorems, we prove several additional theorems for real half lightlike submanifold $M$ of $\bar{M}(c)$ such that $M$ is totally umbilical or $S(TM)$ is totally umbilical in $M$. Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and $P$ the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingarten formulas are given by

\begin{align}
\bar{\nabla}_XY &= \nabla_XY + B(X, Y)N + D(X, Y)L, \\
\bar{\nabla}_XN &= -A_NX + \tau(X)N + \rho(X)L,
\end{align}
\[ \nabla_X L = -A_L X + \phi(X) N, \]
\[ \nabla_X P Y = \nabla^*_X P Y + C(X, P Y) \xi, \]
\[ \nabla_X \xi = -A^*_\xi X - \tau(X) \xi, \]

for all \( X, Y \in \Gamma(TM) \), where \( \nabla \) and \( \nabla^* \) are induced linear connections on \( TM \) and \( S(TM) \) respectively, \( B \) and \( D \) are called the \textit{local second fundamental forms} of \( M \), \( C \) is called the \textit{local second fundamental form} on \( S(TM) \). \( A_N, A^*_\xi \) and \( A_L \) are linear operators on \( TM \) and \( \tau, \rho \) and \( \phi \) are 1-forms on \( TM \). We say that \( h(X, Y) = B(X, Y) N + D(X, Y) L \) is the \textit{second fundamental tensor} of \( M \).

Since \( \nabla \) is torsion-free, \( \nabla \) is also torsion-free and both \( B \) and \( D \) are symmetric. From the facts \( B(X, Y) = \bar{g}(\nabla_X Y, \xi) \) and \( D(X, Y) = \epsilon \bar{g}(\nabla_X Y, L) \) for all \( X, Y \in \Gamma(TM) \), we know that \( B \) and \( D \) are independent of the choice of a \( S(TM) \) and

\[ B(X, \xi) = 0, \quad D(X, \xi) = -\epsilon \phi(X), \]

for all \( X \in \Gamma(TM) \). The induced connection \( \nabla \) of \( M \) is not metric and satisfies

\[ (\nabla_X g)(Y, Z) = B(X, Y) \eta(Z) + B(X, Z) \eta(Y), \]

for all \( X, Y, Z \in \Gamma(TM) \), where \( \eta \) is a 1-form on \( TM \) such that

\[ \eta(X) = \bar{g}(X, N), \]

for all \( X \in \Gamma(TM) \). But the connection \( \nabla^* \) on \( S(TM) \) is metric. The above three local second fundamental forms are related to their shape operators by

\[ B(X, Y) = g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \]
\[ C(X, P Y) = g(A_N X, P Y), \quad \bar{g}(A_N X, N) = 0, \]
\[ \epsilon D(X, P Y) = g(A_L X, P Y), \quad \bar{g}(A_L X, N) = \epsilon \rho(X), \]
\[ \epsilon D(X, Y) = g(A_L X, Y) - \phi(X) \eta(Y), \quad \forall X, Y \in \Gamma(TM). \]

By (1.12) and (1.13), we show that \( A^*_\xi \) and \( A_N \) are \( \Gamma(S(TM)) \)-valued shape operators related to \( B \) and \( C \) respectively and \( A^*_\xi \) is self-adjoint on \( TM \) and

\[ A^*_\xi \xi = 0. \]

But \( A_N \) and \( A_L \) are not self-adjoint on \( S(TM) \) and \( TM \) respectively.

