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MULTIPLICITY RESULTS FOR SOME FOURTH ORDER ELLIPTIC EQUATIONS

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Abstract. In this paper we consider the Dirichlet problem for an fourth order elliptic equation on a open set in R^N. By using variational methods we obtain the multiplicity of nontrivial weak solutions for the fourth order elliptic equation.

1. Introduction

In recent years, multiplicity of solutions for fourth order elliptic equations have been widely studied. In [5] the authors Lazer and McKenna proved the existence of $2k-1$ solutions when $\Omega \subset R$ is an interval and $b > \lambda_k(\lambda_k - c)$, for the assumption of $f(x,u) = b(u+1)^+ - 1$ by global bifurcation method, for the same $f(x,u)$.Tarantello [10] showed by degree theory that if $b \geq \lambda_1(\lambda_1 - c)$, then fourth order elliptic equation has a solution u such that $u(x) < 0$ in Ω, for $f(x,u) = (u+1)^+ - 1$ when $c < \lambda_1$. Choi and Jung [2] showed that fourth order elliptic equation has only the trivial solution when $\lambda_k < c < \lambda_{k+1}$ and the nonlinear term is $bu^+(b < \lambda_1(\lambda_1 - c))$. Micheletti and Pistoia [5] showed that fourth order elliptic equation has at least two solutions when $c > \lambda_1$ and the nonlinear term is $b[(u+1)^+ - 1](b < \lambda_1(\lambda_1 - c))$. The other authors in [1,3,4,6,7,8,9] studied the existence of multilple solutions of the semilinear problems with Dirichlet boundary condition.

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In this paper we will study fourth order elliptic problem, when the nonlinearity is replaced by a more general function $\alpha u + f(u)$, by using a variational method.

2. Preliminary results

We consider the problem of the multiplicity of solutions of the fourth order elliptic equation:

$$\begin{aligned} &\Delta^2 u + c\Delta u = \alpha u + f(u) \qquad \text{in } \Omega, \\ &u = 0, \quad \Delta u = 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a smooth open boundary set in R^N, $f : \Omega \times R \to R$ is a Caratheodory's function and $c, \alpha \in R$. We will consider the Hilber space $H = H^2(\Omega) \cap H_0^1(\Omega)$ and for every u and v in H we will set $(u,v)_H = \int \Delta u \Delta v + \int \nabla u \nabla v$. Then H is a closed subspace of $H^2(\Omega)$.

In order to study problem (2.1), we will follow a variational approach. Consider

$$I(u) := \frac{1}{2}\left(\int (\Delta u)^2 - c\int |\nabla u|^2\right) - \frac{\alpha}{2}\int u^2 + \int F(u) \tag{2.2}$$

where $F(u) = \int_0^u f(\sigma)d\sigma$.

Let $C^1(H,R)$ denote the set of all functionals which are Fréchet differentiable and whose Fréchet derivatives are continuous on H. It is easy to prove that I is a C^1 functional and its critical points are weak solutions of problem (2.1). We respectively denote by $(\Lambda_k)_{k\in N}$ and by $(e_k)_{k\in N}$ the eigenvalues and the eigenfunctions of the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = \Lambda u \qquad \text{in } \Omega, \\ &u = 0, \quad \Delta u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

Linking Theorem is of importance in critical point theory. Let E be a Banach space. We introduce the set Φ of mapping $\Gamma(t) \in C(E\times[0,1], E)$ with the following properties:

- (a) for each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of E onto itself and $\Gamma(t)^{-1}$ is continuous on $E \times [0,1)$
- (b) $\Gamma(0) = I$
- (c) for each $\Gamma(t) \in \Phi$ there is a $u_0 \in E$ such that $\Gamma(1)u = u_0$ for all $u \in E$ and $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of E.

A subset A of E links a subset B of E if $A \cap B = \emptyset$ and for each $\Gamma(t) \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$. We define the following sets.

