ON THE TOPOLOGY OF DIFFEOMORPHISMS OF SYMPLECTIC 4-MANIFOLDS

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Abstract. For a closed symplectic 4-manifold $X$, let $\text{Diff}_0(X)$ be the group of diffeomorphisms of $X$ smoothly isotopic to the identity, and let $\text{Symp}(X)$ be the subgroup of $\text{Diff}_0(X)$ consisting of symplectic automorphisms. In this paper we show that for any finitely given collection of positive integers $\{n_1, n_2, \ldots, n_k\}$ and any non-negative integer $m$, there exists a closed symplectic (or Kähler) 4-manifold $X$ with $b_2^+(X) > m$ such that the homologies $H_i$ of the quotient space $\text{Diff}_0(X)/\text{Symp}(X)$ over the rational coefficients are non-trivial for all odd degrees $i = 2n_1 - 1, \ldots, 2n_k - 1$.

The basic idea of this paper is to use the local invariants for symplectic 4-manifolds with contact boundary, which are extended from the invariants of Kronheimer for closed symplectic 4-manifolds, as well as the symplectic compactifications of Stein surfaces of Lisca and Matić.

1. Introduction

The purpose of this paper is to investigate the existence of closed symplectic smooth 4-manifolds having non-trivial homotopy groups of certain diffeomorphisms of symplectic 4-manifolds, which was first initiated by P. B. Kronheimer in [9]. To be precise, let $(X, \omega_0)$ be a closed symplectic 4-manifold. Let $\text{Diff}_0(X)$ be the group of diffeomorphisms of $X$ smoothly isotopic to the identity, and let $\text{Symp}(X)$ be the subgroup of $\text{Diff}_0(X)$ consisting of symplectic automorphisms. Let $\Lambda_0$ be the space of 2-forms on $X$ that are symplectic and cohomologous to $\omega_0$. Now consider the map

$$\Psi : \text{Diff}_0(X) \to \Lambda_0, \quad f \mapsto (f^{-1})^* \omega_0.$$ 

Then $\Psi$ induces an injection from the quotient space $\text{Diff}_0(X)/\text{Symp}(X)$ to the space $\Lambda_0$. Since $\text{Diff}_0(X)$ is connected, it follows from a well-known theorem of Moser in [13] that the image of $\Psi$ is the connected component of $\Lambda_0$ containing $\omega_0$.

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Using families of the Seiberg-Witten solutions, Kronheimer in [9] defined an interesting invariant which is an obstruction to some higher homotopy groups of the quotient space \( \text{Diff}_0(X)/\text{Symp}(X) \) or equivalently \( \Delta_0 \). One of the key ingredients in [9] is a result of C. Taubes about the constraints on symplectic forms [23, 24]. By the work of Taubes, Kronheimer showed the following interesting theorem.

**Theorem 1.1.** For each positive integer \( n \), there exists a closed symplectic 4-manifold \( (Y_n, \omega_n) \) with \( b_2^+ (Y_n) > 2n + 1 \) such that both homotopy group \( \pi_{2n-1} \) and homology group \( H_{2n-1} \) of the quotient space \( \text{Diff}_0(Y_n)/\text{Symp}(Y_n) \) over the rational coefficients are non-trivial.

His explicit examples are algebraic surfaces of general type. Kronheimer’s result in [9] was motivated by a result of P. Seidel [19]. By using symplectic Floer homology, Seidel showed that there are diffeomorphisms in \( \text{Diff}_0(X) \) on an open symplectic 4-manifold which cannot be symplectically isotopic to the identity.

In view of Kronheimer’s results on the homotopy groups of diffeomorphisms in symplectic 4-manifolds, the following general question seems to be worthy of further investigation:

**Question 1.2.** Is there a closed symplectic (or Kähler) 4-manifolds \( (X, \omega_0) \) having the non-trivial homologies

\[
H_i(\text{Diff}_0(X)/\text{Symp}(X); \mathbb{Q})
\]

for all positive integers \( i \)?

Another interesting motivation for this question can also be found in a series of Ruberman’s applications of the Seiberg-Witten theory to an obstruction to smooth isotopy in dimension 4 and the topology of the space of all metrics with positive scalar curvature. In particular, Ruberman, among other things, showed in [16] that \( \pi_0 \) of the diffeomorphism group of certain 4-manifolds is infinitely generated (see [15], [16], and [17] for more applications). It is also worth mentioning some related works of McMullen and Taubes, Smith, and Vidussi. They independently showed that the moduli spaces of symplectic forms modulo diffeomorphisms on certain simply connected 4-manifolds are disconnected (see [12], [20], and [25]).

Let \( M \) be a smooth closed 3-manifold. A contact structure on \( M \) is a distribution \( \xi \) of tangent 2-planes locally defined by a 1-form \( \theta \) such that \( \theta \wedge d\theta \) is nowhere vanishing. Clearly \( \theta \wedge d\theta \) defines an orientation on \( Y \) and this orientation does not depend of the choice of the sign of \( \theta \). When \( M \) has a fixed orientation, we say that \( \xi \) is positive (negative, respectively) if the orientation on \( M \) induced by \( \xi \) coincides with (is opposite to, respectively) the given orientation. From now on, we assume that every contact structure of this paper is positive. A 4-manifold with contact boundary is a pair \( (X, \xi) \) consisting of a connected oriented smooth 4-manifold with boundary and a positive contact
structure $\xi$ on the boundary $\partial X$. A symplectic structure $\omega$ on the oriented 4-manifold $X$ with a contact structure $\xi$ on $\partial X$ is compatible if the symplectic 2-form $\omega$ satisfies $\omega|_\xi > 0$ at every point of the boundary (see [1] for more details on contact structures).