Denote by \( \bar{R}, R \) and \( R^* \) the curvature tensors of \( \nabla, \nabla \) and \( \nabla^* \) respectively. Using the Gauss-Weingarten equations for \( M \) and \( S(TM) \), for all \( X, Y, Z, W \in \Gamma(TM) \),
we obtain the Gauss-Codazzi equations for $M$ and $S(TM)$:

\begin{align}
\tag{1.17}
\bar{g}(\bar{R}(X,Y)Z, PW) &= g(R(X,Y)Z, PW) \\
&\quad + B(X,Z)C(Y, PW) - B(Y,Z)C(X, PW) \\
&\quad + \epsilon\{D(X,Z)D(Y, PW) - D(Y,Z)D(X, PW)\}, \\
\tag{1.18}
\bar{g}(\bar{R}(X,Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
&\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\
&\quad + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \\
\tag{1.19}
\bar{g}(\bar{R}(X,Y)Z, N) &= \bar{g}(R(X,Y)Z, N) \\
&\quad + \epsilon\{D(X,Z)\rho(Y) - D(Y,Z)\rho(X)\}, \\
\tag{1.20}
\bar{g}(\bar{R}(X,Y)Z, L) &= \epsilon\{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \\
&\quad + B(Y,Z)\rho(X) - B(X,Z)\rho(Y)\}, \\
\tag{1.21}
\bar{g}(\bar{R}(X,Y)PZ, PW) &= g(R^*(X,Y)PZ, PW) \\
&\quad + C(X,PZ)B(Y, PW) - C(Y,PZ)B(X, PW), \\
\tag{1.22}
g(R(X,Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
&\quad + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X).
\end{align}

2. **REAL HALF LIGHTLIKE SUBMANIFOLDS**

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real $2m$-dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2\nu$, $0 < \nu < m$ and $J$ is an almost complex structure on $\bar{M}$ satisfying, for all $X, Y \in \Gamma(TM)$,

\begin{align}
\tag{2.1}
J^2 &= -I, \quad \bar{g}(JX, JY) = \bar{g}(X, Y), \quad (\bar{\nabla}_X J)Y = 0.
\end{align}

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature $c$ such that

\begin{align}
\tag{2.2}
\bar{R}(X,Y)Z &= \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX \\
&\quad - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\}, \quad \forall X, Y, Z \in \Gamma(TM).
\end{align}

**Definition 1.** Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. We say that $M$ is a **CR-lightlike submanifold**[2] of $\bar{M}$ if the following two conditions are fulfilled:
(A) $J(\text{Rad}(TM))$ is a distribution on $M$ such that

$$\text{Rad}(TM) \cap J(\text{Rad}(TM)) = \{0\}.$$ 

(B) There exist vector bundles $H_o$ and $H'$ over $M$ such that

$$S(TM) = \{J(\text{Rad}(TM)) \oplus H'\} \oplus_\text{orth} H_o, \quad J(H_o) = H_o, \quad J(H') = K_1 \oplus_\text{orth} K_2,$$

where $H_o$ is a non-degenerate almost complex distribution on $M$, and $K_1$ and $K_2$ are vector subbundles of $ltr(TM)$ and $S(TM^\perp)$ respectively.

**Theorem 2.1** ([7]). Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}$. Then $M$ is a CR-lightlike submanifold of $\tilde{M}$.

**Proof.** Let $\xi$, $N$ and $L$ be local sections of $\text{Rad}(TM)$, $ltr(TM)$ and $S(TM^\perp)$ respectively. From $\tilde{g}(J\xi, \xi) = 0$ and $\text{Rad}(TM) \cap J(\text{Rad}(TM)) = \{0\}$, we show that $J(\text{Rad}(TM))$ is a vector subbundle of $S(TM)$ or $S(TM^\perp)$ of rank 1. Also, from $\tilde{g}(JN, N) = 0$ and $\tilde{g}(JN, \xi) = -\tilde{g}(N, J\xi) = 0$, $J(ltr(TM))$ is also a vector subbundle of $S(TM)$ or $S(TM^\perp)$ of rank 1. Since $J\xi$ and $JN$ are null vector fields satisfying $\tilde{g}(J\xi, JN) = 1$ and both $S(TM)$ and $S(TM^\perp)$ are non-degenerate, we see that either $\{J\xi, JN\} \in \Gamma(S(TM))$ or $\{J\xi, JN\} \in \Gamma(S(TM^\perp))$. If $\{J\xi, JN\} \in \Gamma(S(TM^\perp))$, since $J(\text{Rad}(TM))$, $J(ltr(TM))$ and $S(TM^\perp)$ are non-degenerate of rank 1, we have $J(\text{Rad}(TM)) = J(ltr(TM)) = S(TM^\perp)$. It is contradiction. Thus we choose a screen distribution $S(TM)$ that contains $J(\text{Rad}(TM))$ and $J(ltr(TM))$. For $L \in \Gamma(S(TM^\perp))$, as $\tilde{g}(JL, L) = 0$, $\tilde{g}(JL, \xi) = -\tilde{g}(L, J\xi) = 0$ and $\tilde{g}(JL, N) = -\tilde{g}(L, JN) = 0$, $J(S(TM^\perp))$ is also a vector subbundle of $S(TM)$ such that

