INVERSE POLYNOMIAL MODULES INDUCED BY AN R-LINEAR MAP

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Abstract. In this paper we show that the flat property of a left R-module does not imply (carry over) to the corresponding inverse polynomial module. Then we define an induced inverse polynomial module as an R[x]-module, i.e., given an R-linear map \( f : M \to N \) of left R-modules, we define \( N + x^{-1}M[x^{-1}] \) as a left \( R[x] \)-module. Given an exact sequence of left R-modules

\[
0 \to N \to E^0 \to E^1 \to 0,
\]

where \( E^0, E^1 \) injective, we show \( E^1 + x^{-1}E^0([x^{-1}]) \) is not an injective left \( R[x] \)-module, while \( E^0([x^{-1}]) \) is an injective left \( R[x] \)-module. Make a left R-module \( N \) as a left \( R[x] \)-module by \( xN = 0 \). We show

\[
\text{inj dim}_R N = n \quad \text{implies} \quad \text{inj dim}_{R[x]} N = n + 1
\]

by using the induced inverse polynomial modules and their properties.

1. Introduction

If \( R \) is a left Noetherian ring, then for an injective left \( R \)-module \( E, E[x^{-1}] \) is an injective left \( R[x] \)-module ([2], [3]). But for a projective left \( R \)-module \( P, P[x^{-1}] \) is not a projective left \( R[x] \)-module, in general ([5]). We extend this question to the flat module and we show that for a flat left \( R \)-module \( F, F[x^{-1}] \) is not a flat left \( R[x] \)-module, in general. Then we construct an induced inverse polynomial as an \( R[x] \)-module. Let \( M \) and \( N \) be left \( R \)-modules and \( f : M \to N \) be an \( R \)-linear map. Then we can define \( N + x^{-1}M[x^{-1}] \) as a left \( R[x] \)-module defined by

\[
x(b_0 + a_1 x^{-1} + \cdots + a_n x^{-n}) = b_1 + a_2 x^{-1} + \cdots + a_n x^{-n+1},
\]

where \( f(a_1) = b_1, b_0 \in N, \) and \( a_i \in M \). Given an exact sequence of \( R \)-modules

\[
0 \to N \to E^0 \to E^1 \to 0,
\]

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where \( E^0, E^1 \) are injective, we show \( E^1 + x^{-1}E^0[x^{-1}] \) is not an injective left \( R[x] \)-module, while \( E^0[x^{-1}] \) is an injective left \( R[x] \)-module. Make a left \( R \)-module \( N \) as a left \( R[x] \)-module by \( xN = 0 \). We show
\[
\text{inj dim}_R N = n \implies \text{inj dim}_{R[x]} N = n + 1
\]
by using the inverse polynomial modules. Inverse polynomial modules were developed in ([1], [6], [7], [8]) recently.

**Definition 1.1 ([4]).** Let \( R \) be a ring and \( M \) be a left \( R \)-module. Then \( M[x^{-1}] \) is a left \( R[x] \)-module defined by
\[
x(m_0 + m_1x^{-1} + \cdots + m_ix^{-i}) = m_1 + m_2x^{-1} + \cdots + m_ix^{-i+1}
\]
and such that
\[
r(m_0 + m_1x^{-1} + \cdots + m_ix^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_ix^{-n},
\]
where \( r \in R \). We call \( M[x^{-1}] \) as an inverse polynomial module.

Similarly, we can define \( M[[x^{-1}]], M[x, x^{-1}], M[[x, x^{-1}]], M[x, x^{-1}] \) and \( M[[x, x^{-1}]] \) as left \( R[x] \)-modules where, for example, \( M[[x, x^{-1}]] \) is the set of Laurent series in \( x \) with coefficients in \( M \), i.e., the set of all formal sums \( \sum_{k \geq n_0} m_kx^k \) with \( n_0 \) any element of \( \mathbb{Z} \) (\( \mathbb{Z} \) is the set of all integers).

**Lemma 1.2 ([8]).** Let \( E \) be a left \( R \)-module. Then \( E[[x^{-1}]] \) is an injective left \( R[x] \)-module.