In [10], Kronheimer and Mrowka introduced monopole invariants for smooth 4-manifolds with contact boundary, and extended the results of Taubes in [22] and [23] to this non-compact settings. Moreover, using the monopole invariants for 4-manifolds with contact boundary in [10], Kronheimer also obtained local versions of the results in Theorem 1.1. More precisely, let $B^4$ be the unit ball in $\mathbb{C}^2$, $X'$ be the quotient $B^4/\mathbb{Z}_{n+1}$, and let $\xi$ be the contact structure on the lens space at the boundary obtained from the embedding in $\mathbb{C}^2/\mathbb{Z}_{n+1}$. Let $X_{n+1}$ be the resolution obtained from $X'$ by resolving the singular point with a sphere $C$ of self-intersection $-(n+1)$. Then the boundary of $X_{n+1}$ is a lens space $L(n+1,1)$. It is well-known that any lens space $L(p,q)$ is obtained by $-\frac{p}{q}$-surgery on the unknot, with $-\frac{p}{q} < -1$ except for $S^3$ and $S^1 \times S^2$.

By Proposition 5.3 in [7], $X_{n+1}$ admits a Stein structure with $L(n+1,1)$ as its oriented boundary. Thus if we denote by $J^*$ the dual of an almost complex structure $J$, there exists a smooth strictly pluri-subharmonic function $\phi : X_{n+1} \rightarrow \mathbb{R}$ such that the 2-form $\omega_\phi = dJ^*d\phi$ is non-degenerate and closed (so $X_{n+1}$ admits a Kähler form $\omega_\phi$) such that its boundary admits a contact structure $\xi$ obtained by a 1-form $-J^*d\phi$. In his paper [9], Kronheimer stated the following theorem whose proof was omitted.

**Theorem 1.3.** Let $\Lambda$ be the space of symplectic 2-forms that are cohomologous to $\omega_\phi$. Then there is a family $\omega_u$ parameterized by $u \in S^{2n-1}$ which represents a non-trivial class in homology of the space $\Lambda$. In particular, the family cannot be extended to a family parameterized by the ball. Indeed, if $\omega_\nu (\nu \in B^{2n})$ is any family of symplectic forms compatible with a contact structure $\xi$ and extending the given family on $S^{2n-1}$, then there exists at least one $\nu \in B^{2n}$ for which the pairing of $\omega_\nu$ with the sphere $C$ is positive.

For the sake of completeness, we provide a proof of this theorem in Sections 2 and 3, relatively in detail.

On the other hand, using the symplectic compactifications of Stein surfaces and the Seiberg-Witten theory, Lisca and Matić showed, among other things, in [11] that given any positive integer $n$, there exists homology 3-spheres with at least $n$ homotopic, but non-isomorphic tight contact structures. Their proof of the result seems to give many implications to answer Question 1.2. Indeed, combining Kronheimer’s local results with the plumbing construction using disk bundles over a sphere, in this paper we give a positive partial answer to Question 1.2 as follows.

**Theorem 1.4.** For any given collection of positive integers $\{n_1, n_2, \ldots, n_k\}$ and any non-negative integer $m$, there exists a closed symplectic 4-manifold $X$ (or closed Kähler minimal surface of general type) with $b_+(X) > m$ having the
non-trivial homologies

\[ H_i(\text{Diff}_0(X)/\text{Symp}(X); \mathbb{Q}) \]

for all \( i = 2n_1 - 1, \ldots, 2n_k - 1 \).

As a simple corollary, we have:

**Corollary 1.5.** For any positive odd integer \( k \), there exists a closed symplectic 4-manifold \( X \) (or closed Kähler minimal surface of general type) having the non-trivial homologies

\[ H_i(\text{Diff}_0(X)/\text{Symp}(X); \mathbb{Q}) \]

for all odd degrees \( i \) between 0 and \( k \) inclusive.

We organize this paper as follows. In Section 2, we set up and review detail constructions for the local results by Kronheimer which was stated without proof. In Section 3, we prove Theorem 1.3. In Section 4, we construct examples of closed symplectic 4-manifolds to give a partial answer to Question 1.2. It seems that we are able to extend the result of Theorem 1.4 to even degrees using the global invariants of even degrees which can be constructed analogously. We hope we return this issue elsewhere.

### 2. Local invariants

In this section, we set up and review the facts necessary to detect the local versions of Kronheimer’s results stated in Theorem 1.3, in detail.

#### 2.1. 4-manifolds with contact boundary

Let \((X, \xi)\) be a compact oriented smooth 4-manifold with a contact structure \( \xi \) on the boundary \( \partial X \) which is compatible with the boundary orientation, and let \( X^+ \) be the smooth manifold obtained from \( X \) by attaching the open cylinder \([1, \infty) \times \partial X\). According to [10], since \( \partial X \) is a contact 3-manifold, we can give a symplectic structure \( \omega_0 \) and its compatible Riemannian metric \( g_0 \) to \([1, \infty) \times \partial X\) for which \( \omega_0 \) has length \( \sqrt{2} \) and is self-dual. These two in turn give \( X^+ \) a metric and a symplectic structure outside a compact set. Now extend the metric \( g_0 \) to all of \( X^+ \), also called \( g_0 \). In [10], the triple \((X^+, \omega_0, g_0)\) is called an AFAK (asymptotically flat almost Kähler) manifold (see Subsection 2(iii) in [10] for more detailed constructions). By Lemma 2.1 in [10], \( \omega_0 \) provides a canonical \( spin^c \) structure \( s_0 = (W^+, W^-, \rho) \), a spinor \( \Phi_0 \) of unit length, and a unique spin connection \( A_0 \) on \( X^+ \setminus X \) satisfying \( D^{A_0} \Phi = 0 \). As in [10], we write \( Spin^c(X, \xi) \) for the set of isomorphisms of \( spin^c \) structures \( s \) on \( X^+ \), equipped with an isomorphism \( s \to s_0 \) on \( X^+ \setminus X \). For the sake of convenience, we use the same notations \( A_0 \) and \( \Phi_0 \) for arbitrary extensions of them to all of \( X^+ \).