$$J(S(TM^\perp)) \oplus_\text{orth} \{J(\text{Rad}(TM)) \oplus J(ltr(TM))\}.$$

We choose $S(TM)$ to contain $J(S(TM^\perp))$ too. Thus the screen distribution $S(TM)$ is expressed as follow:

$$S(TM) = \{J(\text{Rad}(TM)) \oplus J(ltr(TM))\} \oplus_\text{orth} J(S(TM^\perp)) \oplus_\text{orth} H_o,$$

where $H_o$ is a non-degenerate almost complex distribution on $M$ with respect to $J$, i.e., $J(H_o) = H_o$. Denote $H' = J(ltr(TM)) \oplus_\text{orth} J(S(TM^\perp))$. Thus (2.3) gives $S(TM)$ as in condition (B) and $J(H') = K_1 \oplus_\text{orth} K_2$, where $L_1 = ltr(TM)$ and $K_2 = S(TM^\perp)$. Hence $M$ is a CR-lightlike submanifold of $\tilde{M}$. \hfill $\square$

From Theorem 2.1, the general decompositions (1.1) and (1.3) reduce to

$$TM = H \oplus H', \quad T\tilde{M} = H \oplus H' \oplus tr(TM),$$
where $H$ is a 2-lightlike almost complex distribution on $M$ such that
\[ H = \text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM)) \oplus_{\text{orth}} H_0. \]

Consider null vector fields $\{U, V\}$ and non-null vector field $W$ such that
\begin{equation}
U = -JN, \quad V = -J\xi, \quad W = -JL.
\end{equation}

Denote by $S$ the projection morphism of $TM$ on $H$. Then, by the first equation of (2.4)[denote (2.4)-1], any $X \in \Gamma(TM)$ is expressed as follows
\begin{equation}
X = SX + u(X)U + w(X)W, \quad JX = FX + u(X)N + w(X)L,
\end{equation}
where $u$, $v$ and $w$ are 1-forms locally defined on $M$ by
\begin{equation}
u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = \varepsilon g(X, W)
\end{equation}
and $F$ is a tensor field of type $(1,1)$ globally defined on $M$ by
\[ FX = JSX, \quad \forall X \in \Gamma(TM). \]

Differentiating (2.5) with $X \in \Gamma(TM)$ and using the local Gauss and Weingartan formulas (1.4)~(1.8), (2.1), (2.6) and (2.7), we have
\begin{equation}
B(X, U) = C(X, V), \quad C(X, W) = \varepsilon D(X, U), \quad B(X, W) = \varepsilon D(X, V).
\end{equation}

We say that two vectors $X$ and $Y$ on $M$ are conjugate with respect to the second fundamental tensor $h$ if $h(X, Y) = 0$. A self-conjugate vector is said to be an asymptotic vector field. Then by (2.8) we get

**Theorem 2.2.** Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}$. Then the vector fields $\xi$ and $V$ are conjugate with respect to the second fundamental forms $C$ and $D$.

