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Cumulative Impulse Response Functions for a Class of Threshold-Asymmetric GARCH Processes

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Abstract

A class of threshold-asymmetric GRACH(TGARCH, hereafter) models has been useful for explaining asymmetric volatilities in the field of financial time series. The cumulative impulse response function of a conditionally heteroscedastic time series often measures a degree of unstability in volatilities. In this article, a general form of the cumulative impulse response function of the TGARCH model is discussed. In particular, We present formula in their closed forms for the first two lower order models, *viz.*, TGARCH(1, 1) and TGARCH(2, 2).

Keywords: Cumulative impulse response function, persistent, asymmetric-TGARCH.

1. Introduction

The volatility (defined as a conditional variance) of financial time series data exhibits characteristics such as volatility clustering, time varying, heavy-tailed distribution and leverage effect. In order to accommodate these characteristics in time series modeling, autoregressive conditional heteroscedastic (ARCH) model was introduced by Engle (1982) and Bollerslev (1986) extended the ARCH class by introducing generalized-ARCH(GARCH). Both ARCH and GARCH have served as useful models for analyzing symmetric models for which volatility is a linear function of the *squared* past values. Since volatilities in GARCH do not discriminate whether the past values are positive or negative, as long as they are of the same magnitude, the GARCH class fails to capture asymmetric volatilities. Therefore, there has been growing interest in asymmetric GARCH modelling in a response to the empirical evidences of asymmetric volatilities arising mainly from the financial time series. Refer to, Li and Li (1996), Rabemananjara and Zakoian (1993), Hwang and Basawa (2004), Pan *et al.* (2008) and Park *et al.* (2009) with references therein.

A general class of threshold GARCH(TGARCH) is formulated by

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{1.1}$$
$$h_t^\delta - \sum_{j=1}^{q} \beta_j h_{t-j}^\delta = \omega + \sum_{i=1}^{p} \left[\alpha_{i1} \left(\varepsilon_{t-i}^+\right)^{2\delta} + \alpha_{i2} \left(\varepsilon_{t-i}^-\right)^{2\delta} \right],$$

where $\delta > 0$ denotes a power transformation and the notation

$$\varepsilon^+ = \max(\varepsilon, 0) \quad \text{and} \quad \varepsilon^- = \min(-\varepsilon, 0)$$

will be used. Here the innovation $\{e_t\}$ stands for a sequence of *iid* random variables with mean zero and variance unity. The time series $\{\varepsilon_t\}$ given by (1.1) is referred to as power-transformed TGARCH(p, q)

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model. Refer to Pan *et al.* (2008). With the case of the first order model with $\delta = 1$, TGARCH(1, 1) is of the form

$$h_t = \omega + \alpha_{11}\left(\varepsilon_{t-1}^{+}\right)^2 + \alpha_{12}\left(\varepsilon_{t-1}^{-}\right)^2 + \beta_1 h_{t-1}. \tag{1.2}$$

When the distribution of $\{e_t\}$ is symmetric, it is well known for (1.2) that $\{\varepsilon_t\}$ is second order stationary provided

$$\phi_1 = \frac{1}{2}(\alpha_{11} + \alpha_{12}) + \beta_1 < 1. \tag{1.3}$$

Refer to Hwang and Basawa (2004) and Liu (2006).

Most of the research on TGARCH models has been directed to second order stationary case where l-step ahead volatility converges almost surely to a constant of the stationary moment $E(h_t)$ as the lead time l goes to infinity. As regards unstable cases, Park *et al.* (2009) recently proposed a "integrated" TGARCH process (I-TGARCH, for short) for which $\phi_1 = 1$. They showed that I-TGARCH(1, 1) exhibits "persistent" properties in the sense that the current volatility continues to remain in the future volatilities for *all-step* ahead volatilities. In addition, see Hwang *et al.* (2010) for the explosive case of $\phi_1 > 1$. In this article, we are concerned with the limiting cumulative impulse response function for TGARCH(p, q) processes in order to measure long-run effect of current shocks to the future volatilities. A general solution for the cumulative impulse response function is presented and specific examples on TGARCH(1, 1) and TGARCH(2, 2) are discussed for illustration.