- $S_\rho(Y) = \{x \in Y \mid \|x\| = \rho\}$,
- $\Delta_\rho(k,s) = \{u+v | u \in H_k, v \in span(e_k, \cdots e_s), \|u+v\| \leq \rho\}$,
- $\Sigma_\rho(k,s) = \{u+v | u \in H_k, v \in span(e_k, \cdots e_s), \|u+v\| = \rho\} \cup \{v \mid u \in H_k,\ \|u\| \leq \rho\}$.

Then the set $S_\rho(H_s)$ and $\Sigma_\rho(k,s)$ is linking set.

We will use the following assumptions:

- (f1) $\frac{F(u)}{u^2} \to 0$ as $|u| \to \infty$ uniformly for $x \in \Omega$;
- (f2) $\lim\limits_{\|u\|_H \to 0} \int \frac{F(u)}{\|u\|^2{}_H} = 0$:

The following is the main result of this paper.

Theorem 2.1. *Assume that* (f1),(f2). *Suppose that* $\Lambda_k \leq \alpha < \Lambda_{k+1}$ *and* $c < \Lambda_1$. *Then there exists a nontrivial critical point* u *of* I *which is a forcing solution of problem* (2.1).

Theorem 2.2. *Assume that* (f1),(f2). *Suppose that for a given* k *in* N *one has* $\Lambda_k < \Lambda_{k+1} \leq \Lambda_1$. *Then there exist positive constant* δ *such that if* $\Lambda_k - \delta < \alpha < \Lambda_k$, *problem* (2.1) *has at least 2 nontrivial solutions.*

3. Proof of Theorem 2.1 and Theorem 2.2

Definition 3.1. *We say* G *satisfies the (PS) condition if any sequence* $\{u_k\} \subset H$ *for which* $G(u_k)$ *is bounded and* $G'(u_k) \to 0$ *as* $k \to \infty$ *possesses a convergent subsequence.*

The (PS) condition is a convenient way to build some "compactness" into the functional G. Indeed observe that (PS) implies that $K_c \equiv \{u \in H \mid G(u) = c \text{ and } G'(u) = 0\}$, i.e. the set of critical points having critical value c, is compact for any $c \in R$. In this problem the functional I satisfies the (PS) conditions.

Lemma 3.2. *Assume that* $\alpha \neq \Lambda_i$.*Then* $I(u)$ *satisfies the* $(PS)_c$ *condition for every* $c \in R$.

Proof. Let (u_k) be a sequence in H with $DI(u_k) \to 0$ and $I(u_k) \to c$. It is enough to show that $||u_k||$ is bounded, since $\forall u \in H$

$$\nabla I(u) = u + i^*[(1+c)\Delta u - \alpha u + g(u)]. \tag{3.1}$$

where $i^* : L^2(\Omega) \to H$, the adjoint of the immersion $i : H \to L^2(\Omega)$ is a compact operator. In fact, if $\{u_k\}_{k=1}^{\infty} \subset H$, then u_k converses strongly in $L^k(\Omega)$ By contradiction we suppose that $\lim\limits_k \|u_k\|_H = +\infty$. Up to a subsequence we can assume that $\lim\limits_k \frac{u_k}{\|u_k\|_H} = u$ weakly in H, strongly in $L^2(\Omega)$ and pointwise in Ω. Note that dividing $I(u_n)$ by $||u_n||$ and passing to the limit, we get $\int u^- dx = 0$, and so $u \geq 0$ a.e. in Ω and $u \not\equiv 0$. On the other hand from $\nabla I(u_k) \to 0 \quad in \quad H$, we get

$$\lim_{k \to \infty} \frac{\nabla I(u_k)}{\|u_k\|_H} = 0 \quad as \quad n \to \infty.$$