**Proof.** Replacing $X$ with $\xi$ in the first and third equations in (2.8) by turns and using the equation (1.9), we have $C(\xi, V) = 0$ and $D(\xi, V) = 0$. \qed

**Definition 2.** A half lightlike submanifold $(M, g, S(TM))$ is said to be irrotational[8] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

**Note 1.** Since $B(X, \xi) = 0$ due to the first equation of (1.9), the above definition is equivalent to $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

**Theorem 2.3.** Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite Kaehler manifold $\tilde{M}$. Then $M$ is irrotational. Moreover, if $M$ is totally geodesic,
i.e., $h = 0$, then $H$ is an integrable and parallel distribution with respect to the induced connection $\nabla$ on $M$.

Proof. Take $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Then we show that $FY = JY \in \Gamma(H)$ due to $u(Y) = w(Y) = 0$. Apply $J$ to (1.4) and use (1.4), (2.1), (2.5) and (2.6), we have

$$\nabla_X FY + B(X, FY)N + D(X, FY)L$$

$$= F(\nabla_X Y) + u(\nabla_X Y)N + w(\nabla_X Y)L - B(X, Y)U - D(X, Y)W.$$ 

Taking the scalar product with $\xi$ and $L$ in this equation, we have

$$(2.9) \quad B(X, FY) = g(\nabla_X Y, V), \quad D(X, FY) = \epsilon g(\nabla_X Y, W),$$

$$(2.10) \quad (\nabla_X F)Y = -D(X, Y)U - D(X, Y)W.$$  

Apply the operator $\tilde{\nabla}_X$ to (2.7) and then, to (2.6)-2 and use (1.12)- (1.14), (2.1), (2.6)-2 and (2.7) and Gauss-Weingarten equations for $M$, we deduce

$$(2.11) \quad (\nabla_X u)(Y) = -u(Y)\tau(X) - w(Y)\phi(X) - B(X, FY),$$

$$(2.12) \quad (\nabla_X v)(Y) = v(Y)\tau(X) + \epsilon w(Y)\rho(X) - g(ANX, FY),$$

$$(2.13) \quad (\nabla_X w)(Y) = -u(Y)\rho(X) + \epsilon v(Y)\phi(X) - D(X, FY),$$

$$(2.14) \quad (\nabla_X F)(Y) = u(Y)ANX + w(Y)ALX - B(X, Y)U - D(X, Y)W$$

$$- \epsilon v(Y)\phi(X)L, \quad \forall X, Y \in \Gamma(TM).$$

Take $Y \in \Gamma(H)$ in (2.14) and use (2.10), we have $v(Y)\phi(X) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Replace $Y$ by $V$ in this equation, we have $\phi(X) = 0$ for all $X \in \Gamma(TM)$. Thus $M$ is irrotational. Moreover, if $M$ is totally geodesic, then, by (2.9), $H$ is an integrable and parallel distribution with respect to $\nabla$.

### 3. Totally Umbilical Half Lightlike Submanifolds

**Definition 3.** We say that $M$ is *totally umbilical* [3] in $\tilde{M}$ if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$(3.1) \quad h(X, Y) = \mathcal{H} g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

In case $\mathcal{H} \neq 0$ on $\mathcal{U}$, we say that $M$ is *proper totally umbilical*.

It is easy to see that $M$ is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions $\beta$ and $\delta$ such that

$$(3.2) \quad B(X, Y) = \beta g(X, Y), \quad D(X, Y) = \delta g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$
Theorem 3.1. Let \((M,g,S(TM))\) be a totally umbilical real half lightlike submanifold of an indefinite Kaehler manifold \(\bar{M}\). Then \(M\) is totally geodesic.

