THE ALTERNATIVE DUNFORD-PETTIS PROPERTY IN
SUBSPACES OF OPERATOR IDEALS

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Abstract. For several Banach spaces $X$ and $Y$ and operator ideal $U$, if $U(X, Y)$ denotes the component of operator ideal $U$; according to Freedman’s definitions, it is shown that a necessary and sufficient condition for a closed subspace $M$ of $U(X, Y)$ to have the alternative Dunford-Pettis property is that all evaluation operators $\phi_x : M \rightarrow Y$ and $\psi_{y^*} : M \rightarrow X^*$ are DP1 operators, where $\phi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ for $x \in X$, $y^* \in Y^*$ and $T \in M$.

1. Introduction

A Banach space $X$ has the Dunford-Pettis property (DP) if for each weakly convergent sequences $x_n \rightarrow x$ in $X$ and $x_n^* \rightarrow 0$ in $X^*$, we have $x_n^*(x_n) \rightarrow 0$ as $n \rightarrow \infty$. But if under the additional condition $\|x_n\| = \|x\| = 1$ for all integer $n$, the conclusion $x_n^*(x_n) \rightarrow 0$ is obtained, we say that $X$ has the alternative Dunford-Pettis property (DP1).

As an easy consequence of definition, the Banach space $X$ has the DP1 if and only if for each weakly null sequences $(x_n)$ in $X$ and $(x_n^*)$ in $X^*$ and each $x \in X$ with $\|x_n + x\| = \|x\| = 1$, we have $x_n^*(x_n) \rightarrow 0$. An straightforward computation also shows that one can replace the condition $\|x_n\| = \|x\| = 1$, in the definition, by the weaker condition $\|x_n\| \rightarrow \|x\|$. For example, the standard sequence spaces $c_0$, $l_1$, $l_{\infty}$ and all $L^1(\mu)$ and $C(K)$ spaces, for each compact Hausdorff $K$, have the DP and so DP1 property [10].

The concept of DP1 which has introduced by Freedman in [12], is in general weaker than the DP; for example every (infinite dimensional) Hilbert space has DP1, but does not have the DP [10, 12]. Also there are Banach spaces such as von Neumann algebras, that the DP1 and the DP on them are coincide [12].

Another concept which has introduced by Freedman is the concept of DP1 operators, that is weaker than the concept of completely continuous operators. A bounded linear operator $T : X \rightarrow Y$ between Banach spaces $X$ and $Y$ is called...
completely continuous or Dunford-Pettis operator, if $T$ maps weakly convergent sequences to norm convergent sequences, and the operator $T$ is said to be a DP1 operator if $T$ carries weakly convergent sequences on the unit sphere of $X$ to norm convergent sequences. This means that for every weakly convergent sequence $x_n \to x$ in $X$ with $\|x_n\| = \|x\| = 1$, we have $\|Tx_n - Tx\| \to 0$. We refer the reader to [5], [6] and [15] for valuable results on DP1.

In [14], the authors have obtained some characterizations of arbitrary Banach space $X$ which contains no copy of $l_1$ and has the DP (or equivalently, the dual $X^*$ of $X$ has the Schur property, i.e., weak and norm convergence of sequences in $X^*$ are coincident), with respect to compactness of every weakly compact operator $T$ from $X$ into arbitrary Banach space $Y$. A similar result, among other things, will be found about Banach spaces with the DP or DP1 property in [10, Theorem 1] and [12, Theorem 1.4]. Specially, a Banach space $X$ has the DP property if and only if every weakly compact operator $T : X \to Y$ is completely continuous; while $X$ has the DP1 property if and only if every weakly compact operator $T : X \to Y$ is DP1. If $M$ is a closed subspace of some operator ideals, there is a well known refinement of it about the Dunford-Pettis property.

If $U$ is a (Banach) operator ideal, by meaning of [9] or [16], let $U(X, Y)$ be any its component and $M$ be a closed subspace of it. For each $x \in X$ and $y^* \in Y^*$, we denote the evaluation operators at $x$ and $y^*$ respectively, by $\phi_x : M \to Y$ and $\psi_{y^*} : M \to X^*$ where, $\phi_x(T) = Tx$, $\psi_{y^*}(T) = T^*y^*$ and $T \in M$. We also use the standard notations $K_w^*(X^*, Y)$ and $K(X, Y)$ for the Banach spaces of all compact weak$^*$-weak continuous operators and all compact operators between related Banach spaces. $K(X)$ is an abbreviation of $K(X, X)$; $\langle x, x^* \rangle$ denotes the duality between $x \in X$ and $x^* \in X^*$ and $T^*$ refers to the adjoint of the operator $T$.