So the bounded sequence $\lim\{\frac{u_k}{\|u_k\|_H}\}_{k \in N}$ converges strongly in H. Hence

$$u - i^*[(1+c)\Delta u - \alpha u] = 0.$$

Here $i^* : L^2(\Omega) \to H$ is a compact operator. This implies that $u \geq 0$ is a nontrivial solution of

$$\Delta^2 u + c\Delta u = \alpha u, \tag{3.2}$$

which contradicts to the equation (3.2) ($\alpha \neq \Lambda_i(c), \alpha \neq 0$) that has only the trivial solution. So we discovered that $\{u_k\}_{k=1}^{\infty}$ is bounded in H, hence there exists a subsequence $\{u_{kj}\}_{kj=1}^{\infty}$ and $u \in H$ with $u_{kj} \to u$ in H. □

Proof of Theorem 2.1. Since $I(u) \leq \frac{\Lambda_k - \alpha}{2} \int u^2 dx$ for $\forall u$ in H_k and $I(0) = 0$. So we have $\sup\limits_{H_k} I(u) = 0$. For any $\epsilon > 0$ there exists $\rho > 0$ such that, if $\|u\| \leq \rho$,

$$I(u) \geq C\|u\|^2 - \epsilon\|u\|^2,$$

where

$$C = \inf_{n \geq k+1} \frac{\lambda_n^2 - c\lambda_n - \alpha}{\lambda_n^2}.$$

So we have

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \inf_{u \in H_k^{\perp}, \|u\| = \rho} i(u) \geq C.$$

This implies that there exist R and ρ such that $R > \rho > 0$ and

$$\inf_{S_k(\rho)} I(u) > \sup_{\Sigma(H_k, e_1)} I(u).$$

In this way the hypotheses of the Linking theorem are satisfied, so there exists a critical point u such that

$$0 < \inf_{s_k(\rho)} I(u) < I(u) < \underset{\Delta(H_k,e_1)}{I(u)} .$$

$\square$

LEMMA 3.3. *Suppose that for given s and k in N $\Lambda_k < \Lambda_{k+1} \leq \ldots \leq \Lambda_s < \Lambda_{s+1} \leq \Lambda_1$ and $(f2)$, then*

$$\sup_{\|u\|=\rho, u\in H_s} I(u) < 0.$$

Proof. For sufficiently small $\|u\|$ we have,

$$\begin{aligned} I(u) &\leq \frac{1}{2}\left(\int (\Delta u)^2 - c|\nabla u|^2\right) - \frac{1}{2}\int \alpha u^2 + O\|u\| \\ &\leq \frac{1}{2}(\Lambda_s(c)u^2 - \alpha)\int u^2 + O\|u\| \end{aligned}$$

for some positive constant $\alpha > \Lambda_s(c)$. The norms $\|\cdot\|_{H_s}$ and $\|\cdot\|_{L^2(\Omega)}$ in H_s are equivalent, since dim $H_s = s$. Condition $\alpha > \Lambda_s(c)$ implies that $\Lambda_s(c)u^2 - \alpha < 0$. So, for small $\rho > 0$ we have

$$\sup_{\|u\|=\rho, u\in H_s} I(u) < 0.$$

$\square$

LEMMA 3.4. *Suppose that for given s and k in N, $\Lambda_k < \Lambda_{k+1} \leq \ldots \leq \Lambda_s < \Lambda_{s+1} \leq \Lambda_1$ and $\Lambda_k \leq \alpha < \Lambda_{s+1}$, $f(1)$ and set $X_{(k,s)} = H_k \oplus {H_s}^{\perp}$. Then for every $\delta > 0$,if $\Lambda_k + \delta \leq \alpha \leq \Lambda_{s+1} - \delta$,*

$$\sup_{\|u\|=R, u\in\Sigma_{\rho(k,s)}\subset X_{(k,s)}} I(u) < 0.$$

Proof. Set $K_\phi = \{u \in H | u \geq \phi\}$. There exists $\rho > 0$ such that, if $u \in K_\phi \cap X_{(k,s)}$, $\|u\| < \rho$, $u \neq 0$ and $\Lambda_s \leq \alpha < \Lambda_{k+1}$, then u is not an upper critical point for I_α on $X_{(s,k)}$.