This article is organized as follows. Section 2 reviews the impulse response function often measuring persistency in conditional variance. A general formula with specific examples of the cumulative impulse response functions for a class of TGARCH(p, q) is presented in Section 3.

2. Impulse Response Function as a Measure of Persistency in Volatility

The topic of long memory and persistence has recently attracted considerable attention in terms of the second moment of a process. Many evidences of long memory processes have appeared in studies of financial times series. Therefore, Baillie *et al.* (1996), in analogy with the fractionally difference process for the level of a time series, introduced a class of FIGARCH(Fractionally Integrated Generalized AutoRegressive Conditional Heteroscedasticity) models. This class provides a slow hyperbolic rate of decay for lagged squared innovations instead of the usual exponential rates as for the standard GARCH models. In order to measure the persistence of shocks to the conditional variance, Baillie *et al.* (1996) investigated "impulse response function" for the optimal forecast of the future conditional variance as a function of the current innovation and the cumulative impulse response weights. Baillie *et al.* (1996) measure the persistent of shocks to the conditional variance using impulse response functions for the first order FIGARCH. Also, Conrad and Karanasos (2006) extend the results in Baille *et al.* (1996). Along with Baillie *et al.* (1996), we define the impulse response function and cumulative impulse response function as follows.

Definition 1. *The impulse response function* γ_k *is defined by* $\gamma_0 = 1$ *and*

$$\gamma_k = \frac{\partial h_t(k)}{\partial \eta_t} - \frac{\partial h_t(k-1)}{\partial \eta_t}, \quad k \geq 1, \tag{2.1}$$

where $h_t(l)$ *is* l*-step ahead volatility and* $\{\eta_t\}$ *denotes innovation process for the squared process* $\{\varepsilon_t^2\}$*, i.e.,*

$$\eta_t = \varepsilon_t^2 - E\left(\varepsilon_t^2 \middle| \mathcal{F}_{t-1}\right) = \varepsilon_t^2 - h_t. \tag{2.2}$$

Note that $\{\eta_t\}$ forms a sequence of martingale differences.

Definition 2. *The cumulative impulse response function is defined by*

$$\lambda_l = \sum_{k=0}^{l} \gamma_k. \tag{2.3}$$

Note that the cumulative impulse response function λ_l measures a certain contribution of innovation η_t (at time t) to the l-step ahead volatility $h_t(l)$. The long-run effect of current shocks to the future volatilities can be assessed in terms of the limit of λ_l as $l \to \infty$, *i.e.*,

$$\lambda_\infty = \lim_{l\to\infty} \lambda_l. \tag{2.4}$$

As noted by Baillie *et al.* (1996), the cumulative impulse response function of standard second order stationary GARCH(1, 1) is given by $\lambda_l = (\phi_1 - \beta_1)\phi_1^{l-1}$ for $l > 1$ and hence, the effect of current shocks to the future conditional variance decreases exponentially to zero. However, in the IGARCH(1, 1) model, $\lambda_l = (1-\beta_1)$ for all lags ($l > 1$) which is obviously a nonzero constant, implying IGARCH(1, 1) is a persistent process.

3. The Impulse Response Function of TGARCH(p,q) Model

Consider the time series $\{\varepsilon_t\}$ following TGARCH(p, q) model defined by

$$\begin{aligned} \varepsilon_t &= \sqrt{h_t} \cdot e_t, \\ h_t &= \omega + \sum_{i=1}^{p} \left(\alpha_{i1}\varepsilon_{t-i}^{+2} + \alpha_{i2}\varepsilon_{t-i}^{-2}\right) + \sum_{j=1}^{q} \beta_j h_{t-j}, \end{aligned} \tag{3.1}$$

where $\omega > 0$, $\alpha_{i1}, \alpha_{i2} > 0 (i = 1, 2, \ldots, p)$, $\beta_j \geq 0 (j = 1, \ldots, q)$. Also $\{e_t\}$ is *iid* with mean zero and variance unit.

(C1) $\{e_t\}$ is symmetrically distributed with support $(-\infty, \infty)$.