In [18], A. Ülger proved that for any Hilbert space $H$, if $M \subseteq K(H)$ is a closed subspace, then $M$ has the DP (or equivalently $M^*$ has the Schur property) if and only if all evaluation operators $\phi_x : M \to H$ and $\psi_{y^*} : M \to H$ are compact operators if and only if all evaluation operators are completely continuous. The same conclusion has obtained by E. Saksman and H. O. Tylli in [17] for closed subspaces of $K(l_p)$ with $1 < p < \infty$.

In [14], the authors extend these conclusions to closed subspaces of several operator ideals. They proved that for a large class of Banach spaces $X$ and $Y$, the Schur property of the dual $M^*$ of closed subspace $M$ of arbitrary operator ideal $\mathcal{U}(X, Y)$, is a sufficient condition for compactness and so complete continuity of all evaluation operators $\phi_x$ and $\psi_{y^*}$.

On the opposite direction, they have shown that for several Banach spaces $X$ and $Y$ with Schauder decompositions, if $M$ is a closed subspace of $K(X, Y)$ or $K_w^*(X^*, Y)$, then the Schur property of $M^*$ is a necessary condition for compactness of all point evaluations.

Also, in [1], the authors study the DP1 property for closed subspaces of $K(X, Y)$, where $X$ and $Y$ admit Schauder basis and the basis of $X$ is shrinking;
and they proved some necessary and sufficient conditions for the DP1 property of suitable subspaces of $K(X,Y)$.

Here, we will show that similar consequences of [14], that extend some results of [1], remain valid for the DP1 property and a suitable class of closed subspaces of some operator ideals, where in this case the evaluation operators must be assumed DP1 operators.

2. Main results

Recall that, by Freedman’s Theorem [12], an arbitrary Banach space $X$ has the DP1 property if and only if every weakly compact operator $T$ from $X$ into arbitrary Banach space $Y$ is DP1. So, in order to prove a key result of this article, one can give a necessary and sufficient condition among Banach spaces containing no copy of $l_1$ to have the DP1.

**Theorem 2.1.** A Banach space $X$ containing no copy of $l_1$ has the DP1 property if and only if for every weakly sequentially complete (wsc) Banach space $Y$, every operator $T : X \to Y$ is DP1.

**Proof.** Suppose that $X$ has the DP1 and $T : X \to Y$ is an operator into wsc Banach space $Y$. If $(x_n)$ is an arbitrary sequence in the unit ball of $X$, then by Rosenthal’s $l_1$-Theorem [11], $(x_n)$ has a weakly Cauchy subsequence $(x_{nk})$. This shows that $(Tx_{nk})$ is weakly Cauchy and so is weakly convergent. Therefore the operator $T$ is weakly compact and the hypothesis of DP1 property of $X$ implies that $T$ is DP1.

On the other hand, since by Davis-Figiel-Johnson-Pelczynski’s Theorem [11], every weakly compact operator factors through a reflexive (and so wsc) Banach space, the opposite implication is also clear. □

**Theorem 2.2.** Suppose that $X^*$ and $Y$ are wsc and $M \subseteq U(X,Y)$ is a closed subspace containing no copy of $l_1$. If $M$ has the DP1, then all of the evaluation operators $\phi_x$ and $\psi_{y^*}$ are DP1 operators.

**Proof.** Since $X^*$ and $Y$ are wsc, by Theorem 2.1, the bounded linear operators $\phi_x$ and $\psi_{y^*}$ are DP1. □

Notice that if $X$ and $Y$ are two reflexive Banach spaces, a referring to Freedman’s Theorem imply the same conclusion, without any assumption on non containment of $l_1$ by $M$. This assertion also treated in [1].