In fact if ρ is small enough then $B(0,\rho) \subset K_\phi$. On the other hand the unique upper critical point for I on $X_{(k,s)}$ is o, since $\Lambda_k \leq \alpha < \Lambda_{s+1}$. So the argument holds for some large $\rho > 0$. $\square$

Proof of Theorem 2.2. Since $\Lambda_s \leq \alpha < \Lambda_{k+1}$ and f satisfies (f1),(f2) by Lemma 3.3 and 3.4 there exist $R > \rho > 0$ such that

$$\sup_{\|u\|=\rho, u\in H_s} I(u) < 0 < \sup_{\|u\|=R, u\in \Sigma_{\rho(k,s)}\subset X_{(k,s)}} I(u),$$

where $\Sigma_\rho(k,s) = \{u+v | u \in H_k, v \in span(e_k, \cdots e_s), \|u+v\| = \rho\} \cup \{v \mid u \in H_s,\ \|u\| \leq \rho\}$. By the Variational Linking Theorem $I(u)$ has at least two nonzero critical values c_1, c_2 such as

$$c_1 \leq \sup_{\|u\|=\rho, u\in H_s} I(u) < 0 < \sup_{\|u\|=R, u\in \Sigma_{\rho(k,s)}\subset X_{(k,s)}} I(u) \leq c_2.$$

Therefore, (2.1) has at least two nontrivial solutions. This implies that (2.1) has at least three solutions. □

4. Variational setting

We introduce a variational linking theorem.

THEOREM 4.1 (a Variation of Linking). *Let X be a Hilbert space which is topological direct sum of the subspaces X_1, X_2. Let $f \in C^1(X,R)$. Moreover assume*

(a) $dim X_1 < +\infty$,

(b) there exist $\rho > 0$, $R > 0$ and $e \in X_1$, $e \neq 0$ such that $\rho < R$ and $\sup_{S_\rho(X_1)} f < \inf_{\Sigma_R(e,X_2)} f$,

(c)$-\infty < a = \inf_{\Delta_R(e,X_2)} f$,

(d) $(PS)_c$ condition holds for any $c \in [a,b]$ where $b = \sup_{B_\rho(X_1)} f$. Then there exist at least two critical levels c_1 and c_2 for the functional f such that

$$\inf_{\Delta_R(e,X_2)} f \leq c_1 \leq \sup_{S_\rho(X_1)} f < \inf_{\Sigma_R(e,X_2)} f \leq c_2 \leq \sup_{B_\rho(X_1)} f.$$

Let $0 < \delta < R$, $e_1 \in M_1$ moreover, consider

$$Q_R = \{se_1 + u : u \in M_2, s \geq 0 \|se_1 + u\| \leq R\},$$

$$S_\delta = B_\delta \cap M_1,$$

then ∂Q_R links ∂S_δ.

We recall a theorem of existence of two critical levels for a functional which is a linking theorem on product space.

Theorem 4.2. *Suppose*

$$\sup_{\partial S_\delta \times V} I < \inf_{\partial Q_R \times V} I$$

$$\inf_{Q_R \times V} I > -\infty, \qquad \sup_{S_\delta \times V} I < +\infty,$$

and that I satisfies $(PS)^*_c$ with respect to X, for every

$$c \in [\inf_{Q_R \times V} I, \sup_{S_\delta \times V} I].$$

Then I admits at least two distinct critical values c_1, c_2 such that

$$\inf_{Q_R \times V} I \leq c_1 \leq \sup_{\partial S_\delta \times V} I < \inf_{\partial Q_R \times V} I \leq c_2 \leq \sup_{S_\delta \times V} I,$$

and at least $2 + 2$ cuplength(V) distinct critical points.

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