Proof. From the third equation of (2.8) and (3.2), we show that
\[
\beta g(X,W) = \epsilon \delta g(X,V), \quad \forall X \in \Gamma(TM).
\]
Replacing \(X\) by \(W\) and \(U\) in this equation by turns, we have \(\beta = 0\) and \(\delta = 0\) respectively. Thus \(B = D = 0\) and \(M\) is totally geodesic in \(\bar{M}\). \(\square\)

Corollary 1. We have the following assertions:

1. There exist no proper totally umbilical real half lightlike submanifolds of an indefinite Kaehler manifold \(\bar{M}\).
2. The second fundamental form \(C\) of the screen distribution \(S(TM)\) is degenerate on \(\Gamma(S(TM))\).
3. The vector fields \(V\) and \(W\) are conjugate to any vector field on \(M\) with respect to \(C\). In particular, \(V\) and \(W\) are asymptotic vector fields.
4. \(A_N = \Gamma(J(Rad(TM))) \otimes_{\text{orth}} H_o\)-valued shape operator related to \(C\).

Proof. From the first two equations of (2.8) and the fact that \(B = D = 0\), we have \(C(X,V) = C(X,W) = 0\) for any \(X \in \Gamma(TM)\). Therefore we have (2) and (3). From these equations and (1.13), we get \(g(A_NX, V) = g(A_NX, W) = 0\) for all \(X \in \Gamma(TM)\), which proves the assertion (4). \(\square\)

Combining Theorem 2.3 and 3.1, we have the following theorem:

Theorem 3.2. Let \((M,g,S(TM))\) be a totally umbilical real half lightlike submanifold of an indefinite Kaehler manifold \(\bar{M}\). Then \(H\) is a parallel distribution with respect to \(\nabla\) and \(M\) is locally a product manifold \(L_u \times L_w \times M^4\), where \(L_u\) and \(L_w\) are null and non-null curves tangent to \(J(\text{itr}(TM))\) and \(J(S(TM^\perp))\) respectively and \(M^4\) is a leaf of \(H\).

Theorem 3.3. Let \((M,g,S(TM))\) be a totally umbilical real half lightlike submanifold of an indefinite complex space form \(\bar{M}(c)\). Then we have \(c = 0\).

Proof. Using (1.18) and the fact that \(B = D = 0\), we get
\[
\frac{c}{4}\{u(X)\bar{g}(JY,Z) - u(Y)\bar{g}(JX,Z) + 2u(Z)\bar{g}(X,JY)\} = 0,
\]
for all \(X,Y,Z \in \Gamma(TM)\). Replace \(Y\) by \(\xi\) and use (2.5) and (2.7), we show that \(\frac{3c}{4}u(X)u(Z) = 0\) for all \(X, Z \in \Gamma(TM)\). Take \(X = Z = U\), we get \(c = 0\). \(\square\)
Corollary 2. There exist no totally umbilical real half lightlike submanifolds of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$.

Theorem 3.4. Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then $M$ and each leaf $M^*$ of $S(TM)$ are spaces of constant curvature 0.

Proof. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on $M$ such that $\text{Rad}(TM) = \text{Span}\{\xi\}$ and $S(TM) = \text{Span}\{W_a\}$. Using this quasi-orthonormal frame field, we obtain
\begin{equation}
R(X, Y)Z = \sum_{a=1}^{2m-3} \epsilon_a g(R(X, Y)Z, W_a)W_a + g(R(X, Y)Z, N)\xi,
\end{equation}
for any $X, Y \in \Gamma(TM)$ and $\epsilon_a = g(W_a, W_a)$. Using (1.17), (1.19) and the last equation, we have $R(X, Y)Z = 0$ for any $X, Y, Z \in \Gamma(TM)$, due to the facts that $c = 0$ and $B = D = 0$. Thus $M$ is a lightlike manifold of constant curvature 0. Also, from (1.17) and (1.21), we also have $R^*(X, Y)Z = 0$ for any $X, Y, Z \in \Gamma(S(TM))$. Thus $M^*$ is also a semi-Euclidean space.

Combining Theorem 3.2 and 3.4, we have the following theorem:

Theorem 3.5. Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then $M$ is locally a product manifold $L_u \times L_w \times M^s$, where $L_u$ and $L_w$ are null and non-null curves respectively and $M^s$ is a 2-lightlike manifold of constant curvature 0.