From now on, assume that we have provided suitable function spaces on \( X^+ \) to define a moduli space of pairs which solves the monopole equations and which are asymptotic to \((A_0, \Phi_0)\) on the ends of \( X^+ \).
Let \( \eta \in L^2_{l-1}(\mathfrak{isu}(W^+)) \) for \( l > 4 \). The Seiberg-Witten equations, perturbed by \( \eta \), are the following pair of equations for a spin connection \( A \) and a section \( \Phi \) of \( W^+ \):

\[
\rho(F_A^+) - \{ \Phi \otimes \Phi^* \} = \rho(F_{A_0}^+) - \{ \Phi_0 \otimes \Phi_0^* \} + \eta,
\]

\[
D_A^+ \Phi = 0,
\]

where \( \hat{A} \) means the induced connection on \( \text{det}(W^+) \) and \( \{ \Phi \otimes \Phi^* \} \) denotes the traceless part of the endomorphism \( \Phi \otimes \Phi^* \).

Let \( R(X^+) \) denote the space of all Riemannian metrics \( g \) on \( X^+ \) of class \( C^l \), and let \( N(X) = e^{-\varepsilon_0 \epsilon^r} C^r(\mathfrak{isu}(W^+)) \) with norm \( ||\eta||_{N(X)} = ||e^{\varepsilon_0 \epsilon^r} \eta||_{C^r} \) for some fixed \( r \geq l \), where \( \varepsilon_0 > 0 \) and \( \epsilon^r \) is an extension of the function \( \epsilon \) on \([1, \infty) \times \partial X\) to all of \( X^+ \). As in the case of closed 4-manifolds, we let \( P \) be the subset of \( R(X^+) \times N(X^+) \) consisting of pairs \((g, \eta)\) such that \( \eta \) is self-dual with respect to \( g \). Let \( L^2_l \) and \( L^2_{l,A_0} \) (\( l > 4 \)) be the Sobolev spaces of imaginary 1-forms and sections of \( W^+ \). We define

\[
\mathcal{C} = \{(A, \Phi) \mid (A - A_0) \in L^2_l \text{ and } (\Phi - \Phi_0) \in L^2_{l,A_0}\}
\]

and

\[
\mathcal{G} = \{u : X^+ \to \mathbb{C} \mid |u| = 1 \text{ and } 1 - u \in L^2_{l+1}\}.
\]

Then \( \mathcal{G} \) is a Hilbert Lie group acting freely on \( \mathcal{C} \). Thus, unlike the Seiberg-Witten equations on closed 4-manifolds, there is no Banach sub-manifold such as \( \mathcal{P}_{\text{red}} \) for which the corresponding Seiberg-Witten equations have a reducible solution. This is the reason why no restriction on \( b_2^+ \) is necessary to define monopole invariants for 4-manifolds with contact boundary.

We now have a family of the Seiberg-Witten equations (2.1) parameterized by \( \mathcal{P} \), and write \( \mathcal{M} \) for the parameterized space of solutions modulo the gauge group \( \mathcal{G} \) as follows.

\[
\mathcal{M} = \{([A, \Phi], (g, \eta)) \mid (2.1) \text{ hold}\}.
\]

The transversality and compactness results of Theorem 2.4 in [10] say that \( \mathcal{M} \) is a Banach manifold of \( \mathcal{C}/\mathcal{G} \times \mathcal{P} \) and that the projection

\[
\pi : \mathcal{M} \to \mathcal{P}
\]

is a smooth and proper Fredholm map of index

\[
\text{ind } \pi = \langle e(W^+; \Phi_0), [X, \partial X] \rangle
\]

and has orientable index bundle. An orientation of the index bundle can be specified by a choice of homology orientation of \((X, \xi)\) described in [10]. From now on, we assume that such an orientation is chosen.

In order to define local invariants derived from the Seiberg-Witten equations, we suppose that the index of \( \pi \) is negative, and thus we write \( \text{ind } \pi = -d \) (\( d > 0 \)). Let us denote by \( \Delta \) the image of \( \pi \).
The proof is very similar to it in [9]. For the sake of convenience, we briefly explain it. Since \( \mathcal{P} \) is contractible, any closed chain \( S \) in \( \mathcal{P}\setminus \Delta \) of dimension \( d-1 \) is the boundary of a singular chain \( T^d \) in \( \mathcal{P} \) of dimension \( d \). Now arrange that \( T^d \) is transverse to \( \pi \), and we define \( Q_{d-1}(S) \) to be an integer \( \langle [\mathcal{M}]_{d-1}, \pi^{-1}(T^d) \rangle \) obtained by counting the points of \( \mathcal{M} \) over \( T^d \). It is straightforward to show that this definition is independent of the choice of \( T^d \). This completes the proof. \( \square \)

2.2. Special cases

In this subsection, we apply the above constructions to special cases of symplectic 4-manifolds with contact boundary \((X, \xi)\) which are compatible with \( \xi \).