**Lemma 1.3.** If \( E[[x^{-1}]] \) is an injective left \( R[x] \)-module, then \( E \) is an injective left \( R \)-module.

**Proof.** Let \( I \) be a left ideal of \( R \) and \( f : I \to E \) be an \( R \)-linear map. Then since \( E[[x^{-1}]] \) is an injective left \( R[x] \)-module, we can complete the following diagram by \( g \)

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & I[[x^{-1}]] \\
\downarrow & & \downarrow f[[x^{-1}]] \\
E[[x^{-1}]] & \xrightarrow{g} & R[[x^{-1}]]
\end{array}
\]

as a commutative diagram, where \( f[[x^{-1}])(\sum_{i=0}^{\infty} r_ix^{-i}) = \sum_{i=0}^{\infty} f(r_i)x^{-i} \). Since \( xR = 0, xg(R) = 0 \) in \( E[[x^{-1}]] \). But this implies \( g(R) \subset E \). Thus \( E | g|_R : R \to E \) can complete the following diagram.
as a commutative diagram. Hence, $E$ is an injective left $R$-module. \hfill \Box

**Lemma 1.4.** Let $M$ be a left $R$-module. Then

\[ \text{inj} \dim_{R[x]} M[[x^{-1}]] = \text{inj} \dim_{R} M. \]

**Proof.** Suppose $\text{inj} \dim_{R} M = n$ and

\[ 0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0 \]

is an injective resolution of $M$. Then by Lemma 1.2, for each $i$, $E^i[[x^{-1}]]$ is an injective left $R[x]$-module and

\[ 0 \to M[[x^{-1}]] \to E^0[[x^{-1}]] \to E^1[[x^{-1}]] \to \cdots \to E^n[[x^{-1}]] \to 0 \]

is an injective resolution of $M[[x^{-1}]]$. Let $K^i = \ker(E^i \to E^{i+1})$ for $0 \leq i < n$. Then $K^i$ is not an injective left $R$-module for $0 \leq i < n$. So by Lemma 1.3, $K^i[[x^{-1}]]$ is not an injective left $R[x]$-module. So then we get $\text{inj} \dim_{R[x]} M[[x^{-1}]] = n$. Suppose $\text{inj} \dim_{R} M = \infty$ and

\[ 0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to \cdots \]

is an injective resolution of $M$. Then

\[ 0 \to M[[x^{-1}]] \to E^0[[x^{-1}]] \to E^1[[x^{-1}]] \to \cdots \to E^n[[x^{-1}]] \to \cdots \]

is an injective resolution of $M[[x^{-1}]]$. But $K^i$ is not an injective left $R$-module for all $i$. Thus $K^i[[x^{-1}]]$ is not an injective left $R[x]$-module for all $i$. Therefore, $\text{inj} \dim_{R[x]} M[[x^{-1}]] = \infty$. Similarly, if $\text{inj} \dim_{R[x]} M[[x^{-1}]] = n$, then $\text{inj} \dim_{R} M = n$, and if $\text{inj} \dim_{R[x]} M[[x^{-1}]] = \infty$, then $\text{inj} \dim_{R} M = \infty$. Hence, $\text{inj} \dim_{R[x]} M[[x^{-1}]] = \text{inj} \dim_{R} M$. \hfill \Box

2. Flat module

**Lemma 2.1.** Let $M$ be a left $R$-module. Then $R[x] \otimes_{R[x]} M[x^{-1}] \cong M[x^{-1}]$.

**Proof.** Define $\phi : M[x^{-1}] \to R[x] \otimes M[x^{-1}]$ by $\phi(f) = 1 \otimes f$ and $\psi : R[x] \otimes M[x^{-1}] \to M[x^{-1}]$ by $\psi(x \otimes f) = xf$. Then $\phi$ and $\psi$ are $R[x]$-linear maps. And

\[ (\phi \circ \psi)(x \otimes f) = \phi(\psi(x \otimes f)) = \phi(xf) = 1 \otimes xf = x \otimes f, \]

\[ (\psi \circ \phi)(f) = \psi(\phi(f)) = \psi(1 \otimes f) = f. \]

Hence, $R[x] \otimes_{R[x]} M[x^{-1}] \cong M[x^{-1}]$. \hfill \Box
Similarly, we can get $R[x] \otimes_{R[x]} M[[x^{-1}]] \cong M[[x^{-1}]]$.