4. Totally Umbilical Screen Distributions

Definition 4. We say that (each leaf $M^*$ of) $S(TM)$ is totally umbilical [3] in $M$ if, on any coordinate neighborhood $U \subset M$, there is a smooth function $\gamma$ such that $A_N X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,
\begin{equation}
C(X, PY) = \gamma g(X, Y), \quad \forall X, Y \in \Gamma(TM).
\end{equation}

In case $\gamma = 0$ (or $\gamma \neq 0$) on $U$, we say that (each leaf $M^*$ of) $S(TM)$ is totally geodesic (or proper totally umbilical) in $M$.

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

Theorem 4.1 ([1]). Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following are equivalent:
(1) \( S(TM) \) is integrable.
(2) \( C \) is symmetric on \( \Gamma(S(TM)) \).
(3) \( A_N \) is self-adjoint on \( \Gamma(S(TM)) \) with respect to \( g \).

**Note 2.** If \( S(TM) \) is totally umbilical in \( M \), then \( C \) is symmetric on \( \Gamma(S(TM)) \). Thus \( S(TM) \) is integrable and \( M \) is locally a product manifold \( L_\xi \times M^* \), where \( L_\xi \) is a null curve tangent to \( \text{Rad}(TM) \) and \( M^* \) is a leaf of \( S(TM) \) [2].

**Theorem 4.2.** Let \( (M, g, S(TM)) \) be a real half lightlike submanifold of an indefinite complex space form \( \tilde{M}(c) \). If \( S(TM) \) is totally umbilical in \( M \), then we have \( c = 0 \) and \( C = 0 \), on any coordinate neighborhood \( U \subset M \). Moreover,

(1) \( c = 0 \) implies the ambient space \( \tilde{M}(c) \) is a semi-Euclidean space,

(2) \( C = 0 \), on any \( U \subset M \), implies \( S(TM) \) is totally geodesic in \( M \).

**Proof.** Using the first two equations of (2.8) and (4.1), we have

\[
B(X, U) = \gamma g(X, V), \quad D(X, U) = \epsilon \gamma g(X, W),
\]

for all \( X \in \Gamma(TM) \). Using (1.19), (1.22), (2.2), (2.5), (2.7) and (4.1), we get

\[
\gamma \{B(Y, PZ)\eta(X) - B(X, PZ)\eta(Y)\}
+ \epsilon \{D(Y, PZ)\rho(X) - D(X, PZ)\rho(Y)\}
= \{X[\gamma] - \gamma \tau(X) - \frac{c}{4}\eta(X)\}g(Y, PZ)
- \{Y[\gamma] - \gamma \tau(Y) - \frac{c}{4}\eta(Y)\}g(X, PZ)
+ \frac{c}{4}\{\tilde{g}(JX, PZ)v(Y) - \tilde{g}(JY, PZ)v(X) - 2\tilde{g}(X, JY)v(PZ)\},
\]

for any \( X, Y, Z \in \Gamma(TM) \). Replacing \( X \) by \( \xi \) in this equation, we have

\[
\gamma B(Y, PZ) + \epsilon D(Y, PZ)\rho(\xi)
= \{\xi[\gamma] - \gamma \tau(\xi) - \frac{c}{4}\}g(Y, PZ) - \frac{c}{4}\{u(PZ)v(Y) + 2u(Y)v(PZ)\},
\]

for all \( Y, Z \in \Gamma(TM) \). Taking \( Y = U, PZ = V \); \( Y = V, PZ = U \) and \( Y = PZ = U \) in (4.4) by turns and using (2.7) and (4.2), we have

\[
\xi[\gamma] - \gamma \tau(\xi) - \frac{3c}{4} = 0, \quad \xi[\gamma] - \frac{c}{2} = 0, \quad \gamma^2 = 0,
\]

respectively. This shows that \( \gamma = 0 \) and \( c = 0 \). Thus we have our theorem. \( \square \)

**Corollary 3.** We have the following assertions:

(1) There exist no real half lightlike submanifolds of \( \tilde{M}(c) \) with \( c \neq 0 \) such that \( S(TM) \) is totally umbilical in \( M \).
(2) There exist no real half lightlike submanifolds of \( \tilde{M}(c) \) such that \( S(TM) \) is proper totally umbilical.