Let \((X, \xi)\) be a symplectic 4-manifold with contact boundary, and let \( \omega \) be a symplectic form on \( X \), compatible with \( \xi \). As before, let \( X^+ \) be the 4-manifold obtained by attaching the cone \([1, \infty) \times \partial X\) to \( X \), together with a metric \( g_0 \) and symplectic form \( \omega_0 \) defined outside a compact set of \( X^+ \). Then the manifold \( X^+ \) has a symplectic form \( \omega \) on the compact submanifold \( X \), and a symplectic form \( \omega_0 \) on the complement of \( X \) in \( X^+ \). Note that the compatibility condition between \( \omega \) and \( \xi \) does not guarantee that \( \omega \) and \( \omega_0 \) match on \( \partial X \). However, by Lemma 4.1 in [10] using the argument of patching two symplectic forms, there exists a symplectic form, also denoted by \( \omega \), on all of \( X^+ \) which is an extension of the symplectic form \( \omega \) on \( X \setminus U \), where \( U \) is a collar neighborhood of \( \partial X \) of \( X \), and is asymptotic to \( \omega_0 \) on the end of \( X^+ \).

Let \((A_0, \Phi_0)\) be the canonical spin connection and spinor for the \( spin^c \) structure \( s_0 \) defined on all of \( X^+ \). Let \( E \to X^+ \) be a line bundle with a trivialization outside a compact set and the first Chern class \( c \in H^2_\mathbb{Z}(X^+, \mathbb{Z}) \) so that the spinor bundles \( W^+ \) and \( W^+_0 \) for \( s \) and \( s_0 \), respectively, are related by \( W^+ = W^+_0 \otimes E \).

In order to apply the construction in the previous subsection, we assume that the index \(-d\) of the \( spin^c \) structure \( s \) is negative (so \( d > 0 \)). Since \( d \) is non-zero, \( s \) and \( s_0 \) are distinct and so \( e \) is non-zero.

We will need the following dimension formula later.

Lemma 2.2. The formula \( d(s) \) is given by

\[
d(s) = -\langle c^2 + c_1(W^+_0; \Phi_0) \cup c_1[X, \partial X] \rangle.
\]

Proof. By the relationship \( ch(W^+; \Phi_0) = ch(W^+_0; \Phi_0)ch(E) \), it is easy to see

\[
c_1(W^+; \Phi_0) = 2c_1(E) + c_1(W^+_0; \Phi_0),
\]

\[
\frac{1}{2}c_1(W^+; \Phi_0)^2 - c_2(W^+; \Phi_0) = c_1(E)^2 + c_1(W^+_0; \Phi_0) \cup c_1(E) + \frac{1}{2}c_1(W^+_0; \Phi_0)^2 - c_2(W^+_0; \Phi_0).
\]
Thus we get
\[
c_2(W^+; \Phi_0) = -c_1(E)^2 - c_1(W_0^+; \Phi_0) \cup c_1(E) - \frac{1}{2} c_1(W_0^+; \Phi_0)^2 + c_2(W_0^+; \Phi_0)
\]
\[+ \frac{1}{2}(2c_1(E) + c_1(W_0^+; \Phi_0))^2
\]
\[= c_1(E)^2 + c_1(W_0^+; \Phi_0) \cup c_1(E) + c_2(W_0^+; \Phi_0).
\]
Since \( d(s) = -\langle c_2(W^+; \Phi_0), [X, \partial X] \rangle \) and \( \langle c_2(W_0^+; \Phi_0), [X, \partial X] \rangle = 0 \), we have the formula, as required. \( \Box \)

2.3. Local invariants

In this subsection, we finish the constructions of local invariants which are obstructions to the non-triviality of the homologies of the space \( \Lambda_0 \) or equivalently \( \text{Diff}_0(X)/\text{Symp}(X) \).

Let \( \Lambda = \Lambda(e, s_0) \) be the space of 2-forms \( \omega \) in \( \Omega^2(X^+) \) satisfying the following three conditions: (1) \( \omega \) is symplectic, (2) \( |\omega| \cup e, [X, \partial X] | \leq 0 \), and (3) \( s_\omega \equiv s_0 \).

Note that the space \( \Lambda \) is smaller than \( \Lambda_0 \) defined in Section 1. The purpose of this subsection is to construct homomorphisms \( \tau_* \) from the homology \( H_*(\Lambda; \mathbb{Z}) \) to the homology \( H_*(\mathcal{P}\setminus \Delta; \mathbb{Z}) \). To do so, we again use the principle established in [22] and [10] that the basic classes of a 4-manifold which were defined using the Seiberg-Witten equations constrain the cohomology class of a symplectic form. By composing \( \tau_* \) with \( Q \), we then can have homomorphisms, also called \( Q_{d-1} \), from the homology \( H_{d-1}(\Lambda; \mathbb{Z}) \) to \( \mathbb{Z} \), which is the purpose of this section.

More precisely, for each compact subset \( K \subset \Lambda \), we give a homotopy class of maps
\[
\tau_K : K \to \mathcal{P}\setminus \Delta
\]
such that if \( K \subset K' \subset \Delta \) are compact subsets, then \( \tau_{K'}|_K \) is homotopic to \( \tau_K \). Thus we will have an element \( \tau \in \text{lim}_{K\downarrow K'} \mathcal{P}\setminus \Delta \). To define \( \tau_K \), we use Theorem 4.2 in [10]. As in the proof of Theorem 4.2, any element \( (A, \Psi) \) of \( \mathcal{C}(X^+, s) \) can be written in terms of a triple \( (a, \alpha, \beta) \), where \( a \) is a connection in \( E \), \( \alpha \in \Omega^{0,0}(E) \), and \( \beta \in \Omega^{0,2}(E) \). As in [10], we also consider the following perturbed Seiberg-Witten equations by introducing a parameter \( r \geq 1 \) with \( \eta = 0 \) as follows.