**Theorem 2.2.** If $F$ is a flat left $R$-module, then $F[x^{-1}]$ is not a flat left $R[x]$-module, in general.

**Proof.** Let $R = \mathbb{R}$ (the ring of real numbers). Let $\phi : R[x] \to R[x]$ be defined by $\phi(f) = xf$. Then $\phi$ is a left $R[x]$-linear map. Consider $\phi \otimes_{R[x]} \text{id}_{F[x^{-1}]} : R[x] \otimes_{R[x]} F[x^{-1}] \to R[x] \otimes_{R[x]} F[x^{-1}]$ defined by $(g \otimes ax)(bx^{-1}) = abx^{-1}$, where $a, b \in \mathbb{R}$. Since $R[x] \otimes_{\mathbb{R}} F[x^{-1}] \cong F[x^{-1}]$, we have the following commutative diagram. But $(h \circ f)(ax \otimes bx^{-1}) = (g \circ \phi)(ax \otimes bx^{-1})$ implies $h(ab) = 0$. Thus $h : F[x^{-1}] \to F[x^{-1}]$ is not injective, so that $\phi \otimes_{R[x]} \text{id}_{F[x^{-1}]}$ is not injective. Hence, $F[x^{-1}]$ is not a flat left $R[x]$-module. □

**Remark 1.** Since $R[x] \otimes_{\mathbb{R}} M[[x^{-1}]] \cong M[[x^{-1}]]$, we also see that $F[[x^{-1}]]$ is not a flat left $R[x]$-module.

3. Induced inverse polynomial modules

**Definition 3.1.** Let $f : M \to N$ be an $R$-linear map. Then $N + x^{-1}M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(b_0 + a_1x^{-1} + \cdots + a_nx^{-n}) = b_1 + a_2x^{-1} + \cdots + a_nx^{-n+1},$$

where $f(a_1) = b_1$, $b_0 \in N$, $a_i \in M$.

Similarly, we can define $N + x^{-1}M[[x^{-1}]]$ as a left $R[x]$-module.

**Note.** Given a left $R$-module $M$, we can make $M$ as a left $R[x]$-module by defining $xM = 0$.

**Lemma 3.2.** If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of $R$-modules, then

$$0 \to L \to M[x^{-1}] \to N + x^{-1}M[x^{-1}] \to 0$$

is a short exact sequence of $R[x]$-modules.
Proof. Let $f[x^{-1}] : L \to M[x^{-1}]$ be defined by $f[x^{-1}](n) = f(n)$ for $n \in L$. Then since $f$ is an injective $R$-linear map, $f[x^{-1}]$ is an injective $R[x]$-linear map. Let $g[x^{-1}] : M[x^{-1}] \to N + x^{-1}M[x^{-1}]$ be defined by

$$g[x^{-1}](e_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i}) = g(e_0) + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i}.$$ 

Then easily $g[x^{-1}]$ is an $R[x]$-linear map. Let $b_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i} \in N + x^{-1}M[x^{-1}]$. Then since $g$ is a surjective $R$-linear map, there exists $e_0 \in M$ such that $g(e_0) = b_0$. So, $g[x^{-1}]$ is a surjective $R[x]$-linear map. Now

$$(g[x^{-1}] \circ f[x^{-1}](n) = g[x^{-1}](f(n)) = g(f(n)) = 0.$$ 

And if $e_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i} \in \ker g[x^{-1}]$, where $e_i \in M$, then

$$g[x^{-1}](e_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i}) = g(e_0) + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i} = 0.$$ 

So $g(e_0) = 0$, $e_1 = e_2 = \cdots = e_i = 0$, which implies $e_0 \in \ker g = \text{Im} f = f(L)$. Hence,

$$0 \to L \to M[x^{-1}] \to N + x^{-1}M[x^{-1}] \to 0$$

is a short exact sequence of $R[x]$-modules. \hfill \Box

Similarly, given a short exact sequence $0 \to L \to M \to N \to 0$ of $R$-modules, we get a short exact sequence $0 \to L \to M[[x^{-1}]] \to N + x^{-1}M[[x^{-1}]] \to 0$ of $R[x]$-modules.