(3) The second fundamental form tensor \( h \) is degenerate on \( M \).

(4) The vector field \( U \) is conjugate to any vector field on \( M \) with respect to \( h \). In particular, \( U \) is an asymptotic vector field.

Proof. From the two equations of (4.2) with \( \gamma = 0 \), we obtain

\[
(4.5) \quad h(X, U) = 0, \quad \forall X \in \Gamma(TM).
\]

Therefore \( h \) is degenerate on \( M \) and we get (3). By (4.5), we have (4).

**Theorem 4.3.** Let \( (M, g, S(TM)) \) be a real half lightlike submanifold of an indefinite complex space form \( \tilde{M}(c) \) such that \( S(TM) \) is totally umbilical in \( M \). Then the curvatures \( R \) and \( R^* \) are related by

\[
(4.6) \quad R(X, Y)Z = R^*(PX, PY)PZ, \quad \forall X, Y, Z \in \Gamma(TM).
\]

Proof. From (1.9) with \( \phi = 0 \) and (4.3) with \( \gamma = c = 0 \), we have

\[
(4.7) \quad D(Y, Z)\rho(X) = D(X, Z)\rho(Y), \quad \forall X, Y, Z \in \Gamma(TM).
\]

From this and (1.19), we obtain \( \bar{g}(R(X, Y)Z, N) = 0 \). Thus we see that the equation (4.6) of this theorem is equivalent with the following equation:

\[
(4.8) \quad g(R(X, Y)Z, PW) = g(R^*(PX, PY)PZ, PW),
\]

for all \( X, Y, Z, W \in \Gamma(TM) \). Due to (1.17) with \( \gamma = c = 0 \), we show that \( g(R(X, Y)\xi, Z) = 0 \). Thus we see that (4.8) is true for \( Z = \xi \). Using (1.9), (1.17) and (1.21) satisfy \( \gamma = c = 0 \), we derive (4.8). \( \square \)

The induced Ricci type tensor \( R^{(0, 2)} \) of \( M \) is defined by

\[
(4.9) \quad R^{(0, 2)}(X, Y) = \text{trace}\{Z \rightarrow R(Z, X)Y\},
\]

for any \( X, Y \in \Gamma(TM) \). Consider the induced quasi-orthonormal frame field \( \{\xi; W_a\} \) on \( M \) such that \( \text{Rad}(TM) = \text{Span}\{\xi\} \) and \( S(TM) = \text{Span}\{W_a\} \). Using this quasi-orthonormal frame field and the equation (4.9), we obtain

\[
(4.10) \quad R^{(0, 2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N),
\]

for any \( X, Y \in \Gamma(TM) \) and \( \epsilon_a = g(W_a, W_a) \) is the sign of \( W_\beta \). In general, the induced Ricci type tensor \( R^{(0, 2)} \), defined by the method of the geometry of the non-degenerate submanifolds [9], is not symmetric [2, 3, 5]. A tensor field \( R^{(0, 2)} \) of half
lightlike submanifolds $M$ is called its induced Ricci tensor of $M$ if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

**Theorem 4.4.** Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in $M$. Then the Ricci type tensor $R^{(0,2)}$ is a symmetric Ricci tensor. Moreover, if $M$ is an Einstein manifold, then $M$ is Ricci flat.