\[
\partial_a \alpha + \partial_a^* \beta = 0,
\]
\[
2i F_a^\omega - \frac{r}{4} (1 - |\alpha|^2 + |\beta|^2) = 0,
\]
\[
2 F_a^{0,2} - \frac{r}{2} \alpha \beta = 0,
\]
where \( F_a^\omega = \frac{1}{2} (F_a, \omega) \). Following [22], Kronheimer and Mrowka proved the following lemma:

**Lemma 2.3.** Under the hypothesis \( e \neq 0 \) and \( \langle [\omega] \cup e, [X, \partial X] \rangle \leq 0 \), there exists a constant \( r_0 = r_0(\omega, g_\omega, e) \) such that for all \( r \geq r_0 \) the equations \( \text{(2.2)} \) have no solutions for the metric \( g_\omega \) on \( X^+ \) with perturbing term \( \eta = 0 \).
As a consequence, we can say that \((g_\omega, 0) \in \mathcal{P}\setminus \Delta\), when \(r \geq r_0\). As in closed symplectic 4-manifolds, this \(r_0\) depends on the geometry of \(X^+\) and its almost complex structure. Thus this implies that we can choose a single sufficiently large \(r_0\) so that the lemma above holds for all \(\omega \in K\). This in turn gives a map \(\tau_K\) from the compact set \(K\) to \(\mathcal{P}\setminus \Delta\), as required. Clearly the homotopy class of the map does not depend on the choice of \(g_\omega\), and so on. Now the family of the maps on compact subsets induces a well-defined map on homologies

\[\tau_* : H_i(\Lambda; \mathbb{Z}) \to H_i(\mathcal{P}\setminus \Delta; \mathbb{Z}).\]

3. Local results: Proof of Kronheimer’s Theorem 1.3

The purpose of this section is to give a proof of Theorem 1.3 by Kronheimer, for the sake of reader’s convenience. As before, let \(X_{n+1}\) be the resolution obtained from \(X'\) by resolving the singular point with a sphere \(C\) of self-intersection \(- (n + 1)\). In addition, as in [9] we assume the following holds:

- We are given an analytic family \(X\) of Kähler manifolds \(X_u\) with the same contact boundary as \(X'\).
- \(X_u\) are the fibers of a map \(p : X \to U\), where \(U\) is an open ball about \(0 \in U \subset \mathbb{C}^n\).
- All fibers \(X_u\) are smooth, except for \(X_0 = X'\) which has a single quotient singularity at \(x_0\).
- All \(X_u\) \((u \neq 0)\) are embedded in \(\mathbb{C}^N\), with the Kähler form inherited from the Fubini-Study metric on \(\mathbb{C}^N\).
- We have a smooth family of manifolds \(\tilde{X}\) and a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\sigma} & X \\
\downarrow \tilde{p} & & \downarrow p \\
U & \xrightarrow{\pi} & U
\end{array}
\]

such that \(\tilde{X}_0 = X_{n+1}\) is a minimal resolution of \(X_0\).
- The family \(\tilde{X}\) has a \(C^\infty\) trivialization \(\tilde{X} \to \tilde{X}_0 \times U\).

Note that using the trivialization we can regard the Fubini-Study forms on \(X_u\) as giving a family of exact symplectic forms \(\omega_u\) on the fixed manifold \(X_{n+1} = \tilde{X}_0\). Clearly all the forms \(\omega_u\) in the family as exact symplectic forms are cohomologous. Furthermore, let \(e\) denote the Poincaré dual of the homology class represented by the exceptional 2-sphere \(C\) in \(\tilde{X}_0\). All the pairings \(\langle [\omega_u] \cup e, [X, \partial X] \rangle\) are zero. Therefore, we have a \((2n - 1)\)-sphere \(S = S^{2n-1}\) in the space \(\Lambda\). By Lemma 2.2, \(C^2 = -(n + 1)\), and the adjunction equality, the index \(-d\) for the Seiberg-Witten equations with the \(\text{spin}^c\) structure \(s = s_0 + e\) is given by

\[d = -(e^2 + c_1(W^+_0; \Phi_0) \cup e, [X, \partial X]) = 2n.\]

Thus it makes sense to evaluate the homomorphism \(Q_{2n-1}\) on \(S\).
To complete the proof of Theorem 1.3, we need to show that $S^{2n-1}$ is an essential homology class in $H_{2n-1}(\Lambda; \mathbb{Z})$. To do so, it suffices to show that $Q_{2n-1}(S^{2n-1}) = \pm 1$ as follows. Let $\mu_u$ be any smooth family of Kähler forms on the fibers $\tilde{X}_u$ of $\tilde{p}$. Let $\sigma^* \omega_u$ be the pull-back of the forms $\omega_u$ under $\sigma$. Since the form $\sigma^* \omega_0$ is degenerate along the complex exceptional curve $C$, we define a new family of Kähler forms

$$\tilde{\omega}_u = \sigma^* \omega_u + \psi(u) \mu_u,$$

where $\psi: U \to \mathbb{R}$ is a non-negative $C^\infty$ bump function supported near 0 and equal to zero on the small sphere $S^{2n-1}$. Since $\tilde{\omega}_u$ coincides with $\sigma^* \omega_u = \omega_u$ for $u$ in the small sphere $S^{2n-1}$, we can regard the family $\tilde{\omega}_u$, parameterized by a ball $B^{2n}$ which bounds $S^{2n-1}$, as an extension of the family $\omega_u$ of symplectic forms on the fixed manifold $\tilde{X}_0$. Let us denote $\tilde{g}_u$ for the Kähler metric with respect to the Kähler form $\tilde{\omega}_u$. Then we have a map

$$T^{2n}: B^{2n} \to \mathcal{P}, \quad u \mapsto (\tilde{g}_u, 0).$$

Now we need the following proposition which is analogous to Proposition 4.1 in [9].