Define the process $\{\phi_t\}$ given by

$$\phi_{i,t} = \alpha_{i,t} + \beta_i, \quad (i = 1, 2, \ldots, p)$$

where $\alpha_{i,t} = \alpha_{i1} + (\alpha_{i2} - \alpha_{i1})I_{[\varepsilon_t<0]}$ and $I_{[\cdot]}$ stands for standard indicator function. It is immediate from (C1) that

$$\phi_i = E(\phi_{i,t}) = \left(\frac{\alpha_{i1} + \alpha_{i2}}{2}\right) + \beta_i, \quad \text{for all } t. \tag{3.2}$$

Definition 3. (I-TGARCH(p,q)) *The TGARCH(p, q) defined in (3.1) is called I-TGARCH(p, q) when $\phi = 1$ for which $\phi = \sum_{i=1}^{p} \phi_i$.*

The squared process $\{\varepsilon_t^2\}$ from TGARCH(p, q) can be expressed in terms of a ARMA representation with random coefficients. See Park *et al.* (2009) for the TGARCH(1, 1) case.

Lemma 1. *For $\{\varepsilon_t\}$ following TGARCH(p, q) with $\phi \leq 1$ ($\phi = \sum_{i=1}^{p} \phi_i$), $\{\varepsilon_t^2\}$ can be represented as a form of ARMA(*$\max(p, q)$*, q) with random coefficient $\{\phi_{i,t}\}$. That is,*

$$\varepsilon_t^2 = \omega + \sum_{i=1}^{\max(p,q)} \phi_{i,t-i}\varepsilon_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j}.$$

Proof: Replace h_t by $\varepsilon_t^2 - \eta_t$ in the Equation (3.1) and obtain

$$\varepsilon_t^2 - \eta_t = \omega + \sum_{i=1}^{p}\left(\alpha_{i1}\varepsilon_{t-i}^{+2} + \alpha_{i2}\varepsilon_{t-i}^{-2}\right) + \sum_{j=1}^{q}\beta_j\left(\varepsilon_{t-j}^2 - \eta_{t-j}\right)$$

concluding the proof. □

Fix n and define l-step ahead volatility given by $h_n(l) = E(\varepsilon_{n+l}|\mathcal{F}_n)$, $l \geq 1$. Cumulative impulse response of TGARCH(p, q) will be addressed in the following theorem. Assume that the data consist of $\{\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_1\}$ and let $\mathcal{F}_n$ denote σ-field generated by $\varepsilon_n, \varepsilon_{n-1}, \ldots$.

Theorem 1. *Under (C1), fix the data size n and consider the random process $\{\phi_{i,t}\}$ with mean ϕ_i in (3.2). For a general TGARCH(p,q) process, we identify the cumulative impulse response λ_l as*

(i)
$$h_n(l) = \omega + \sum_{i=1}^{\max(p,q)} \phi_i h_n(l-i). \tag{3.3}$$

(ii)
$$\lambda_l = \sum_{i=1}^{\max(p,q)} \phi_i \lambda_{l-i}. \tag{3.4}$$

(iii) *The general solution of linear difference Equation (3.4) is given by*

$$\lambda_l = \sum_{i=1}^{m}\sum_{j=0}^{d_i-1} c_{ij} k^j \xi_i^l,$$

where $\Phi_p(L) = 1 - \sum_{i=1}^{p}\phi_i L^i = \prod_{i=1}^{m}(1-\xi_i L)^{d_i}$ and $\xi_i^{-1}(i = 1, 2, \ldots, m)$ are the (possibly complex) roots of $\Phi_p(L) = 0$ with multiplicity d_i. Here L denotes the backward operator.

Proof: Without loss of generality, assume $p \geq q$. Note that (i) is verified by considering, for $l > p$,

$$\begin{aligned} h_n(l) &= E\left(\varepsilon_{t+l}^2|\mathcal{F}_n\right) \\ &= E\left(\omega + \sum_{i=1}^{p}\phi_{i,n+l-i}\varepsilon_{n+l-i}^2 + \eta_t - \sum_{j=1}^{q}\beta_j\Big|\mathcal{F}_n\right) \\ &= \omega + \phi_1 h_n(l-1) + \cdots + \phi_p h_n(l-p). \end{aligned}$$