**Proof.** By Theorem 2.3, since $M$ is an irrotational real half lightlike submanifold of $\bar{M}(c)$, then, using (1.17) and (1.19), the equation (4.10) reduces to

\[(4.11) \quad R^{(0,2)}(X, Y) = D(X, Y)tr A_L - \epsilon g(A_LX, A_LY),\]

where $tr A_L$ is the trace of $A_L$. Thus $R^{(0,2)}$ is a symmetric Ricci tensor $\text{Ric}$. Let $M$ be an Einstein manifold, that is, $R^{(0,2)} = \kappa g$ for a constant $\kappa$. Replacing $Y$ by $U$ in (4.11) and using the fact that $D(X, U) = g(A_LU, X) = 0$ for any $X \in \Gamma(TM)$, we obtain $\kappa g(X, U) = 0$ for all $X \in \Gamma(TM)$. Replacing $X$ by $V$ in this equation, we have $\kappa = 0$. Thus $M$ is Ricci flat. \qed

**Definition 5.** A vector field $X$ on $\bar{M}$ is said to be conformal Killing [5] if there exists a smooth function $\alpha$ such that $\mathcal{L}_X \bar{g} = -2\alpha \bar{g}$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to $X$. In particular, if $\alpha = 0$, then $X$ is called a Killing. A distribution $\mathcal{G}$ on $\bar{M}$ is said to be conformal Killing (or Killing) if each vector field belonging to $\mathcal{G}$ is a conformal Killing (or Killing).

**Theorem 4.5.** Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in $M$. If $S(TM^{\perp})$ is a conformal Killing distribution, then we have $D = 0$.

**Proof.** Using the equations (1.6) and (1.15), we have

\[
(\tilde{\mathcal{L}}_X \bar{g})(X, Y) = \bar{g}(\tilde{\nabla}_X L, Y) + \bar{g}(X, \tilde{\nabla}_Y L), \quad \forall X, Y \in \Gamma(TM),
\]

\[
\bar{g}(\tilde{\nabla}_X L, Y) = -g(A_LX, Y) + \phi(X)\eta(Y) = -\epsilon D(X, Y).
\]

Thus $(\tilde{\mathcal{L}}_X \bar{g})(X, Y) = -2\epsilon D(X, Y)$ for any $X, Y \in \Gamma(TM)$. We show that if $S(TM^{\perp})$ is a conformal Killing distribution, then there exists a smooth function $\delta$ such that $D(X, Y) = \epsilon \delta g(X, Y)$ for all $X, Y \in \Gamma(TM)$. Using this and the second equation of (4.2) with $\gamma = 0$, we have $0 = D(X, U) = \epsilon \delta g(X, U)$ for any $X \in \Gamma(TM)$. Replace $X$ by $V$ in this equation, we obtain $\delta = 0$. \qed
Theorem 4.6. Let \((M, g, S(TM))\) be a real half lightlike submanifold of an indefinite complex space form \(M(c)\) such that \(S(TM)\) is totally umbilical in \(M\). If \(S(TM^\perp)\) is a conformal Killing, then \(M\) and each leaf \(M^*\) of \(S(TM)\) are spaces of curvature 0. Moreover, \(M\) is locally a product manifold \(L_\xi \times M^*\), where \(L_\xi\) is a null curve and \(M^*\) is a semi-Euclidean space.

Proof. Using (1.17), (1.19), (3.3) and (4.7) with \(\gamma = c = 0\), we have

\[ R(X, Y)Z = R^*(PX, PY)PZ = D(Y, Z)A_LX - D(X, Z)A_LY, \]

for any \(X, Y, Z \in \Gamma(TM)\). Thus, if \(S(TM^\perp)\) is a conformal Killing, then we have \(R(X, Y)Z = R^*(PX, PY)PZ = 0\) due to \(D = 0\). Thus \(M\) and \(M^*\) are semi-Euclidean spaces of constant curvature 0. By Note 2, we have our assertion. \(\square\)

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Department of Mathematics, Dongguk University, Gyeongju, Gyeongbuk 780-714, Korea
Email address: jindh@dongguk.ac.kr