**Proposition 3.1.** When $r$ is sufficiently large, the solutions of the perturbed Seiberg-Witten equations (2.2) on the Kähler manifold $X^+$ correspond to algebraic curves in $X^+$ homologous to $C$, whose fundamental class is Poincaré dual to $e$.

**Proof.** To show this, first act on the first equation of (2.2) with $\bar{\partial_a}$ and use the last equation of (2.2) to get

$$\bar{\partial_a} \partial^*_a \beta + \frac{r}{4}|\partial_a^\alpha|^2 \beta = 0.$$ 

Thus we get $|\partial^*_a \beta|^2 = -\frac{r}{4}|\partial_a^\alpha|^2 |\beta|^2$, and so either $\alpha = 0$ or $\beta = 0$. On the other hand, it follows from the proof of Theorem 4.2 in [10] that for sufficiently large $r$ we have the following inequality

$$\int_{X^+} \frac{r}{2} F_{a^*} \geq \int_{X^+} \left( \frac{1}{4} |\nabla_a \alpha|^2 + \frac{1}{2} |\tilde{\nabla}_a \beta|^2 + \frac{r^2}{32} (1 - |\alpha|^2 - |\beta|^2)^2 + \frac{r^2}{16} |\beta|^2 \right).$$

As in [10], a gauge transformation can be chosen so that the left hand side of the inequality is the pairing $r\pi(\omega \cup e, [X, \partial X])$. Thus if $\alpha = 0$, then we get

$$r\pi(\omega \cup e, [X, \partial X]) \geq \int_{X^+} \left( \frac{1}{2} |\tilde{\nabla}_a \beta|^2 + \frac{r^2}{32} (1 + |\beta|^4) \right) \geq \int_{X^+} \frac{r^2}{32},$$

which implies that $\pi(\omega \cup e, [X, \partial X])$ is infinite. This is a contradiction. Therefore $\beta = 0$, and the zero set of $\alpha$ is a curve $C$ whose fundamental class is
Poincaré dual to $e$. Conversely, it is a well-known procedure that from the algebraic curve $C$ we can obtain the holomorphic bundle $(E, \bar{\partial})$ and the section $\alpha$ up to isomorphism (see e.g. [3]). This completes the proof.

Now we return to the proof of Theorem 1.3. Note that there is only one such curve in our manifold $\tilde{X}_0$ by construction and no such curve for $u \neq 0$. Thus for sufficiently large $r$ the image of $T^{2n}$ meets $\Delta = \pi(M)$ only at $T^{2n}(0)$, and there is only one solution in $M$ over $T^{2n}(0)$.

As a final step, we need to show that the map $\pi$ is transverse to $T^{2n}$, as in Proposition 3.1 in [9]. However, in Section 4 of [9], Kronheimer provided a detailed argument showing that for the examples in his paper the map $\pi$ is transverse to $T^{2n}$ at $T^{2n}(0)$ by comparing the deformation theory of the solutions $(a, \alpha, \beta)$ of the Seiberg-Witten equations with the deformation theory of the curve $C$ given by the zero set of $\alpha$ (see [14] for related discussions). Certainly his argument goes through in our case, too. Thus this completes the proof of Theorem 1.3.

4. Global results: Proof of Theorem 1.4

The main purpose of this section is to give a partial answer to Question 1.2 positively.

In order to get global results on closed symplectic (or Kähler) manifolds from the local result of Theorem 1.3, we need to use the symplectic compactifications of Stein surfaces of P. Lisca and G. Matić in [11].

**Theorem 4.1** (Theorem 3.2 or Corollary 3.3 in [11]). Let $X$ be a Stein surface, and let $\phi: X \to \mathbb{R}$ be a smooth strictly pluri-subharmonic function such that the boundary of $X$ is the regular level set of $\phi$. Then there exist a holomorphic embedding $j$ of $X$ as a domain inside a closed Kähler minimal surface $S$ of general surface such that the pull-back of the Kähler form of $S$ to $X$ equals $\omega = \omega_\phi = dJ^* d\phi$.

Moreover, we can refine the statement in Theorem 4.1 as follows (see [21] and [2] for a similar argument).

**Lemma 4.2.** Let $X$ be a Stein surface. For any non-negative integer $m$, there exists a holomorphic embedding $j$ of $X$ as a domain inside a closed Kähler minimal surface $S$ of general type with $b_2^+(S \setminus j(X)) > m$ (so $b_2^+(S) > m$).

**Proof.** Suppose that $b_2^+(S \setminus j(X)) = m$ in Theorem 4.1. Then we first extend $X$ into $X'$ by attaching a 2-handle with framing $tb(K) - 1$ along a Legendrian knot satisfying $tb(K) > 1$ contained in a standard 3-ball $D^3$ in $\partial X$ (see [7] for the definitions of Legendrian knots and $tb(K)$). By the choice of such a knot with the framing, the well-known Eliashberg's theorem in [5] and [6] and its Gompf's refinement in [7] imply that $X'$ is also a Stein surface. Now we embed $X'$ into a closed Kähler minimal surface $S'$ of a general type. Since we have attached a 2-handle with positive framing, we have a second homology class with positive
self-intersection number in $S' \setminus X$. Thus clearly we have $b_2^+(S' \setminus j(X)) > m$, so that $b_2^+(S') > m$. This completes the proof. □

We also need the following symplectic gluing result of symplectic forms, provided that the two corresponding contact structures on the glued region are isomorphic.