**Lemma 3.3.** Let $0 \to N \xrightarrow{f} E^0 \xrightarrow{g} E^1 \to 0$ be a short exact sequence of $R$-modules, where $E^0$, $E^1$ are injective with $\text{injdim}_R N = 1$. Then $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module.

**Proof.** Suppose $E^1 + x^{-1}E^0[[x^{-1}]]$ is an injective left $R[x]$-module. Then there exists a $R[x]$-linear map $\phi$ which completes the following diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & E^1 \\
\downarrow{id} & & \downarrow{\phi} \\
E^1 & \longrightarrow & E^1 + x^{-1}E^1
\end{array}
$$

\hfill \Box
Then there exists an $R$-linear map $h : E^1 \to E^0$ such that $g \circ h = \text{id}_{E^1}$. But since $\text{inj} \dim_R N = 1$, $0 \to N \xrightarrow{f} E^0 \xrightarrow{g} E^1 \to 0$ is not split, which implies a contradiction. Hence, $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module.

Similarly, given a short exact sequence $0 \to N \to E^0 \to E^1 \to 0$ of $R$-modules with $E^0$, $E^1$ injective and $\text{inj} \dim_R N = 1$, we see that $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module.

**Theorem 3.4.** Let $\text{inj} \dim_R N = n$ (with $N \neq 0$). Make $N$ into an left $R[x]$-module so that $xN = 0$. Then

$$\text{inj} \dim_{R[x]} N = n + 1.$$

**Proof.** Let $N$ be a left $R$-module. Then

$$\text{inj} \dim_R N = \text{inj} \dim_{R[x]} N[[x^{-1}]] = n.$$

And we have the short exact sequence of $R[x]$-modules

$$0 \to N \to N[[x^{-1}]] \to N[[x^{-1}]] \to 0.$$

Then $\text{inj} \dim_{R[x]} N \leq (\text{inj} \dim_R N) + 1 = n + 1$. Since if $N$ is an injective $R[x]$-module, then $N$ is an injective $R$-module so that

$$\text{inj} \dim_R N \leq \text{inj} \dim_{R[x]} N \leq (\text{inj} \dim_R N) + 1.$$

Now by induction on $n$, if $n = 0$, then we want to show $\text{inj} \dim_{R[x]} N = 1$. But $\text{inj} \dim_R N = 0$ means that $N$ is an injective $R$-module. If $N$ is an injective $R[x]$-module, then $N$ is divisible by $x$. But $xN = 0$. Thus $N$ is not divisible by $x$. Thus $N$ is an injective $R[x]$-module. Therefore, $\text{inj} \dim_{R[x]} N \neq 0$, i.e., $\text{inj} \dim_{R[x]} N = 1$.

If $n = 1$, then we have a short exact sequence $0 \to N \to E^0 \to E^1 \to 0$ of $R$-modules with $E^0$, $E^1$ injective. Then by Lemma 3.3, $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module and by Lemma 3.2, $0 \to N \to E^0[[x^{-1}]] \to E^1 + x^{-1}E^0[[x^{-1}]] \to 0$ is a short exact sequence. Therefore, $\text{inj} \dim_{R[x]} N = 2$.

Now we suppose $\text{inj} \dim_{R[x]} N = n > 1$ and make the obvious induction hypothesis. Let $0 \to N \to E \to L \to 0$ be an exact sequence of left $R$-modules with $E$ injective. Then $\text{inj} \dim_R L = n - 1$. Now make $N$, $E$, $L$ into $R[x]$-modules with $xN = 0$, $xE = 0$, $xL = 0$. Then $\text{inj} \dim_{R[x]} E = 1$ and by the induction hypothesis we know $\text{inj} \dim_{R[x]} L = n$. Using the long exact sequence of $\text{Ext}_{R[x]}(A, -)$ where $A$ is any left $R$-module, we get that $\text{inj} \dim_{R[x]} N = n + 1$. \hfill \Box

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