(ii) Following the line in Park *et al.* (2009) and Hwang *et al.* (2010), a combination of (2.1) and (2.3) gives

$$\lambda_l = \frac{\partial h_n(l)}{\partial \eta_n}, \quad \text{for } l > p.$$

Thus, (ii) can be easily obtained.
(iii) Equation (3.4) can be rewritten as $\Phi_p(L)\lambda_l = 0$, with $\Phi_p(L) = 1 - \sum_{i=1}^{p}\phi_i L^i$ in a form of linear difference equation. A general solution of the linear difference equation $\Phi_p(L)\lambda_l = 0$ can be obtained as in (iii) via employing Theorem 3.6.2 and Corollary 3.6.2 in Brockwell and Davis (1991). □

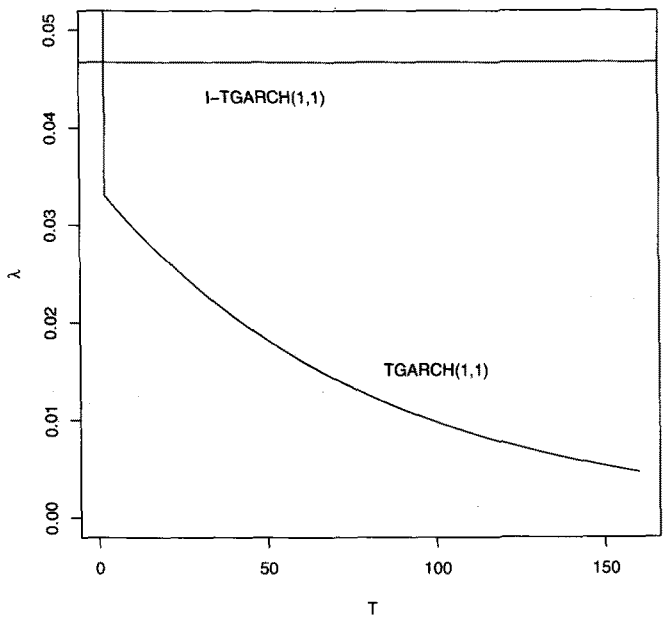

Figure 1: *Estimated cumulative impulse response functions for KOSPI data*

It is mentioned that the general solution (iii) in the theorem may not be easy to solve explicitly. To illustrate how to obtain a closed solution of (iii), it may suffice only to consider the first two lower order examples of TGARCH(1, 1) and TGARCH(2, 2).

Example 1. **TGARCH(1, 1) process**
Consider the following first order model specified by

$$h_t = \omega + \left(\alpha_{11}\varepsilon_{t-1}^{+2} + \alpha_{12}\varepsilon_{t-1}^{-2}\right) + \beta_1 h_{t-1}.$$

We may readily conclude via Theorem 1 that

(i) TGARCH(1, 1) with $\phi_1 < 1$; $\lambda_l = \phi_1^{l-1}(\phi_{1,n} - \beta_1)$, $l \geq 1$ and thus $\lambda_\infty = 0$.

(ii) I-TGARCH(1, 1); $\lambda_\infty = \alpha_{11} I_{[\varepsilon_n \geq 0]} + \alpha_{12} I_{[\varepsilon_n < 0]}$ which is random.

Proofs are omitted. We refer to Park *et al.* (2009) for details.

The impulse response functions can be used to distinguish between TGARCH and I-TGARCH specifications. Park *et al.* (2009) analyzed the conditional variance of the Korea stock prices index(KOSPI) returns for the period from Jan. 5, 2000 to Jun. 29, 2007 using both TGARCH(1, 1) and I-TGARCH(1, 1). Figure 1 shows the estimated cumulative impulse response functions λ_l both for TGARHC(1, 1) and I-TGARCH processes. It is noted in Figure 1 that λ_l decreases to zero along with lead time l for TGARCH(1, 1) model whereas λ_l is "persistent" for the case of I-TGARCH process.

Example 2. **TGARCH(2, 2) process**
The second order model TGARCH(2, 2) is defined by

$$h_t = \omega + \sum_{i=1}^{2}\left(\alpha_{i1}\varepsilon_{t-i}^{+2} + \alpha_{i2}\varepsilon_{t-i}^{-2}\right) + \sum_{j=1}^{2}\beta_j h_{t-j}.$$

It will be shown that

(i) TGARCH(2, 2) with $\phi < 1$; $\lambda_l = \phi_1\lambda_{l-1} + \phi_2\lambda_{l-2}$, $l \geq 3$ and thus $\lambda_\infty = 0$.