**Lemma 4.3** (Lemma 4.1 in [11]). Let $X_1$ and $X_2$ be two Stein surfaces with boundary $\partial X_i$ ($i = 1, 2$), and suppose that $\partial X_i$ are diffeomorphic to the connected 3-manifold $M$. Let $\phi : X_1 \to \mathbb{R}$ be a smooth strictly pluri-subharmonic function having $\partial X_1$ as a level set, and let $X_1$ have the symplectic structure $\omega_1 = d J^* d \phi$. Suppose that the contact structures $\xi_1$ and $\xi_2$ induced on $M$ are isomorphic. Then there exist a $J$-compatible symplectic form $\omega_2$ on the interior of $X_2$ and a symplectic embedding of the interior of a collar $U_1 \subset X_1$ around $\partial X_1$ as a subcollar of the interior of a collar $U_2 \subset X_2$ around $\partial X_2$.

**4.1. Simple cases**

We first show the following simple theorem which is a slight generalization of Theorem 1.1 of Kronheimer, in the sense that we do not need any restriction on $b_2^+(X)$.

**Theorem 4.4.** For any positive integer $n$, there exists a closed symplectic (Kähler) 4-manifold $X$ having the non-trivial homology $H_{2n−1}(\text{Diff}_0(X)/\text{Symp}(X); \mathbb{Q})$.

**Proof.** Once again, let $B^4$ be the unit ball in $\mathbb{C}^2$, $X'$ be the quotient $B^4/\mathbb{Z}_{n+1}$, and let $X_{n+1}$ be the resolution obtained from $X'$ by resolving the singular point with a sphere $C$ of self-intersection $−(n+1)$. Then the boundary is a lens space $L(n+1, 1)$ and admits the contact structure $\xi$ obtained from the embedding in $\mathbb{C}^2/\mathbb{Z}_{n+1}$. Moreover, $X_{n+1}$ becomes a Stein manifold with contact boundary. By Theorem 1.3, we also have a family $\omega_u$ parameterized by $u \in S^{2n−1}$ which represents a non-trivial class in homology of the space consisting of symplectic 2-forms that are cohomologous to $\omega_\phi$. According to Theorem 4.1, we can consider the Kähler embedding $j : X_{n+1} \to S$, where $S$ is a minimal surface of general type.

Our aim now is to extend the family $\omega_u$ of symplectic forms on $X_{n+1}$ to a family of symplectic forms on all of $S$ which represents a non-trivial class in homology of the space $\Lambda$. To do so, we first choose a symplectic form $\omega_u$ from the family $\omega_u$ for $u \in S^{2n−1}$. Then, by Lemma 4.3, we can extend $\omega_u$ to a symplectic form $\tilde{\omega}_u$ defined on all of $S$. Now it remains to extend the rest of the family $\omega_u$ defined on $X_{n+1}$ to all of $S$. Since $\omega_u$ and $\omega_{\tilde{u}}$ are isotopic, they are connected by a path of cohomologous symplectic forms between $\omega_u$ and $\omega_{\tilde{u}}$. Let $\Omega^u$ be such a smooth path over $X_{n+1}$ of cohomologous symplectic forms such that $\Omega^u_0 = \omega_{\tilde{u}}$ and $\Omega^u_1 = \omega_u$. Assuming the collar $U$ of $\partial X_{n+1}$ is diffeomorphic to $\partial X_{n+1} \times [0, 1)$, we can define a new symplectic form $\tilde{\omega}_u$ as
follows:
\[
\tilde{\omega}_u = \begin{cases} 
\omega_{u}, & \text{on } X_{n+1} \setminus U \\
\Omega^u_t(y, t), & (y, t) \in U \cong \partial X_{n+1} \times [0, 1) \\
\tilde{\omega}_{u_0}, & \text{on } S \setminus X_{n+1}.
\end{cases}
\]

Then all new symplectic forms \(\tilde{\omega}_u\) are continuous and further cohomologous, since two forms \(\tilde{\omega}_{u_0}\) and \(\tilde{\omega}_u\) are connected by a path of symplectic forms given by
\[
\tilde{\Omega}_u = \begin{cases} 
\Omega^u_{1-s}, & \text{on } X_{n+1} \setminus U \\
\Omega^u_{(1-s)t}, & \text{on } U \\
\tilde{\omega}_{u_0}, & \text{on } S \setminus X_{n+1}.
\end{cases}
\]

We next claim that the new family \(\tilde{\omega}_u\) for \(u \in S^{2n-1}\) is an essential element in the space of \(\Lambda\). Indeed, suppose that there exists a singular chain \(T^{2n}\) in the space \(\Lambda\) of dimension \(2n\) which bounds the family \(\tilde{\omega}_u\). Then the restriction of \(T^{2n}\) to the submanifold \(X_{n+1}\) of \(S\) induces a singular chain of dimension \(2n\), also denoted \(T^{2n}\), in the space consisting of symplectic 2-forms that are cohomologous to \(\omega_\phi\), which bounds the family \(\omega_u\) for \(u \in S^{2n-1}\). But this is a contradiction. This completes the proof. \(\square\)

### 4.2. General cases

The purpose of this subsection is to prove Theorem 1.4 which is a partial answer to Question 1.2.