If $X$ and $Y$ are Banach lattices, $X$ contains no complemented copy of $l_1$ and $Y$ contains no copy of $c_0$, then $X^*$ and $Y$ are wsc [13, V.II] and we can apply Theorem 2.2 for any closed subspace $M \subseteq U(X,Y)$. As another corollary, if instead of $X$ and $Y$, the closed subspace $M \subseteq U(X,Y)$ is a Banach lattice, we have the following corollary which can be proved by the same method applied in the proof of Corollary 2.4 of [14]:

**Corollary 2.3.** Suppose that $X$ contains no complemented copy of $l_1$ and $Y$ contains no copy of $c_0$. If $M \subseteq U(X,Y)$ is a Banach lattice containing no copy
of $l_1$ and satisfying the DP1, then all of the evaluation operators $\phi_x$ and $\psi_y^*$ are DP1 operators.

Here we use similar techniques to those in [1] and [14] to obtain some characterizations of the DP1 property for suitable closed subspaces of some compact operator ideals between Banach spaces that extend some results of [1]. We need some notations.

If $V$ is a complemented subspace of a Banach space $X$, the projection of $X$ onto $V$ is denoted by $P_V$ and $P_{V'} = I - P_V$ is the projection onto complementary subspace $V'$ of $V$. As mentioned in [14], if $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ are Schauder decompositions of $X$ and $Y$ respectively, and $M \subseteq U(X, Y)$ is a closed subspace, we say that $M$ has the $\mathcal{P}$-property if for all integers $m_0$ and $n_0$ and every operators $T, S \in M$,

$$
\|P_WTP_V + P_WSP_V\| \leq \max\{\|P_WTP_V\|, \|P_WSP_V\|\},
$$

where $V = X_1 \oplus \cdots \oplus X_{m_0}$ and $W = Y_1 \oplus \cdots \oplus Y_{n_0}$. Finally, if $(X_n)_{n=1}^\infty$ is a shrinking Schauder decomposition for $X$ [13], we denote the corresponding Schauder decomposition of $X^*$ by $(X_n^*)_{n=1}^\infty$.

**Theorem 2.4.** Let $X$ and $Y$ have monotone finite dimensional Schauder decompositions (abb. FDD) such that the decomposition of $X$ is shrinking. Let $M$ be a closed subspace of $K_{w^*}(X^*, Y)$ which has the $\mathcal{P}$-property. If all of the evaluation operators $\phi_x^*$ and $\psi_y^*$ are DP1 operators, then $M$ has the DP1 property.

**Proof.** Suppose that $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ are finite dimensional Schauder decompositions of $X$ and $Y$ respectively. Since the decompositions of $X^*$ and $Y$ are monotone $\|P_V\| = \|P_W\| = 1$, and $\|P_{W'}\| \leq 2$ for all $V = X_1^* \oplus \cdots \oplus X_{m_0}$ and $W = Y_1^* \oplus \cdots \oplus Y_{n_0}$.

Let $(K_n) \subseteq M$ and $(\Gamma_n) \subseteq M^*$ be weakly null sequences in $M$ and $M^*$ respectively and $K \in M$ such that $\|K\| = \|K_n + K\| = 1$ and there exists $r > 0$ such that for all integer $n$, $|(K_n, \Gamma_n)| \geq r$. Let $(\varepsilon_n)$ be a sequence of positive numbers such that $\sum \varepsilon_n < \infty$.

We shall construct by induction, subsequences $(\Lambda_n)$ of $(\Gamma_n)$ and $(S_n)$ of $(K_n)$ such that for all $n$, there exist (finite dimensional) subspaces $V$ and $W$ of $X^*$ and $Y$ respectively, satisfying the following properties:

$$
\|S_iP_V\| \leq \varepsilon_{n+1}\text{ and } \|P_{W^*}S_i\| \leq \varepsilon_{n+1} \text{ for all } i = 1, 2, \ldots, n,
$$

$$
|\langle S_i, \Lambda_{n+1}\rangle| < r2^{-(n+1)}, \text{ } i = 1, 2, \ldots, n,
$$

$$
|\langle S_{n+1}, \Lambda_{n+1}\rangle| > r \text{ and } |\langle S_{n+1}, \Lambda_i\rangle| \leq r2^{-(n+1)}, \text{ } i = 1, 2, \ldots, n,
$$

$$
\|S_{n+1}P_V\| \leq \varepsilon_{n+1}\text{ and } \|P_{W^*}S_{n+1}\| \leq \varepsilon_{n+1}.
$$