(ii) I-TGARCH(2, 2); $\lambda_l = \lambda_1 + (\lambda_2 - \lambda_1)[\{1 - (-\phi_2)^{l-1}\}/(1 + \phi_2)]$, $l \geq 3$ and therefore

$$\lambda_\infty = \frac{(\alpha_{11} + \alpha_{12})I_{[\varepsilon_n \geq 0]} + (\alpha_{21} + \alpha_{22})I_{[\varepsilon_n < 0]}}{1 + \phi_2}$$

which is a non-zero random quantity.

To verify (i) and (ii), using Lemma 1, notice first that $\{\varepsilon_t^2\}$ can be represented as a form of ARMA(2, 2) with random coefficients, *viz.*,

$$\varepsilon_{n+1}^2 = \omega + \phi_{1,n}\varepsilon_n^2 + \phi_{2,n-1}\varepsilon_{n-1}^2 + \eta_{n+1} - \beta_1\eta_n - \beta_2\eta_{n-1},$$

where $\phi_{1,n} = \alpha_{11} + (\alpha_{12} - \alpha_{11})I_{[\varepsilon_n<0]} + \beta_1$ and $\phi_{2,n} = \alpha_{21} + (\alpha_{22} - \alpha_{21})I_{[\varepsilon_n<0]} + \beta_2$.

Taking expectation on both sides conditionally on $\mathcal{F}_n$, we have

$$h_n(1) = E\left(\varepsilon_{n+1}^2 \middle| \mathcal{F}_n\right) = \omega + \phi_{1,n}\varepsilon_n^2 + \phi_{2,n-1}\varepsilon_{n-1}^2 - \beta_1\eta_n - \beta_2\eta_{n-1}, \tag{3.5}$$

which leads to via partial derivatives with respect to η_n

$$\lambda_1 = \frac{\partial h_n(1)}{\partial \eta_n} = \phi_{1,n} - \beta_1. \tag{3.6}$$

For λ_2, we have

$$h_n(2) = E\left(\varepsilon_{n+2}^2 \middle| \mathcal{F}_n\right) = \omega + \phi_1 h_n(1) + \phi_{2,n}\varepsilon_n^2 - \beta_2\eta_n \tag{3.7}$$

and it then follows from (3.7) that

$$\lambda_2 = \frac{\partial h_n(2)}{\partial \eta_n} = \phi_1\lambda_1 + (\phi_{2,n} - \beta_2). \tag{3.8}$$

Due to Theorem 1, it can be shown that

$$h_n(l) = \omega + \phi_1 h_n(l-1) + \phi_2 h_n(l-2), \quad \text{for } l \geq 3, \tag{3.9}$$

and it then follows, for $l \geq 3$

$$\lambda_l = \frac{\partial h_n(l)}{\partial \eta_n} = \phi_1\lambda_{l-1} + \phi_2\lambda_{l-2}. \tag{3.10}$$

(i) For $\phi = \phi_1 + \phi_2 < 1$, it follows from (3.10) that λ_∞ reduce to zero.

(ii) Note that for $\phi_1 + \phi_2 = 1$, λ_l in (3.10) can be expressed as

$$\lambda_l - \lambda_{l-1} = (-\phi_2)(\lambda_{l-1} - \lambda_{l-2}). \tag{3.11}$$

From (3.11), we can easily obtain

$$\lambda_l = \lambda_1 + (\lambda_2 - \lambda_1)\left(\frac{1 - (-\phi_2)^{l-1}}{1 + \phi_2}\right), \quad \text{for all } l \geq 3,$$

which gives via (3.6) and (3.8)

$$\lambda_\infty = \frac{\phi_2\lambda_1 + \lambda_2}{1 + \phi_2} = \frac{(\phi_{1,n} - \beta_1) + (\phi_{2,n} - \beta_2)}{1 + \phi_2}, \quad \text{concluding (ii).}$$

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