For any rational number \(-p/q \in \mathbb{Q}\), we have a continued fraction expansion of the form
\[
-p/q = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \cdots - \frac{1}{a_k}}},
\]
where \(a_j \in \mathbb{Z}\) for \(0 \leq j \leq k\). We will abbreviate (4.1) by writing
\[
-p/q = [a_0, a_1, \ldots, a_k].
\]

Note that lens spaces are a special case of Seifert fibered spaces. Our orientation convention will be that a lens space \(L(p, q)\) is obtained by \(-\frac{p}{q}\)-Dehn surgery on an unknot in \(S^3\) (see [8], [18], or [4]). Using the continued fraction expansion \(-p/q = [a_0, a_1, \ldots, a_k]\), we get \(L(p, q)\) as the boundary of the 4-manifold obtained by plumbing together \(k + 1\) disk bundles over \(S^2\) with the Euler number \(a_j\), according to the following linear chain \(\Gamma\):

\[
\Gamma: \bullet \rightarrow a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_{k-2} \rightarrow a_{k-1} \rightarrow a_k \rightarrow \bullet.
\]
where each dot represents a sphere $S_j$ ($j = 0, 1, \ldots, k + 1$) which has self-intersection number $a_j$ and intersects only $S_{j-1}$ and $S_{j+1}$.

**Theorem 4.5.** For any given collection of positive integers $\{n_1, n_2, \ldots, n_k\}$ and any non-negative integer $m$, there exists a closed symplectic 4-manifold $X$ (or closed Kähler minimal surface of general type) with $b_2^+ (X) > m$ having the non-trivial homologies

$$H_i(\text{Diff}_0(X)/\text{Symp}(X); \mathbb{Q})$$

for all $i = 2n_1 - 1, \ldots, 2n_k - 1$.

**Proof.** To construct such a closed symplectic 4-manifold, we use the 4-manifold obtained by plumbing together the disk bundles of cotangent bundles over a sphere. As noted earlier, Seifert fibered 3-manifolds are the boundaries of the plumbed 4-manifolds $P(\Gamma)$ associated to the plumbing graph $\Gamma$, and a lens space $L(p, q)$ is a special case of Seifert fibered 3-manifolds, associated to the linear graph $\Gamma$ with weights $-(n_1 + 1), -(n_2 + 1), \ldots, -(n_k + 1)$. In addition, given a 4-manifold $P(\Gamma)$ obtained by plumbing on the linear graph $\Gamma$, let $\Gamma'$ be a sub-chain in $\Gamma$ as shown below:

$$\Gamma': -\,(n_i + 1) \quad -\,(n_{i+1} + 1) \quad \ldots \quad -\,(n_{i+l} + 1)$$

Then the plumbing on $\Gamma'$ gives rise to a sub-manifold $P(\Gamma')$ of $P(\Gamma)$ with the lens space $L(p', q')$ as a boundary, where

$$\frac{-p'}{q'} = [-(n_{i+1} + 1), -(n_{i+2} + 1), \ldots, -(n_{i+l} + 1)]$$

Here $P(\Gamma')$ is taken so that $P(\Gamma')$ is strictly contained in the interior of $P(\Gamma)$ (e.g., see [18] for the construction). Thus the closure of $P(\Gamma) \setminus P(\Gamma')$ is a smooth compact 4-manifold with oriented boundary $-L(p', q') \cup L(p, q)$ in our case. In particular, if we take $\Gamma'$ to be $-\,(n_{i+1} + 1)$, then $P(\Gamma)$ contains the sub-manifold $P(\Gamma')$ with the lens space $L(n_{i+1} + 1)$ as its boundary, which will be used in the proof, later.

Moreover, it is a theorem of Y. Eliashberg or a construction of Gompf in [7] that by attaching 2-handles along Legendrian knots $K$ with framing $tb(K) - 1$ we can assume that the plumbed 4-manifold is a Stein manifold with the lens space as a contact boundary. So, we assume that the plumbed 4-manifold $P(\Gamma)$ associated to the linear graph $\Gamma$ is a Stein manifold with contact boundary. In order to compactify the Stein manifold $P(\Gamma)$, we use Theorem 4.1 of Lisca and Matić and Lemma 4.2 concerning the symplectic compactifications of Stein surfaces to get such a compactification $X$ with $b_2^+ (X) > m$. 

Let $\phi : P(\Gamma) \to \mathbb{R}$ be a smooth strictly pluri-subharmonic function such that the sub-level set of $\phi$ for a regular value is $\partial P(\Gamma)$, and such that the pull-back of the Kähler form of $X$ to $P(\Gamma)$ equals $\omega_\phi = dJ^*d\phi$. Using Lemma 4.2, we next extend the exact Kähler form $\omega_\phi$ to all of $X$. We may also assume that for $i = 1, \ldots, k$, $X$ contains $X_{n_i+1}$ as a Stein sub-manifold whose boundary is a lens space $L(n_i + 1, 1)$. Thus over each Stein sub-manifold $X_{n_i+1}$ we have an essential element cohomologous to $\omega_\phi$, parameterized by a $(2n_i - 1)$-sphere, in the homology $H_{2n_i-1}(\text{Diff}_0(X_{n_i+1})/\text{Symp}(X_{n_i+1}), \mathbb{Q})$.

Finally, we apply again Lemma 4.2 and the argument in the proof of Theorem 4.1 to the essential element in order to obtain a family of symplectic forms defined on all of $X$, parameterized by a $(2n_i - 1)$-sphere. Then an easy argument proving Theorem 4.5 implies that the extended family of symplectic forms induces an essential element in the homology $H_{2n_i-1}(\text{Diff}_0(X)/\text{Symp}(X); \mathbb{Q})$. Since we can apply the same argument to any $X_{n_i+1}$, we have completed the proof for the odd degrees, as stated.

□

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