Suppose that $\Lambda_1 = \Gamma_1$, and $S_1 = K_1$ and inductively, suppose that $\Lambda_1, \ldots, \Lambda_n \in (\Gamma_n)$ and $S_1, \ldots, S_n \in (K_n)$ have been chosen. To obtain $\Lambda_{n+1}$ and $S_{n+1}$, by Lemma 3.2 of [14] we find finite dimensional subspaces $V = X_1^* \oplus \cdots \oplus X_{m_0}$
and $W = Y_1 \oplus \cdots \oplus Y_{n_0}$ of $X^*$ and $Y$ respectively, such that
\[
\|S_i P_V\| \leq \varepsilon_{n+1} \text{ and } \|P_W S_i\| \leq \varepsilon_{n+1} \text{ for all } i = 1, 2, \ldots, n.
\]
Since $P_V$ and $P_W$ are finite rank operators, it is easy to check that the operators $K \mapsto K P_V$ and $K \mapsto P_W K$ from $\mathcal{M}$ into $\mathcal{K}_{w^*}(X^*, Y)$ are DP1 (see for instance, Remark 2.3 of [1]). Thus by hypothesis on $(K_n)$ we have
\[
\|K_n P_V\| \to 0 \text{ and } \|P_W K_n\| \to 0 \text{ as } n \to \infty.
\]
So there exists an integer $N_1 > 0$ such that for all $j \geq N_1$:
\[
\|K_j P_V\| \leq \varepsilon_{n+1} \text{ and } \|P_W K_j\| \leq \varepsilon_{n+1}.
\]
On the other hand, the weak nullity of the sequences $(K_n)$ and $(\Gamma_n)$ imply the existence of two integers $N_2$ and $N_3$ such that
\[
|\langle K_j, \Lambda_i \rangle| < r 2^{-(n+1)} \text{ for all } i = 1, 2, \ldots, n, \text{ and all } j \geq N_2,
\]
\[
|\langle \Gamma_j, \Lambda_i \rangle| < r 2^{-(n+1)} \text{ for all } i = 1, 2, \ldots, n, \text{ and all } j \geq N_3.
\]
Now select an integer $j_0$ bigger than $N_1$, $N_2$ and $N_3$ and set $\Lambda_{n+1} = \Gamma_{j_0}$ and $S_{n+1} = K_{j_0}$. This finishes the induction process. We have constructed a subsequence $(\Lambda_n)$ of $(\Gamma_n)$ and a subsequence $(S_n)$ of $(K_n)$ such that for all integer $n$, there exist finite dimensional subspaces $V$ and $W$ of $X^*$ and $Y$ respectively, that satisfy all conditions of $(\star)$. These properties, as shown in [14], yield that
\[
\left\| P_W \sum_{i=1}^{n} S_i P_V - \sum_{i=1}^{n} S_i \right\| \leq 4n\varepsilon_{n+1} \text{ and } \|P_W S_{n+1} P_V - S_{n+1}\| \leq 5\varepsilon_{n+1}.
\]
Hence
\[
\left\| \sum_{i=1}^{n+1} S_i \right\| \leq \left\| \sum_{i=1}^{n} S_i - P_W \sum_{i=1}^{n} S_i P_V \right\| + \left\| S_{n+1} - P_W S_{n+1} P_V \right\| + \left\| P_W \sum_{i=1}^{n} S_i P_V + P_W S_{n+1} P_V \right\|
\]
\[
\leq (4n + 5)\varepsilon_{n+1} + \max \left\{ \left\| \sum_{i=1}^{n} S_i \right\|, 8 \right\}.
\]
Note that the last inequality holds by the $\mathcal{P}$-property of $\mathcal{M}$ and that $\|S_{n+1}\| \leq 2$. This shows that the sequence $T_n = \sum_{i=1}^{n} S_i$ is bounded and so has a weak*-cluster point $T \in \mathcal{M}^{**}$. For each $j$, choose an integer $n > j$ such that $|\langle T - T_n, \Lambda_j \rangle| < r 2^{-j}$. Therefore
\[
|\langle T, \Lambda_j \rangle| \geq |\langle T_n, \Lambda_j \rangle| - |\langle T - T_n, \Lambda_j \rangle|
\]
\[
\geq \left| \sum_{i=1}^{n} \langle S_i, \Lambda_j \rangle \right| - \frac{r}{2^j}.
\]
\[ |\langle S_j, \Lambda_j \rangle | - \sum_{i=j+1}^{n} |\langle S_i, \Lambda_j \rangle | - \frac{r}{2^j} \geq |\langle S_j, \Lambda_j \rangle | - \sum_{i=1}^{j-1} |\langle S_i, \Lambda_j \rangle | - \frac{r}{2^j} \geq r - \sum_{i=1}^{j-1} \frac{r}{2^i} - \sum_{i=j+1}^{n} \frac{r}{2^i} = r - r(\sum_{i=1}^{j-1} \frac{1}{2^i} + \sum_{i=j+1}^{n} \frac{1}{2^i}) = \frac{r}{2^j} \geq r - r(\sum_{i=2}^{\infty} \frac{1}{2^i}) = \frac{r}{2} - \frac{r}{2} + \frac{r}{3} > \frac{r}{3} > 0 \]

for sufficiently large \( j \). Hence \( \langle T, \Lambda_j \rangle \) and so \( \langle T, \Gamma_j \rangle \) does not tend to zero. Thus the sequence \( (\Gamma_j) \) does not converge weakly to zero, which gives a contradiction. □

Remark 2.5. Note that the proof of Lemma 3.2 of [14] is based on the fact that for each bounded and weak*-weak continuous operator \( K : X^* \to Y \), the adjoint operator \( K^* \) maps elements of \( Y^* \) into \( X \). So we need \( M \subseteq K_{w^*}(X^*, Y) \). In fact, under the same assumptions of Theorem 2.4, if \( M \subseteq K(X^*, Y) \), the conclusion of Lemma 3.2 of [14] is false. However, under the same assumptions on \( X \) and \( Y \), a similar result can be inferred for closed subspaces of \( K(X,Y) \):

**Theorem 2.6.** Let \( X \) and \( Y \) have monotone FDDs, such that the decomposition of \( X \) is shrinking. Let \( M \) be a closed subspace of \( K(X,Y) \) which has the \( \mathcal{P} \)-property. If all of the evaluation operators \( \phi_x \) and \( \psi_y^* \) are DP1 operators, then \( M \) has the DP1 property.

If \( X \) is an \( l_p \)-direct sum and \( Y \) is an \( l_q \)-direct sum of Banach spaces with \( 1 < p \leq q < \infty \), or \( X \) has a Schauder decomposition and \( Y \) is a \( c_0 \)-direct sum of Banach spaces, then the proof of Corollaries 3.5 and 3.6 of [14] shows that \( K(X,Y) \) (resp. \( K_{w^*}(X^*, Y) \)) and so its closed subspace \( M \) has the \( \mathcal{P} \)-property. So we have the following two corollaries:

**Corollary 2.7.** Let \( X \) be an \( l_p \)-direct sum and \( Y \) be an \( l_q \)-direct sum of finite dimensional Banach spaces and \( 1 < p \leq q < \infty \). If \( M \) is a closed subspace of \( K(X,Y) \) such that all evaluation operators \( \phi_x \) and \( \psi_y^* \) are DP1 operators, then \( M \) has the DP1 property.

**Corollary 2.8.** Let \( X \) have a monotone shrinking FDD and \( Y \) be a \( c_0 \)-direct sum of finite dimensional Banach spaces. If \( M \) is either a closed subspace of \( K(X,Y) \) or \( K_{w^*}(X^*, Y) \) such that all of the corresponding evaluation operators are DP1, then \( M \) has the DP1 property.

Remark 2.9. The proof of Theorem 2.4 is based on the facts that for each two Banach spaces \( X \) and \( Y \) with monotone (shrinking) FDDs, suitable closed subspaces of them are complemented and Lemma 3.2 of [14] is valid. By [3], in
the Hilbert space setting, a lemma similar to that lemma is valid; every closed subspace of a Hilbert space is complemented and an inequality similar to that of the definition of $P$-property holds for operators between two Hilbert spaces. So by a proof similar to Theorem 2.4 one can prove the following theorem:

**Theorem 2.10.** Let $H_1$ and $H_2$ be two Hilbert spaces and $M$ be a closed subspace of $K(H_1, H_2)$. Then $M$ has the DP1 property if and only if all of the evaluation operators $\phi_x$ and $\psi_y$ are DP1 operators.

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**References**


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